

# Stochastic Hybrid Systems with Renewal Transitions

Duarte Antunes, João P. Hespanha, and Carlos Silvestre

**Abstract**— We consider Stochastic Hybrid Systems (SHSs) for which the lengths of times that the system stays in each mode are independent random variables with given distributions. We propose an analysis framework based on a set of Volterra renewal-type equations, which allows us to compute any statistical moment of the state of the SHS. Moreover, we provide necessary and sufficient conditions for various stability notions, and determine the exponential decay or increase rate at which the expected value of a quadratic function of the systems' state converges to zero or to infinity, respectively. The applicability of the results is illustrated in a networked control problem considering independently distributed intervals between data transmissions and delays.

## I. INTRODUCTION

A Hybrid System is a system with both continuous dynamics and discrete logic. Its execution is specified by the dynamic equations of the continuous state, a set of rules governing the transitions between discrete modes, and reset maps determining jumps of the state at transition times.

Stochastic Hybrid Systems (SHSs) introduce randomness in the execution of the Hybrid System. As surveyed in [1], various models of SHSs have been proposed differing on where randomness comes into play. In [2] and [3], the continuous dynamics are driven by a Wiener process. In [3] and [4], the transitions are triggered by stochastic events, and [4] allows transitions for which the next state is chosen according to a given distribution. See also [5], [6], [7] and the references therein for additional work on stochastic hybrid systems.

The present work follows closely the definition of a SHS given in [5], where randomness arises solely from the fact that the rate at which the ~~stochastically spaced~~ transitions occur is allowed to depend on both the continuous state and on the discrete modes. In [5], a formula is provided for the extended generator of the Markov process associated with the SHS, which, as explained in [4, Ch.1], completely characterizes the Markov process. The generator can be used to compute expectations and probabilities by using the Dynkin's formula, ~~in some problems with special structure. This approach is followed in [4] for various cases of interest.~~ It is also the starting point of the method proposed in [5] to compute the moments of the sending rate for an on-off TCP flow model.

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D. Antunes and C. Silvestre are with the Dep. of Electrical Eng. and Computer Science, Instituto Superior Técnico, ISR, 1046-001 Lisboa, Portugal. {dantunes, cjs}@isr.ist.utl.pt

João P. Hespanha is with the Dept. of Electrical and Computer Eng., University of California, Santa Barbara, CA 93106-9560, USA. hespanha@ece.ucsb.edu

The approach based on the Dynkin's formula is also used in [8, Ch. 2] to compute the moments of the state of a Markov Jump Linear System (MJLS). This class of systems can be viewed as a special case of a stochastic hybrid system, in which the length of times that the system stays in each discrete mode are independent *exponentially distributed* random variables, with a mean that may depend on the discrete mode.

In the present work we consider stochastic hybrid system with linear dynamics, linear reset maps, and for which the lengths of times that the system stays in each mode are independent *arbitrarily distributed* random variables, whose distributions may depend on the discrete mode. The process that combines the transition times and the discrete mode is called a Markov Renewal process [9], which motivated us to refer to these systems as *stochastic hybrid systems with renewal transitions*. This class of systems can be viewed as a special case of the SHS model in [5], or as a generalization of a MJLS.

A key challenge in the analysis of the class of systems considered here lies in the fact that the discrete component of the state is a semi-Markov process [9], ~~instead of a Markov chain, which would be in the case of MJLS.~~ This prevents the direct use of approaches based on the Dynkin's formula to compute the statistical moments of the continuous state of the system, which is possible for exponential distributions in the durations of each mode [8, Ch. 2].

Inspired by our recent work on impulsive renewal systems [10], the approach followed here to analyze SHS with renewal transitions is based on a set of Volterra renewal-type equations. This allows us to give expressions for any moment of the state of the SHS. Moreover, we characterize the asymptotic behavior of the system by providing necessary and sufficient conditions for various stability notions in terms of LMJ, algebraic and Nyquist criterion conditions. We also determine the decay or increase rate at which the expected value of a quadratic function of the systems' state converges exponentially to zero or to infinity, depending on whether or not the system is mean exponentially stable. The proofs are omitted due to space limitations, but can be found in [11].

The applicability of our theoretical results is illustrated in a networked control problem. We consider independently distributed intervals between data transmissions of the control signal in a feedback-loop, which, as observed in [10], is a reasonable assumption in networked control systems utilizing CSMA protocols. The impulsive renewal systems considered in [10] did not permit us to consider the effect of network induced delays, which is now possible with SHSs with renewal transitions.

The remainder of the paper is organized as follows. SHSs with renewal transitions are defined in Section II. In Section III, we compute the statistical state moments of the SHS, and in Section IV we provide an asymptotic analysis of the SHS. Section V addresses the applicability of the results in a networked control example. In Section VI we draw final conclusions and address future work.

