# Stability of networked control systems with variable sampling and delay

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***Abstract*— We consider Networked Control Systems (NCSs) consisting of a LTI plant; a linear static or dynamic feedback controller; a collection of sensors that provide measurements to the controller; and a collection of actuators that are used to control the plant. The different elements of the control system are spatially distributed, but interconnected through a communication network. Due to the shared and unreliable channel used to connect the subsystems, the sampling intervals are uncertain and variable. Moreover, samples may be dropped and experience uncertain and variable delays before arriving at the destination. We show that the resulting NCSs can be viewed as a MIMO sampled-data system with variable sampling intervals and delay, which can be modeled by linear infinite-dimensional impulsive systems. The infinite dimensionality of the system arises from the existence of delays. We provide conditions for the stability of the closed-loop expressed in terms of LMIs. By solving these LMIs, one can determine positive constants related to each entity sent through the network that determines an upper bound between the sampling time and the next update time at the destination of that entity, for which stability of the closed-loop system is guaranteed.**

## I. INTRODUCTION

*Network Control Systems (NCSs)* are spatially distributed systems in which the communication between plants, sensors, actuators, and controllers occurs through a shared band-limited digital communication network. Using network as a medium to connect spatially distributed elements of the system results in flexible architectures and generally reduces wiring and maintenance cost, since there is no need for point to point wiring. Consequently, NCSs have been finding application in a broad range of areas such as mobile sensor networks, remote surgery, haptics collaboration over the Internet and unmanned aerial vehicles [4]. However, the use of a shared network, in contrast to using several dedicated independent connections, introduces new challenges: the sampling intervals are uncertain and variable, samples may be dropped and experience uncertain and variable delays before arriving at the destination.

We start by considering an abstract single-input single-output (SISO) sampled-data system of the form

$$\dot{x}(t) = Ax(t) + Bx(s_k), \qquad t_k \le t < t_{k+1}, k \in \mathbb{N}, \qquad (1)$$

which models the closed-loop system in Fig. 1, where $s_k$ denotes the $k$-th sampling time instant and $t_k$ is the time at which the $k$-th sample arrives to the destination. If $\tau_k$ denotes the total delay that the $k$-th sample experiences in the loop,

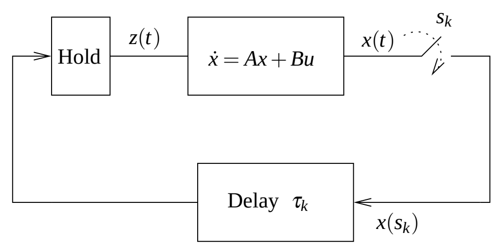


Fig. 1. Sampled-data system with variable sampling intervals and delays where $u(t) = z(t) = x(s_k)$ for $s_k + \tau_k \le t < s_{k+1} + \tau_{k+1}$

then $t_k := s_k + \tau_k$. Equation (1) can be used to model a NCS in which a linear plant $\dot{x}(t) = Ax(t) + B_u u$ is in feedback with static state-feedback remote controller $Kx$ where $B = B_u K$ in (1). We introduce a new *discontinuous Lyapunov functional* to establish stability of the closed-loop based on the theorems in [9] developed for a general nonlinear time-varying delay impulsive systems . The Lyapunov functional is discontinuous at the impulse times but its decrease is guaranteed by construction. The stability test is presented as LMIs that can be solved numerically. By solving these LMIs, one can find a positive constant that determines the upper bound between the sampling time $s_k$ and the next input update time $t_{k+1} = s_{k+1} + \tau_{k+1}$ for which the stability of the closed-loop system is guaranteed, assuming a given lower and upper bound on the total delay in the loop. When there is no delay this upper bound simply determines the maximum sampling interval which is often called $\tau_{MATI}$ in the NCS literature, e.g., in [14]. We use the terminology $\tau_{MATI}$ for the case when there are delays in the systems too, and one can state our result as follows: the system (1) is exponentially stable if $s_{k+1} + \tau_{k+1} - s_k \le \tau_{MATI}$ and $\tau_{\min} \le \tau \le \tau_{\max}$ for $\forall k \in \mathbb{N}$ where $\tau_{MATI}$ appears in our LMIs.