*Notation and Preliminaries:* For dimensionally compatible matrices  $A$  and  $B$ , we define  $(A, B) := [A' B']'$ . The  $n \times n$  identity and zero matrices are denoted by  $I_n$  and  $0_n$ , respectively. For a given matrix  $A$ , its transpose is denoted by  $A'$ , its spectral radius by  $\sigma(A)$ , and an eigenvalue by  $\lambda_i(A)$ . The Kronecker product is denoted by  $\otimes$ . For a complex number  $z$ ,  $\Re[z]$  and  $\Im[z]$  denote the real and complex parts of  $z$ , respectively. The notation  $x(t_k^-)$  indicates the limit from the left of a function  $x(t)$  at the point  $t_k$ . The probability space is denoted by  $(\Omega, \mathbf{B}, \mathbf{P})$  and the expected value is denoted by  $\mathbb{E}(\cdot)$ . The indicator function of a set  $A$  is denoted by  $\mathbf{1}_{x \in A}$ , which equals 1 if  $x \in A$  and 0 otherwise. A probability distribution  $F$  with support on a given interval  $[0, T]$ ,  $T \in \mathbb{R}_{>0} \cup \{+\infty\}$  takes the form  $F = F^c + F^d$ , where  $F^c$  is an absolutely continuous function  $F^c(\tau) = \int_0^\tau f(r)dr$ , for some density function  $f(r) \geq 0, \forall r \geq 0$ , and  $F^d$  is a piecewise constant increasing function that captures possible atom points  $\{b^j > 0\}$  where the cumulative distribution places mass  $\{w^j\}$ . Notice that, the integral with respect to the monotone function  $F$ , in general parametric on some variable  $y$ , is then given by

$$G(y) = \int_0^T W(y, \tau) F(d\tau) = G^c(y) + G^d(y), \quad (1)$$

where  $G^c(y) = \int_0^T W(y, \tau) f(\tau) d\tau$ ,  $G^d(y) = \sum_j w^j W(y, b^j)$ , and  $W(y, \tau)$  is generally a matrix-valued function that depends on the integration variable  $\tau$  and generally also on some parameter  $y$ . We write  $G(y) < \infty$  to say that  $G(y)$  exists for a given  $y$ , i.e., the integral exists as an absolutely convergent Lebesgue integral.

## II. SHS WITH RENEWAL TRANSITIONS

A SHS with renewal transitions, is defined by a linear differential equation

$$\dot{x}(t) = A_{q(t)} x(t), \quad x(0) = x_0, q(0) = q_0, t_0 = 0, \quad (2)$$

where  $x(t) \in \mathbb{R}^n$ ; and  $q(t) \in \mathcal{Q} := \{1, \dots, n_q\}$ ; a family of  $n_l$  discrete transition/reset maps

$$(q(t_k), x(t_k)) = (\xi_l(q(t_k^-)), J_{q(t_k^-), l} x(t_k^-)), \quad (3)$$

$$l \in \mathcal{L} := \{1, \dots, n_l\},$$

where  $\xi_l$  is a map  $\xi_l : \mathcal{Q} \mapsto \mathcal{Q}$  and  $J_{q(t_k^-), l}$  belongs to the set  $\{J_{i,l} \in \mathbb{R}^{n \times n}, i \in \mathcal{Q}, l \in \mathcal{L}\}$ ; and a family of transition distributions

$$F_{i,l}, \quad i \in \mathcal{Q}, l \in \mathcal{L}. \quad (4)$$

Between transition times  $t_k$ , the discrete mode  $q$  remains constant whereas the continuous state  $x$  flows according to (2). At transition times, the continuous state and discrete mode of the SHS are reset according to (3). The intervals

between transition times are independent random variables determined by the transition distributions (4) as follows. A transition distribution can be either a probability distribution or identically zero. In the former case,  $F_{i,l}$  is the probability distribution of the random time that transition  $l \in \mathcal{L}$  takes to trigger in the state  $q(t) = i \in \mathcal{Q}$ . The next transition time is determined by the minimum of the triggering times of the transitions associated with state  $q(t) = i \in \mathcal{Q}$ . When  $F_{i,l}(\tau) = 0, \forall \tau$ , the transition  $l$  does not trigger in the state  $i \in \mathcal{Q}$ , i.e., the reset maps may be specified only for a subset of the states.

When the transition distributions have discrete parts, different transitions may trigger at the same time with probability different from zero, leading to an ambiguity in choosing the next state. Thus, to guarantee that with probability 1, this does not occur, we assume the following. As in (1), let  $F_{ij}$  be decomposed as  $F_{i,l}^c(\tau) = \int_0^\tau f_{i,l}(r)dr$ ,  $f_{i,l}(r) \geq 0, \forall r$ , and  $F_{i,l}^d$ , and let  $F_{ij}$  have support on a given interval  $[0, T_{i,l}]$ ,  $T_{i,l} \in \mathbb{R}_{>0} \cup \{+\infty\}$ .

*Assumption 1:* For every  $i \in \mathcal{Q}$ , the (possible) atom points of the transition distributions have a finite number of (possible) atom points that verify  $b_{i,l_1}^j \neq b_{i,l_2}^k$  for  $l_1 \neq l_2, l_1 \in \mathcal{L}, l_2 \in \mathcal{L}, \forall j, k$ .

We also assume that  $F_{i,l}(0) = 0, \forall i \in \mathcal{Q}, l \in \mathcal{L}$ .

The construction of a sample path of the SHS with renewal transitions can be described as follows.

- 1) Set  $k = 0, t_0 = 0, (q(t_k), x(t_k)) = (q_0, x_0)$ .
- 2) For every  $j \in \mathcal{L}$ , obtain  $\bar{h}_k^j$  as a realization of a random variable distributed according to  $F_{q(t_k), j}$ , if  $F_{q(t_k), j}$  is not identically zero, and set  $\bar{h}_k^j = \infty$  otherwise.
- 3) Take  $h_k = \min\{\bar{h}_k^j, j \in \mathcal{L}\}$  and set the next transition time to  $t_{k+1} = t_k + h_k$ . The state of the SHS in the interval  $t \in [t_k, t_{k+1})$  is given by  $(q(t), x(t)) = (q(t_k), e^{A_{q(t_k)} t} x(t_k))$ .
- 4) In the case where  $t_{k+1} < \infty$ , take the index  $l_k$  of the triggered transition, i.e.,  $l_k = j : h_k = \bar{h}_k^j$  and update the state according to  $(q(t_{k+1}), x(t_{k+1})) = (\xi_{l_k}(q(t_{k+1}^-)), J_{q(t_{k+1}^-), l_k} x(t_{k+1}^-))$ . Set  $k = k + 1$  and repeat the construction from the step 2.

Notice that due to the Assumption 1, in step three the probability of existing two or more indexes  $j$  for which  $h_k = \bar{h}_k^j$  is zero.

## III. STATE MOMENTS

Consider an  $m$ -th order uncentered moment of the state of the SHS with renewal transitions, i.e.,

$$\mathbb{E}[x_{i_1}(t)^{m_1} x_{i_2}(t)^{m_2} \dots x_{i_r}(t)^{m_r}], \quad (5)$$

where  $1 \leq i_1, \dots, i_r \leq n, \sum_{j=1}^r m_j = m, m_j > 0$  and  $x_i(t) \in \mathbb{R}$  denotes the  $i$ th component of  $x(t)$ . Let  $A^{(m)}$ ,  $m \geq 0$ , denote the  $m$ th fold Kronecker product of a matrix  $A$  with itself  $\underbrace{A \otimes A \dots \otimes A}_m$ ,  $A^0 := I$  and recall that (cf. [12])

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD). \quad (6)$$

Using (6), we can write (5) as

$$\mathbb{E}[c'x(t)^{(m)}], \quad (7)$$

where

$$c = e_{i_1}^{(m_1)} \otimes e_{i_2}^{(m_2)} \otimes \dots \otimes e_{i_r}^{(m_r)}, \quad (8)$$

and  $e_j \in \mathbb{R}^n$  is the canonical basis vector whose components equal zero except component  $j$  that equals 1. Furthermore, let  $\Phi_j(t)$  denote the transition matrix of the SHS starting at the discrete mode  $q_0 = j$ , i.e.,  $x(t) = \Phi_{q_0}(t)x_0$  where

$$\Phi_j(t) = e^{A_{q_0}(t-t_r)} \dots J_{\xi_{i_0}(j), \xi_{i_1}(\xi_{i_0}(j))} e^{A_{\xi_{i_0}(j)}} J_{j, \xi_{i_0}(j)} e^{A_j t},$$

$r = \max\{k \in \mathbb{Z}_{\geq 0} : t_k \leq t\}$ , and let

$$z_i^m(t) := \mathbb{E}[(\Phi_i(t)')^{(m)}c], \quad i \in \mathcal{Q}. \quad (9)$$

We can express (5), or equivalently (7), in terms of  $z_i^m(t)$ ,  $i \in \mathcal{Q}$ . In fact, for an initial condition  $(x_0, q_0)$ ,

$$\begin{aligned} \mathbb{E}[c'x(t)^{(m)}] &= \mathbb{E}[(x(t)')^{(m)}c] \\ &= (x_0')^{(m)} \mathbb{E}[(\Phi_{q_0}(t)')^{(m)}c] = (x_0')^{(m)} z_{q_0}^m(t), \end{aligned} \quad (10)$$

where we used (6).