We also consider an abstract multi-input multi-output (MIMO) sampled-data system of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \qquad (2)$$

which models the closed-loop system in Fig. 4. The input and the output are partitioned as $y := [y_1' \cdots y_m']'$ and $u := [u_1' \cdots u_m']'$ and the partition of $u$ matches the partition of $y$. At time $s_k$, $k \in \mathbb{N}$ the $i-$th output of the system, $y_i(t), 1 \le i \le m$ is sampled and $y_i(s_k)$ is sent to update the input $u_i$, to be used as soon as it arrives until the next update arrives. This framework is general enough to model both one-channel NCSs and two-channel NCSs with dynamic output-feedback controllers that may or may not be antecipative [7]. For LTI processes and controllers, we present two stability tests in terms of LMIs. The first one is less conservative but the

This material is based upon work supported by the Institute for Collaborative Biotechnologies through grant DAAD19-03-D-0004 from the U.S. Army Research Office, and by the National Science Foundation under Grant No. CCR-0311084.

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number of LMIs grows exponentially with $m$. The second stability condition is based on the feasibility of a single LMI with dimention that grows linearly with $m$. For small $m$ the first stability test is more desired, because it leads to less conservative results, but the second stability test is more adequate for large $m$.

By solving the LMIs that guarantee stability, one obtains positive constants $\rho_{i\max}, 1 \le i \le m$ that determine the upper bound between the sampling time $s_k$ and the next input update time $t_{k+1} = s_{k+1} + \tau_{k+1}$ for each $i$ which the stability of the closed-loop system is guaranteed for a given lower and upper bound on the total delay in each loop. In practice, these constants produce deadlines for sampling that can be used to design protocols for communication scheduling.

Significant work has been devoted to finding $\tau_{MATI}$ ([4] and references therein). First we review the work which consider NCSs that can be presented as (1). In [3, 16, 8] the effect of delay is ignored and the problem of finding $\tau_{MATI}$ is formulated as LMIs. In the presence of variable delays in the control loop, [2, 7, 15] show that if $\tau_{\min}$, the lower bound on the delay in the control loop, is given, the stability is guaranteed for a less conservative $\tau_{MATI}$. Our result depends not only on $\tau_{\min}$ but also on an upper bound on the delay in the control loop, which we denote by $\tau_{\max}$. Through examples we show that assuming an upper bound $\tau_{\max}$ on the delay reduces the conservativeness greatly.

When the delay in the feedback loop is small ($\tau_{\min}, \tau_{\max} \to 0$), our LMIs reduce to the ones presented in [8] which are less conservative than [3, 16]. The following references consider a more general framework where there are many nodes sending data to the network. To orchestrate network access, Walsh et al. [13, 14] consider Round-Robin (static) and Try-Once-Discard (TOD) (dynamic) protocols and they find $\tau_{MATI}$ which determines the maximum deference between *any consecutive* sampling times. Nesic and Teel [10, 11] study the input-output stability properties of general nonlinear NCSs using an argument based on small gain theorem to find $\tau_{MATI}$ for NCSs.

In section II we consider the sampled-data system in Fig. 1 and we model it as a linear delay impulsive system, then we present stability test. To enlarge the class of NCSs we consider MIMO sampled-data system in Fig. 4 and then we present the stability test and classes of NCSs that can be presented by the MIMO sampled-data system in Fig. 4. We finish by conclusions and future work in section IV.

Notation: We denote the transpose of a matrix $A$ by $A'$ and we write $P > 0$ (or $P < 0$) when $P$ is a symmetric positive (or negative) definite matrix. We write a symmetric matrix $\begin{bmatrix} A & B \\ B' & C \end{bmatrix}$ as $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$. The notations $0_{ij}, I_k$ are used to denote a $i \times j$ matrix with zero entries and a $k \times k$ identity matrix and when there is no confusion about the size of the matrices we drop the dimensions.