The next theorem states that

$$z^m(t) := (z_1^m(t), \dots, z_{n_q}^m(t)) \quad (11)$$

verifies a Volterra renewal-type equation. We define

$$K_m(z^m(t)) := (K_{m,1}(z^m(t)), \dots, K_{m,n_q}(z^m(t))),$$

where  $K_{m,i}$  is a convolution operator defined as

$$K_{m,i}(z^m(t)) := \sum_{l=1}^{n_l} \int_0^t (E_{i,l}(\tau)')^{(m)} z_{\xi_l(i)}^m(t-\tau) \frac{S_i(\tau)}{S_{i,l}(\tau)} F_{i,l}(d\tau), \quad (12)$$

where  $i \in \mathcal{Q}$ ,  $S_i(\tau) := \prod_{l=1}^{n_l} S_{i,l}(\tau)$ ,  $S_{i,l}(\tau) := 1 - F_{i,l}(\tau)$  and  $E_{i,l}(\tau) := J_{i,l} e^{A_i \tau}$ .

*Theorem 2:* The function (11) satisfies

$$z^m(t) = K_m(z^m(t)) + h^m(t), \quad t \geq 0 \quad (13)$$

where  $h^m(t) := (h_1^m(t), \dots, h_{n_q}^m(t))$ , and

$$h_i^m(t) = (e^{A_i t})^{(m)} c S_i(t), \quad i \in \mathcal{Q}. \quad (14)$$

□

As stated in the next result, we can obtain an explicit expression for  $z^m(t)$ , and therefore for any  $m$ th order uncentered moment of the state of the SHS due to (10). We denote by  $K_m^j$  the composite operator obtained by applying  $j$  times  $K_m$ , e.g.,  $K_m^2(y(t)) = K_m(K_m(y(t)))$  and say that a function defined in  $\mathbb{R}_{\geq 0}$  is locally integrable if its integral is finite in every compact subset of  $\mathbb{R}_{\geq 0}$ .

*Theorem 3:* There exists a unique locally integrable solution to (13) given by

$$z^m(t) = \sum_{j=1}^{\infty} K_m^j(h^m(t)) + h^m(t), \quad t \geq 0. \quad (15)$$

□

## IV. ASYMPTOTIC ANALYSIS

Even without explicitly computing the solution to the Volterra equation (13), it is possible to characterize its asymptotic behavior through a frequency-domain analysis. In this section, we restrict our attention to second-order moments and consider the expected value of a quadratic function of the systems' state  $\mathbb{E}[x(t)'x(t)]$ , which is the key quantity in the stability notions that we consider in the present paper.

Let  $\nu$  denote the operator that transforms a matrix into a column vector  $\nu(A) = \nu([a_1 \dots a_n]) = [a_1' \dots a_n']'$ , and recall that

$$\nu(ABC) = (C' \otimes A)\nu(B). \quad (16)$$

Using (16) we can write the quadratic function of interest as

$$\mathbb{E}[x(t)'x(t)] = \mathbb{E}[x(t)' \otimes x(t)']\nu(I_n), \quad (17)$$

That is, we assume  $m = 2$  and  $c = \nu(I_n)$ , in (7).

To achieve the asymptotic analysis of this function, we introduce the following complex function

$$\hat{K}(z) := \begin{bmatrix} \hat{K}_{1,1}(z) & \dots & \hat{K}_{1,n_q}(z) \\ \vdots & \ddots & \vdots \\ \hat{K}_{n_q,1}(z) & \dots & \hat{K}_{n_q,n_q}(z) \end{bmatrix},$$

where

$$\hat{K}_{i,j}(z) := \sum_{l=1}^{n_l} \int_0^{T_{i,l}} (E_{i,l}(\tau)')^{(2)} e^{-z\tau} \mathbf{1}_{\xi_l(i)=j} F_{i,l}(d\tau), \quad (18)$$

which can be partitioned as in (1),  $\hat{K}_{i,j}(z) = \hat{K}_{i,j}^c(z) + \hat{K}_{i,j}^d(z)$ . Likewise, we partition  $\hat{K}(z)$  as  $\hat{K}(z) = \hat{K}_c(z) + \hat{K}_d(z)$ , where  $\hat{K}_c(z)$  is a matrix with blocks  $\hat{K}_{i,j}^c(z)$ , and  $\hat{K}_d(z)$  is a matrix with blocks  $\hat{K}_{i,j}^d(z)$ . It turns out that the asymptotic stability of the Volterra equation (13) is equivalent to the following condition:

$$\det(I - \hat{K}(z)) \neq 0, \quad \forall z \in \mathbb{C} : \Re[z] \geq 0, \quad (19)$$

which will be instrumental in deriving the results that follow. In the sequel, we shall also provide computationally efficient methods to test (19).

To state the main result of this section we need the following technical conditions. We denote by  $\mathcal{C}(R, \epsilon)$  the region in the complex plane  $\mathcal{C}(R, \epsilon) := \{z : |z| > R, \Re[z] > -\epsilon\}$ .

(T1)  $\hat{K}(-\epsilon)$  exists for some  $\epsilon > 0$ .

(T2)  $\inf_{z \in \mathcal{C}(R, \epsilon)} \{|\det(I - \hat{K}_d(z))|\} > 0$  for some  $\epsilon > 0, R > 0$ .

These conditions hold trivially when the transition distributions have bounded support ( $T_{i,l} < \infty$ ) and have no discrete component ( $F_{i,l}^d = 0$ ). However, we shall see below that they hold under much less stringent assumptions.

We consider the following three stability notions for the SHS with renewal transitions.

*Definition 4:* The SHS with renewal transitions is said to be

(i) *Mean Square Stable (MSS)* if for any  $(x_0, q_0)$ ,

$$\lim_{t \rightarrow +\infty} \mathbb{E}[x(t)'x(t)] = 0,$$

(ii) *Stochastic Stable (SS)* if for any  $(x_0, q_0)$

$$\int_0^{+\infty} \mathbb{E}[x(t)'x(t)]dt < \infty,$$

(iii) *Mean Exponentially Stable (MES)* if there exists constants  $c > 0$  and  $\alpha > 0$  such that for any  $(x_0, q_0)$

$$\mathbb{E}[x(t)'x(t)] \leq ce^{-\alpha t} x_0'x_0, \forall t \geq 0. \quad (20)$$

□

The following is the main result of the section. We denote by  $\bar{\lambda}(A_i)$  the real part of the eigenvalues of  $A_i$ ,  $i \in \mathcal{Q}$  with largest real part and  $\bar{m}(A_i)$  the dimension of the largest Jordan block associated with these eigenvalues.

*Theorem 5:* Suppose that (T1) and (T2) hold and consider the following condition

$$\det(I - \hat{K}(z)) \neq 0, \quad \Re[z] \geq 0. \quad (21)$$

The SHS with renewal transitions is

(i) MSS if and only if (21) holds and

$$e^{2\bar{\lambda}(A_i)t} t^{2\bar{m}(A_i)} S_i(t) \rightarrow 0 \text{ as } t \rightarrow \infty, \quad \forall i \in \mathcal{Q}; \quad (22)$$

(ii) SS if and only if (21) holds and

$$\int_0^\infty e^{2\bar{\lambda}(A_i)t} t^{2\bar{m}(A_i)} S_i(t) dt < \infty, \quad \forall i \in \mathcal{Q}; \quad (23)$$

(iii) MES if and only if (21) holds and

$$e^{2\bar{\lambda}(A_i)t} t^{2\bar{m}(A_i)} S_i(t) \leq ce^{-\alpha_1 t}, \text{ for } c > 0, \alpha_1 > 0, \forall i \in \mathcal{Q}. \quad (24)$$

□

As mentioned above, the condition (21) is a stability condition for the Volterra equation (13) and guarantees that the process sampled at the jump times converges to zero, whereas conditions (22), (23) and (24) pertain to the inter-jump behavior of the SHS with renewal transitions. In spite of ~~not being~~ equivalent in general, when the matrices  $A_i$ ,  $i \in \mathcal{Q}$  and the distributions  $F_{i,l}$ ,  $i \in \mathcal{Q}$ ,  $l \in \mathcal{L}$  are such that (24) holds, then (22) and (23) automatically hold and the three notions of stability are equivalent. Note that (T1) holds if  $T_{i,l} < \infty, \forall i \in \mathcal{Q}, l \in \mathcal{L}$  or if all  $A_i$  are Hurwitz, or, more generally, if for some  $\lambda > \bar{\lambda}_i(A)$ ,  $\forall i \in \mathcal{Q}$ , we have that

$$\int_0^{T_{i,l}} e^{2\lambda s} F_{i,l}(ds) < \infty, \quad \forall i \in \mathcal{Q}, l \in \mathcal{L}. \quad (25)$$

Moreover, whenever (T1) holds, the following proposition provides a simple condition to verify if (T2) holds.

*Proposition 6:* ~~Assuming that (T1) holds, the condition (T2) also holds provided that~~

$$\sigma(\hat{K}_d(0)) < 1.$$

□

When the SHS with renewal transitions is MES, we can characterize the exponential decay constant  $\alpha$  in (20) at

~~which the expected value of a quadratic function of the systems' state decreases, as shown in the next Theorem.~~

*Theorem 7:* If the SHS with renewal transitions is MES, the decay constant  $\alpha$  in (20) can be chosen in the interval  $\alpha \in [0, \alpha_{\max})$ , where  $\alpha_{\max} = \min\{\alpha_1, \alpha_2\}$ ,  $\alpha_1$  is such that (24) holds, and

$$\alpha_2 = \begin{cases} \infty, & \text{if } \det(I - \hat{K}(a)) \neq 0, \quad a \in (-\infty, 0), \\ \max\{a < 0 : \det(I - \hat{K}(a)) = 0\}, & \text{otherwise.} \end{cases}$$

□

Suppose that the transition distributions have finite support, in which case  $\alpha_1$  in (24) can be chosen arbitrarily large. Then this theorem tells us that the exponential decay constant of the expected value of the energy of the system is only limited by the largest real negative zero of  $\det(I - \hat{K}(z))$ , which can be obtained by performing a line search on the negative real axis.