## II. Sampled-data systems with variable sampling intervals and delays

Consider a linear delay sampled-data system of the form

$$\dot{x}(t) = Ax(t) + Bx(s_k), \qquad t_k \le t < t_{k+1}, k \in \mathbb{N}, \qquad (3)$$

which models the closed-loop system in Fig. 1 where $s_k$ denotes the $k$-th sampling time and $t_k$ is the instance that the $k$-th sample arrives to the destination. If $\tau_k$ denotes the total delay that the $k$-th sample experiences in the loop, then $t_k := s_k + \tau_k$. The closed-loop system can be modeled by the following impulsive system with delay

$$\dot{\xi}(t) = F\xi(t), \qquad t_k \le t < t_{k+1}, \qquad (4a)$$

$$\xi(t_{k+1}) = \begin{bmatrix} x(t_{k+1}^-) \\ x(s_{k+1}) \end{bmatrix}, \qquad k \in \mathbb{N}, \qquad (4b)$$

where

$$F := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \qquad \xi(t) := \begin{bmatrix} x(t) \\ z(t) \end{bmatrix},$$

$$t_k := s_k + \tau_k, \qquad z(t) := x(s_k), \quad t_k \le t < t_{k+1}.$$

We will seek for a class $\mathscr{S}$ of impulse-delay sequences $\{s_k, \tau_k\}$ such that the infinite-dimensional impulsive system (4) is globally uniformly exponentially stable. We say that the system (4) is *globally uniformly exponentially stable* over a class $\mathscr{S}$ of impulse-delay sequences, if for every sequence of impulse-delay in the $\mathscr{S}$ and every initial condition $x(t_0), x(t_0 - \tau_0)$ the solution to (4) is globally defined and satisfies $|x(t)| \le c \max(|x(t_0)|, |x(t_0 - \tau_0)|)\, e^{-\lambda(t-t_0)}$, for some $c, \lambda > 0$ and $\forall t \ge t_0$.

Consider the Lyapunov functional

$$V := \sum_{i=1}^{8} V_i, \qquad (5)$$

where

$$\begin{aligned}
V_1 &:= x'Px, \\
V_2 &:= \int_{t-\rho}^{t} (\rho_{\max} - t + s)\dot{x}'(s)R_1\dot{x}(s)ds, \\
V_3 &:= \int_{t-\sigma}^{t} (\sigma_{\max} - t + s)\dot{x}'(s)R_2\dot{x}(s)ds, \\
V_4 &:= \int_{t-\tau_{\min}}^{t} (\tau_{\min} - t + s)\dot{x}'(s)R_3\dot{x}(s)ds, \\
V_5 &:= \int_{t-\rho}^{t-\tau_{\min}} (\rho_{\max} - t + s)\dot{x}'(s)R_4\dot{x}(s)ds, \\
V_6 &:= (\rho_{\max} - \tau_{\min}) \int_{t-\tau_{\min}}^{t} \dot{x}'(s)R_4\dot{x}(s)ds, \\
V_7 &:= \int_{t-\tau_{\min}}^{t} x'(s)Zx(s)ds, \\
V_8 &:= (\rho_{\max} - \rho)(x - w)'X(x - w),
\end{aligned}$$

with $P, R_1, R_2, R_3, R_4, X, Z$ symmetric positive definite matrices and

$$\begin{aligned}
&\rho(t) := t - s_k, \quad \sigma(t) := t - t_k, \quad t_k \le t < t_{k+1}, \\
&\rho_{\max} := \sup_{t \ge 0} \rho(t), \qquad \sigma_{\max} := \sup_{t \ge 0} \sigma(t), \\
&w(t) := x(t_k) \quad t_k \le t < t_{k+1}.
\end{aligned}$$

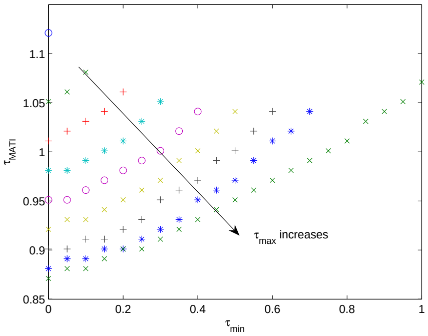


Fig. 2. $\tau_{MATI}$ with respect to $\tau_{min}$ based on Theorem 1 for $\tau_{\max}$ equal to $0, 0.1, \cdots, 0.7$ and 1.

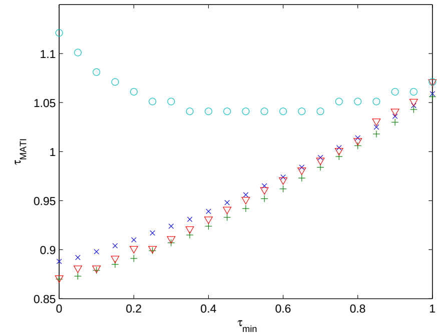


Fig. 3. $\tau_{MATI}$ with respect to $\tau_{\min}$ from [7], [15] are shown by $+$ and $\times$ respectively. The worse case ($\tau_{\max} = \tau_{MATI}$) and the best case ($\tau_{\max} = \tau_{\min}$) from Theorem 1 are shown by $\nabla$ and o respectively.

The LMIs in the next theorem guarantee that the system (4) with the Lyapunov functional (5) satisfies the conditions in [9] of global uniform exponential stability over the set $\mathscr{S}$ such that $s_{k+1} + \tau_{k+1} - s_k \leq \tau_{MATI}$ (or equally $t_{k+1} - t_k + \tau_k \leq \tau_{MATI}$) for a given $\tau_{MATI} > 0$ and that $\tau_{\min} \leq \tau_k \leq \tau_{\max}$. The Lyapunov functional is a positive functional where its decrease at the jump points is guaranteed by the construction and the following LMI conditions guarantee the decrease of the Lyapunov functional along the solution of the system (4). The proof of this result can be found in [9].

*Theorem 1:* The system (4) is exponentially stable for any delay and sampling interval satisfying $\tau_{min} \leq \tau_k \leq \tau_{\max}$ and $s_{k+1} + \tau_{k+1} - s_k \leq \tau_{MATI}$, $\forall k \in \mathbb{N}$, provided that there exist symmetric positive definite matrices $P, R_1, R_2, R_3, R_4, X, Z$ and (not necessarily symmetric) matrices $N_1, N_2, N_3, N_4$ that satisfy the following LMIs: [1]

$$\begin{bmatrix} M_1+\beta_{\max}(M_2+M_3) & \tau_{\max}N_1 & \tau_{\min}N_3 \\ * & -\tau_{\max}R_1 & 0 \\ * & * & -\tau_{\min}R_3 \end{bmatrix} < 0, \tag{6a}$$

$$\begin{bmatrix} M_1+\beta_{\max}M_2 & \tau_{\max}N_1 & \tau_{\min}N_3 & \beta_{\max}(N_1+N_2) & \beta_{\max}N_4 \\ * & -\tau_{\max}R_1 & 0 & 0 & 0 \\ * & * & -\tau_{\min}R_3 & 0 & 0 \\ * & * & * & -\beta_{\max}(R_1+R_2) & 0 \\ * & * & * & * & -\beta_{\max}R_4 \end{bmatrix} < 0, \tag{6b}$$

where $\beta_{max} = \tau_{MATI} - \tau_{\min}$, $\bar{F} := \begin{bmatrix} A & B & 0 & 0 \end{bmatrix}$, and

$$\begin{aligned}
M_1 &:= \bar{F}' \left[ P \ 0 \ 0 \ 0 \right] + \begin{bmatrix} P \\ 0 \\ 0 \\ 0 \end{bmatrix} \bar{F} + \tau_{\min} F'(R_1+R_3)F \\
&- \begin{bmatrix} I \\ 0 \\ -I \\ 0 \end{bmatrix} X \begin{bmatrix} I \\ 0 \\ -I \\ 0 \end{bmatrix}' + \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} Z \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}' - \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} Z \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix}' \\
&- N_1 \left[ I \ -I \ 0 \ 0 \right] - \begin{bmatrix} I \\ -I \\ 0 \\ 0 \end{bmatrix} N_1' - N_2 \left[ I \ 0 \ -I \ 0 \right] - \begin{bmatrix} I \\ 0 \\ -I \\ 0 \end{bmatrix} N_2' \\
&- N_3 \left[ I \ 0 \ 0 \ -I \right] - \begin{bmatrix} I \\ 0 \\ 0 \\ -I \end{bmatrix} N_3' - N_4 \left[ 0 \ -I \ 0 \ I \right] - \begin{bmatrix} 0 \\ -I \\ 0 \\ I \end{bmatrix} N_4', \\
M_2 &:= \bar{F}'(R_1+R_2+R_4)\bar{F}, \\
M_3 &:= \begin{bmatrix} I \\ 0 \\ -I \\ 0 \end{bmatrix} X\bar{F} + \bar{F}'X \left[ I \ 0 \ -I \ 0 \right].
\end{aligned} \tag{7}$$