Theorems 5 and 7 pertain to the asymptotic behavior of (13) when (21) holds. The following result characterizes the asymptotic behavior of the process when (21) does not hold. ~~Since  $m = 2$ , we drop the index  $m$  in the expressions for  $z^m(t)$  and  $h^m(t)$ , i.e.,  $\zeta(t) := z^2(t)$  and  $h(t) := h^2(t)$ , and denote the Laplace transform of  $h(t)$ , by  $\hat{h}(z) := \int_0^\infty h(t)e^{-zt}dt$ .~~

*Theorem 8:* Suppose that (T1), (T2) and (24) hold, but (21) does not hold. Then

(i) there is a finite number  $n_z$  of complex numbers  $z_i$  in  $\Re[z] \geq 0$  that satisfy

$$\det(I - \hat{K}(z_i)) = 0, \quad \forall i = 1, \dots, n_z;$$

(ii) one number, which we label  $z_1$ , is real and verifies  $\Re[z_i] \leq z_1, \forall i$ ;

(iii) the following holds

$$\zeta(t) = \sum_{i=1}^{n_z} \sum_{j=0}^{m_i-1} r_{i,j} t^j e^{z_i t} + \tilde{\zeta}(t) \quad (26)$$

where  $\tilde{\zeta}(t)$  tends to zero exponentially fast and the vectors  $r_{i,j}$  are such that in a neighborhood of  $z_i$ ,

$$[I - \hat{K}(z)]^{-1} \hat{h}(z) = \sum_{j=0}^{m_i-1} r_{i,j} \frac{j!}{(z - z_i)^{j+1}} + y_i(z), \quad (27)$$

where the  $y_i(z)$  are analytic vector-valued functions. Moreover, for at least one  $j$ ,  $r_{1,j} \neq 0$ .

□

In practice, this result shows that when the SHS is not stable with respect to the stability notions considered, the expected energy  $\zeta(t)$  of the SHS grows exponentially fast, according to a sum of exponential terms which are determined by the roots of  $\det(I - \hat{K}(z)) = 0$  and the residues of  $[I - \hat{K}(z)]^{-1} \hat{h}(z)$  in the neighborhood of these roots (singularities). The highest exponential rate is determined by the most positive real part of the roots of  $\det(I - \hat{K}(z_i))$ , which turns out to be a real root, ~~and cannot be a removable singularity.~~

The next result provides computationally efficient tests to verify whether or not the condition (21) holds.

*Theorem 9:* The following are equivalent

- (A)  $\det(I - \hat{K}(z)) \neq 0, \Re[z] \geq 0,$
- (B)  $\sigma(\hat{K}(0)) < 1,$
- (C) There exists a set of matrices  $P := \{P_i > 0, i \in \mathcal{Q}\}$  such that for every  $i \in \mathcal{Q}$

$$L_i(P) - P_i < 0,$$

where

$$L_i(P) := \sum_{j=1}^{n_q} \sum_{l=1}^{n_l} \int_0^{T_{i,l}} E_{i,l}(s)' P_j E_{i,l}(s) \mathbf{1}_{\xi_i(i)=j} \frac{S_i(s)}{S_{i,l}(s)} F_{i,l}(ds),$$

- (D) The solution to  $L_i(P) - P_i = -Q_i, i \in \mathcal{Q},$  (28)

is unique and verifies  $\{P_i > 0, i \in \mathcal{Q}\}$  if  $\{Q_i > 0, i \in \mathcal{Q}\}$  and  $\{P_i \geq 0, i \in \mathcal{Q}\}$  if  $\{Q_i \geq 0, i \in \mathcal{Q}\}$ .

□

The well-known Nyquist criterion can be used to check if (A) holds. The condition (B) is an algebraic condition, and (C) is an LMI condition since the left-hand side of (28) is an affine function of the decision variables.

## V. APPLICATION TO NETWORKED CONTROL

We consider the following simplified version of the networked control set-up that we considered in [10]. Suppose that we wish to control a linear plant

$$\dot{x}_P(t) = A_P x_P(t) + B_P \hat{u}(t). \quad (29)$$

A state feedback controller taking the form  $K_C x_P(t)$  is implemented digitally and the actuation is held constant  $\hat{u}(t) = \hat{u}(s_\kappa), t \in [s_\kappa, s_{\kappa+1})$  between actuation update times denoted by  $\{s_\kappa, \kappa \geq 0\}$ .