*Remark 1:* Suppose that the system matrices $\Omega := \begin{bmatrix} A & B \end{bmatrix}$ are not exactly known and instead they are specified through the following polytopic condition:

$$\Omega \in \Big\{ \sum_{j=1}^{\kappa} f_j \Omega_j, \quad 0 \leq f_j \leq 1, \quad \sum_{j=1}^{\kappa} f_j = 1 \Big\},$$

where the $\kappa$ vertices of the polytope are described by $\Omega_j := \begin{bmatrix} A^j & B^j \end{bmatrix}$. Stability of the system can be checked by solving the LMIs in Theorem 1 for each of the individual vertices with the same matrix variables.

### *A. NCSs modeled by a sampled-data system*

Equation (3) models NCSs in which a linear plant with state-space

$$\dot{x}_p(t) = A_p x_p(t) + B_p u_p$$

where $x_p \in \mathbb{R}^n, u_p \in \mathbb{R}^m$ are the state and the input of the plant, is in feedback with a static feedback gain $K$. At time $s_k$, $k \in \mathbb{N}$ the plant's state, $x(s_k)$, is sent to the controller and the control command $Kx(s_k)$ is sent back to the plant to be used as soon as it arrives until the next control command update. The total delay in the control loop that the $k$-th sample experiences is denoted by $\tau_k$ where $\tau_{\min} \leq \tau_k \leq \tau_{\max}$, $\forall k \in \mathbb{N}$. Then the closed-loop system equations can be written as (3) with

$$x := x_p, \quad A := A_p, \quad B := B_p K, \tag{8}$$

and exponential stability of the system can be concluded from Theorem 1.

*Remark 2:* We only index the samples that get to the destination, which enables us to capture packet drops [16]. Consequently, even if the sampling intervals are constant, because of the packet dropouts in the network, the NCS can be represented by a sampled-data system with variable sampling intervals.

*Example 1:* Consider the state space plant model [1]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u,$$

with state feedback gain $K = -\begin{bmatrix} 3.75 & 11.5 \end{bmatrix}$, for which we have

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \qquad B = -\begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \times \begin{bmatrix} 3.75 & 11.5 \end{bmatrix}.$$

[1] All zero matrices and identity matrices are $n \times n$

By checking the condition that $eig(\begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix} e^{Fh}) < 0$ on a tight grid of $h$ and finding the largest $h$, it is possible to show that the closed-loop system remains stable if and only if the constant sampling interval is smaller than $1.7s$. On the other hand when the sampling interval approaches zero, the system is described by a DDE and we can find the maximum constant delay for which stability is guaranteed. The system is stable if and only if all the roots of characteristic function $\det(sI - A - Be^{-\tau_0 s})$ are negative. We use the Pade approximation $e^{-\tau_0 s} = \frac{1-s\tau_0/2}{1+s\tau_0/2}$ to derive the determinant polynomial. By the Routh-Hurwitz test,one can show that the system is stable for any constant delay smaller than 1.36. Comparing these numbers to the maximum variable sampling interval 1.1137 and maximum variable delay 1.0744 reveals the conservativeness of our method.

*No-delay and variable sampling:* When there is no delay but the sampling intervals are variable, $\tau_{MATI}$ determines the upper bound on variable sampling intervals $s_{k+1} - s_k$ . The upper bound given by [3, 16, 7] (when $\tau_{\min} = 0$) is 0.8696 which is improved to 0.8871 in [15]. Theorem 1 and [8] gives the upper bound equal to 1.1137.