The controller has direct access to the state measurements, but communicates with the plant actuators through a network possibly shared by other users. The controller attempts to do periodic transmissions of data, at a desired sampling period  $T_s$  but these regular transmissions may be perturbed by the medium access protocol. For example, users using CSMA for medium access, may be forced to back-off for a typically random amount of time until the network becomes available. We assume these random times to be i.i.d. and denote by  $F_s$  the associated distribution.

We consider two different cases:

*Case I:* After waiting to obtain network access, the controller (re)samples the sensor, computes the control law and transmits this most recent data. This case is the most reasonable when transmitting dynamic data. Assuming that the transmission delays are negligible, and defining  $x := (x_P, \hat{u})$ , we have

$$\begin{aligned} \dot{x} &= Ax, \quad A = \begin{bmatrix} A_P & B_P \\ 0 & 0 \end{bmatrix}, \\ x(s_k) &= Jx(s_k^-), \quad J = \begin{bmatrix} I & 0 \\ K_C & 0 \end{bmatrix}. \end{aligned} \quad (30)$$

Since in the interval  $s_{k+1} - s_k$  the controller waits a fixed time  $T_s$  and a random amount of time following a distribution  $F(s)$ , the intervals  $\{s_{k+1} - s_k, k \geq 0\}$  are independent and identically distributed according to

$$\tilde{F}(\tau) = \begin{cases} F(\tau - T_s) & \text{if } \tau \geq T_s \\ 0, & t \in [0, T_s). \end{cases}$$

Note that the system (30) is a special case of a SHS with a single state and a single reset map.

*Case II:* After waiting to obtain access to the network, the controller transmits the data collected at the time it initially tried to transmit data. This case is more realistic than Case I since the controller typically sends the sensor data to the network adapter and does not have the option to update this data at the transmission times. We model this by a SHS with the following **two discrete-modes** ( $n_q = 2$ ),

- State  $i = 1$ - The controller waits a random time to gain access to the network.
- State  $i = 2$ - The controller waits for a time  $T_s$ .

Let  $r_k = s_k + T_s, x := (x_P, \hat{u}, v)$  where  $v(t) := u_\kappa, t \in [r_\kappa, r_{\kappa+1})$  is a variable that holds the last computed control value. The transitions between the two discrete modes can be modeled by a single transition function ( $n_l = 1$ ) taking the form (3) which is specified as follows. When in state 2 the SHS transits to state 1 ( $\xi_1(2) = 1$ ) at times  $r_\kappa$ . The state jump models the update of the variable  $v(r_\kappa) = u_\kappa$  that holds the last computed control value and is described by

$$x(r_\kappa) = J_{2,1} x(r_\kappa^-), \quad J_{2,1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ K_C & 0 & 0 \end{bmatrix}.$$

When in state 1 the SHS transits to state 2 ( $\xi_1(1) = 2$ ) at actuation update times  $s_\kappa$ . The state jump models the actuation update  $\hat{u}(s_\kappa) = v(s_\kappa^-) = v(r_\kappa)$  and is described by

$$x(s_\kappa) = J_{1,1} x(s_\kappa^-), \quad J_{1,1} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}.$$

The transition distributions are given by

- $F_{1,1}(\tau) = F_s(\tau).$
- $F_{2,1}(\tau) = \delta(\tau - T_s)$  is a discrete distribution that places mass  $w_i = 1$  at  $T_s$ .

In both discrete-time states, the continuous-time dynamics are described by  $\dot{x} = A_i x, i \in \{1, 2\}, A_1 = A_2 = A$  where

$$A = \begin{bmatrix} A_P & B_P & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

### A. Numerical Example

Suppose that the plant (29) is described by

$$A_P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_P = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which by properly scaling the state and input can be viewed as a linearized model of a damp-free inverted pendulum. Moreover, suppose that the network distribution

$E_s$  is uniform with support on the interval  $[0, T]$ , and fix  $T_s = 0.1s$ . A continuous-time state feedback controller is synthesized assuming that the controller and the plant's actuators are directly connected, i.e., network-free. It is an LQR controller given by  $\hat{u}(t) = K_C x(t)$ ,  $K_C = -[1 + \sqrt{2} \quad 1 + \sqrt{2}]$ , which is the solution to the problem  $\min_{\hat{u}(t)} \int_0^\infty [x_P(t)' x_P(t) + \hat{u}(t)^2] dt$ , yielding  $\lambda_i(A_P + B_P K_C) = \{-1, -\sqrt{2}\}$ . We wish to assert the loss of stability and performance in the closed-loop when instead of the ideal networked-free case we consider the scenarios of Cases I and II. To this effect we define the quantity