*Variable-delay and sampling:* Fig. 2 shows $\tau_{MATI}$ as a function of $\tau_{\min}$ for different values of $\tau_{max}$ obtained form Theorem 1. Note that the curve that shows the largest $\tau_{MATI}$ in Fig. 2 is part of the curve in Fig. 3 shown by o. Fig. 3 shows $\tau_{MATI}$ with respect to $\tau_{min}$ where the results from [7], [15] are shown by $+$, $\times$ respectively. The value of $\tau_{MATI}$ given by [2] lays between $+$ and $\times$ in Fig. 3 and we did not show them. In Theorem 1, $\tau_{MATI}$ is a function of $\tau_{\min}$ and $\tau_{\max}$. To be able to compare our result to others in the literature we consider two values for $\tau_{\max}$ and we obtain $\tau_{MATI}$ as a function of $\tau_{\min}$ based on Theorem 1. Fist we consider $\tau_{\max} = \tau_{\min}$, which is the case that the delay is constant and equal to the value of $\tau_{\min}$. The largest $\tau_{MATI}$ for a given $\tau_{\min}$ for this case provided by Theorem 1 is shown by o in Fig. 3. The second case is when $\tau_{\max} = \tau_{MATI}$, which is the case that there can be very large delays in the loop in compare to the sampling intervals. The largest $\tau_{MATI}$ for a given $\tau_{\min}$ for this case provided by Theorem 1 is shown by $\nabla$ in Fig. 3. One can observe that when the delays in the control loop are small, our method shows a good improvement in compare to other results in the literature.

## III. Multi-Input Multi-Output (MIMO) Sampled-Data Systems

To enlarge the class of NCSs modeled by impulsive systems we consider the MIMO sampled-data system in Fig. 4 with state space of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{9}$$

where $x \in \mathbb{R}^n$ is the overall state of the system, $u \in \mathbb{R}^q$ is the overall input of the system, and $y \in \mathbb{R}^q$ is the overall output of the system. The input is partitioned as $u := [u_1' \cdots u_m']'$ where $u_i \in \mathbb{R}^{q_i}, 1 \le i \le m$ and $\sum_{i=1}^m q_i = q$ and the output is partitioned as $y := [y_1' \cdots y_m']'$ where $y_i \in \mathbb{R}^{q_i}, 1 \le i \le m$ and $\sum_{i=1}^m q_i = q$, so the partition of $y$ is matched with the partion of $u$. In NCSs each $y_i$ represents the output of nodes that can send their information to the network in a single packet. For example sensors spatially close to each other, send their measurements in a single packet or controller outputs sent to actuators spatially close to each other, send the control commands in a single packet.

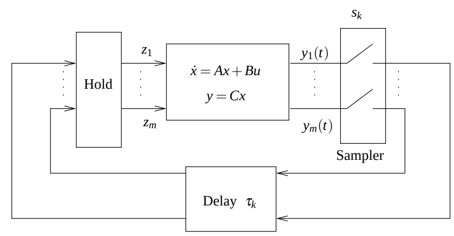


Fig. 4. MIMO sampled-data system with variable sampling intervals and delays where $u_i(t) = z_i(t) = y_i(s_k)$ for $t_k \le t < t_j$ where $k$ and $j$ are the consecutive sampling indexes in $\mathscr{K}_i$, $1 \le i \le m$.

At time $s_k$, $k \in \mathbb{N}$ the $i-$th output of the system, $y_i(t), 1 \le i \le m$ is sampled and $y_i(s_{k_i})$ is sent through the network to update the input $u_i$, to be used as soon as it arrives until the next update arrives. We define sets $\mathscr{K}_i, 1 \le i \le m$ as the set of indexes of sampling times that are related to the output $y_i$;

$$\mathscr{K}_i := \{k \,|\, y_i(t) \textit{ is sampled at } s_k\}.$$

The total delay in the control loop that the $k$-th sample experiences is denoted by $\tau_k$ where $\tau_{i\min} \le \tau_k \le \tau_{i\max}$, $\forall k \in \mathscr{K}_i, 1 \le i \le m$. We define $t_k := s_k + \tau_k$ which is the time instant that the value of $u_i$ is updated. The overall system can be written as an impulsive system of the form

$$\dot{\xi}(t) = F\xi(t), \qquad t_k \le t < t_{k+1}, \tag{10a}$$

$$\xi(t_{k+1}) = \begin{bmatrix} x(t_{k+1}^-) \\ \cdots\cdots \\ z_1(t_{k+1}^-) \\ \vdots \\ z_i(t_{k+1}^-) \\ \vdots \\ z_m(t_{k+1}^-) \end{bmatrix}, \qquad k+1 \in \mathscr{K}_i, 1 \le i \le m, \tag{10b}$$

where $F := \begin{bmatrix} A & \vdots & B \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix}$, $\xi(t) := \begin{bmatrix} x(t) \\ \cdots\cdots \\ z(t) \end{bmatrix}$, and

$$z(t) := \begin{bmatrix} z_1(t) \\ \vdots \\ z_m(t) \end{bmatrix}, \qquad z_i(t) := y_i(s_k), \quad t_k \le t < t_j,$$

where $k$ and $j$ are the consecutive sampling indexes in $\mathscr{K}_i, 1 \le i \le m$.

Since the minimum of delay in the network is typically small and for simplicity of derivations, we assume that $\tau_{i\min} = 0, 1 \le i \le m$. We now present two theorems for the stability of the system (10). The first one is less conservative; however, the number of LMIs grows exponentially with $m$. The second stability condition is based on the feasibility of a single LMI, but its size grows linearly with $m$. For small

$m$ the first stability test is more adequate because it leads to less conservative results, but the second stability test is more desired for large $m$. We present our results for $m = 2$, but deriving the stability conditions for other values of $m$ is straightforward by following the same steps.

Inspired by the Lyapunov functional (5), We employ the Lyapunov functional

$$V := V_1 + V_2 + V_3 + V_4, \tag{11}$$

where

$$\begin{aligned}
V_1 &:= x' P x, \\
V_2 &:= \sum_{i=1}^{2} \int_{t-\rho_i}^{t} (\rho_{i\max} - t + s) \dot{y}_i'(s) R_{1i} \dot{y}_i(s) ds, \\
V_3 &:= \sum_{i=1}^{2} \int_{t-\sigma_i}^{t} (\sigma_{i\max} - t + s) \dot{y}_i'(s) R_{2i} \dot{y}_i(s) ds, \\
V_4 &:= \sum_{i=1}^{2} (\rho_{i\max} - \rho_i)(y_i - w_i)' X_i (y_i - w_i),
\end{aligned}$$

with $P, R_{1i}, R_{2i}, X_i$ symmetric positive definite matrices and

$$\begin{aligned}
&\rho_i(t) := t - s_k, \quad \sigma_i(t) := t - t_k \quad t_k \leq t < t_j, \\
&\rho_{i\max} := \sup_{t \geq 0} \rho_i(t), \quad \sigma_{i\max} := \sup_{t \geq 0} \sigma_i(t), \\
&w_i(t) := y_i(t_k), \quad t_k \leq t < t_j
\end{aligned}$$

where $k$ and $j$ are the consecutive sampling indexes in $\mathscr{K}_i$, $1 \leq i \leq m$. The next theorem guarantees that the Lyapunov functional (11) decreases along the solution to the system (10).

*Theorem 2:* The system (10) is exponentially stable for any delay and sampling interval satisfying $0 \leq \tau_k \leq \tau_{i\max}$ and $s_j + \tau_j - s_k \leq \rho_{i\max}$, where $k$ and $j$ are consecutive sampling indexes in $\mathscr{K}_i$, $i \in \{1, 2\}$ provided that there exist symmetric positive definite matrices $P, R_{1i}, R_{2i}, X_i$ and (not necessarily symmetric) matrices $N_{1i}, N_{2i}$ that satisfy the following LMIs:

$$\begin{bmatrix} M_1 + \rho_{1\max}(M_{21} + M_{31}) + \rho_{2\max}(M_{22} + M_{32}) & \tau_{1\max} N_{11} & \tau_{2\max} N_{12} \\ * & -\tau_{1\max} R_{11} & 0_{q_1 q_2} \\ * & * & -\tau_{2\max} R_{12} \end{bmatrix} < 0, \tag{12a}$$

$$\begin{bmatrix} M_1 + \rho_{1\max} M_{21} + \rho_{2\max}(M_{22} + M_{32}) & \tau_{1\max} N_{11} & \tau_{2\max} N_{12} & G_{11} \\ * & -\tau_{1\max} R_{11} & 0 q_1 q_2 & 0_{q_1 q_1} \\ * & * & -\tau_{2\max} R_{12} & 0_{q_2 q_1} \\ * & * & * & G_{21} \end{bmatrix} < 0, \tag{12b}$$