$$y(t) = x_P(t)' x_P(t) + \hat{u}(t)^2,$$

which can be written as  $y(t) = x' P x$ , where in the network-free case  $P = I_2 + K_C' K_C$ , and  $x = x_P$ ; in case I,  $P = I_3$  and  $x = (x_P, \hat{u})$ ; and in case II,  $P = \text{diag}(I_2, 1, 0)$ , and  $x = (x_P, \hat{u}, v)$ . Note that, in the network-free case,  $y(t)$  is the quantity whose integral is minimized for control synthesis and  $y(t)$  decreases exponentially fast at a rate  $\alpha = 2$ , since the dominant closed-loop eigenvalue equals  $\lambda_i(A_P + B_P K_C) = -1$ . Note also that in cases I and II,  $\mathbb{E}[y(t)]$  tends to zero is equivalent to MSS, which is equivalent to SS and MES since the transition distributions have finite support ( $T$  and  $T_s$  are finite). The conditions of Theorems 5, 9 can be used to assert if for a given  $T$ , the closed-loop in cases I and II is MSS. Moreover, when the closed-loop is MSS, we can determine the exponential decay constant of  $\mathbb{E}[y(t)]$  by Theorem 7. The results are summarized in Table I.

TABLE I  
EXPONENTIAL DECAY RATES  $E[y(t)] \leq ce^{-\alpha t}$

$T$	0.1	0.2	0.3	0.4	0.5	$> 0.521$
$\alpha$	2.000	2.000	2.000	0.849	0.118	NOT MSS

(a) Case I

$T$	0.4	0.6	0.8	1.0	1.2	$> 1.211$
$\alpha$	2.000	2.000	1.969	0.477	$7.63 \times 10^{-5}$	NOT MSS

(b) Case II

The fact that closed-loop stability is preserved for larger values of  $T$  in Case I, confirms what one would expect intuitively, i.e., Case I is more appropriate when transmitting dynamic data, since the most recent sampling information is sent through the network.

Using the state moment expressions provided by Theorems 2 and 3, we can perform a more detailed analysis by plotting the moments of  $y(t)$ . These can be expressed in terms of the moments of the state. For example, the two first moments take the form

$$\begin{aligned} \mathbb{E}[y(t)] &= \mathbb{E}[x(t)' P x(t)] = \mathbb{E}[x(t)'^{(2)}] \nu(P), \\ \mathbb{E}[y(t)^2] &= \mathbb{E}[(x(t)' P x(t))^2] = \mathbb{E}[(x(t)'^{(4)})] (\nu(P) \otimes \nu(P)). \end{aligned}$$

In Figure 1, we plot  $\mathbb{E}[y(t)]$  and  $\mathbb{E}[y(t)] \pm 2\mathbb{E}[(y(t) - \mathbb{E}[y(t)])^2]^{1/2}$  for a network distribution support  $T = 0.4$ . Note that from the Chebyshev inequality

$$\mathbf{P}[|y(t) - \mathbb{E}[y(t)]|] > a(t) \leq \frac{\mathbb{E}[(y(t) - \mathbb{E}[y(t)])^2]}{a(t)^2},$$

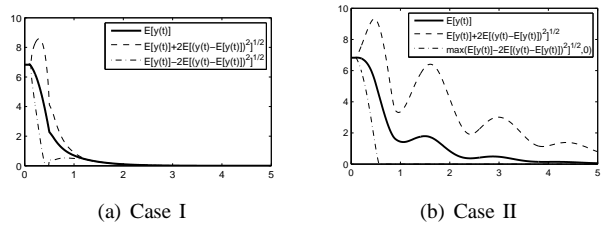


Fig. 1. Plot of  $\mathbb{E}[y(t)]$ , where  $y(t)$  is quadratically state dependent. For a fixed  $t$ ,  $\mathbb{E}[y(t)]$  lies between the dashed curves with probability  $> \frac{3}{4}$ .

one can guarantee that for a fixed  $t$ ,  $y(t)$  lies between the curves  $\mathbb{E}[y(t)] \pm a(t)$ ,  $a(t) = 2\mathbb{E}[(y(t) - \mathbb{E}[y(t)])^2]^{1/2}$  with a probability greater than  $\frac{3}{4}$ . The numerical method used to compute the solution of the Volterra-equation is based on a trapezoidal integration method. In case I, the expected value of the quadratic state function  $y(t)$  tends to zero much faster, and with a much smaller variance than in case II, confirming once again that case I is more appropriate when transmitting dynamic data.

## VI. CONCLUSIONS AND FUTURE WORK

We proposed an approach based on Volterra renewal-type equations to analyze SHSs for which the lengths of times that the system stays in each mode are independent random variables with given distributions. We showed that any statistical  $m$ -th order moment of the state can be computed using this approach, and provided a number of results characterizing the asymptotic behavior of a second-order moment of the system. Due to the large number of problems that fit the stochastic hybrid systems framework, finding more applications where the results can be applied is a topic for future work.

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