$$\begin{bmatrix} M_1 + \rho_{1\max}(M_{21} + M_{31}) + \rho_{2\max} M_{22} & \tau_{1\max} N_{11} & \tau_{2\max} N_{12} & G_{12} \\ * & -\tau_{1\max} R_{11} & 0_{q_1 q_2} & 0_{q_1 q_2} \\ * & * & -\tau_{2\max} R_{12} & 0_{q_2 q_2} \\ * & * & * & G_{22} \end{bmatrix} < 0, \tag{12c}$$

$$\begin{bmatrix} M_1 + \rho_{1\max} M_{21} + \rho_{2\max} M_{22} & \tau_{1\max} N_{11} & \tau_{2\max} N_{12} & G_{11} & G_{12} \\ * & -\tau_{1\max} R_{11} & 0_{q_1 q_2} & 0_{q_1 q_1} & 0_{q_1 q_2} \\ * & * & -\tau_{2\max} R_{12} & 0_{q_2 q_1} & 0_{q_2 q_2} \\ * & * & * & G_{21} & 0_{q_1 q_2} \\ * & * & * & * & G_{22} \end{bmatrix} < 0, \tag{12d}$$

where $\bar{F} := \begin{bmatrix} A & B & 0_{nq} \end{bmatrix}$ and

$$\begin{aligned}
&M_1 := \bar{F}' \begin{bmatrix} P & 0_{nq} & 0_{nq} \end{bmatrix} + \begin{bmatrix} P \\ 0_{qn} \\ 0_{qn} \end{bmatrix} \bar{F} - T_1' X_1 T_1 - T_2' X_2 T_2 - N_{11} T_3 \\
&\quad - T_3' N_{11}' - N_{21} T_1 - T_1' N_{21}' - N_{12} T_4 - T_4' N_{12}' - N_{22} T_2 - T_2' N_{22}', \\
&M_{2i} := \bar{F}' C_i' (R_{1i} + R_{2i}) C_i \bar{F}, \qquad M_{3i} := T_i' X_i C_i \bar{F} + \bar{F}' C_i' X T_i, \\
&G_{1i} := \rho_{i\max}(N_{1i} + N_{2i}), \qquad G_{2i} := \rho_{i\max}(R_{1i} + R_{2i})
\end{aligned} \tag{13}$$

with

$$\begin{aligned}
&C_1 := \begin{bmatrix} I_{q_1} & 0_{q_1 q_2} \end{bmatrix} C, && C_2 := \begin{bmatrix} 0_{q_2 q_1} & I_{q_2} \end{bmatrix} C, \\
&T_1 := \begin{bmatrix} C_1 & 0_{q_1 q} & -\bar{I}_1 \end{bmatrix}', && T_2 := \begin{bmatrix} C_2 & 0_{q_2 q} & -\bar{I}_2 \end{bmatrix}', \\
&T_3 := \begin{bmatrix} C_1 & -\bar{I}_1 & 0_{q_1 q} \end{bmatrix}, && T_4 := \begin{bmatrix} C_2 & -\bar{I}_2 & 0_{q_2 q} \end{bmatrix}, \\
&\bar{I}_1 := \begin{bmatrix} I_{q_1} & 0_{q_1 q_2} \end{bmatrix} && \bar{I}_2 := \begin{bmatrix} 0_{q_2 q_1} & I_{q_2} \end{bmatrix}
\end{aligned} \tag{14}$$

*Proof of Theorem 2.* After taking the derivative of the Lyapunov functional (11) and following the same steps as in the proof of theorem 2 of [9] we conclude that the derivative is negative as long as

$$M_1 + \sum_{i=1}^{2} \rho_{i\max}(M_{2i} + M_{3i}) + \rho_i (M_{4i} - M_{3i}) < 0. \tag{15}$$

Then we can prove that (15) is equivalent to (12). Details of the proof can be found in [6].

It is possible to generalize Theorem 2 for an arbitrary $m$. However, the number of LMIs is $2^m$ and the size of LMIs and the number of scalar variables increases linearly. For complex systems with large number of sending nodes, to have a numerically tractable test, it is crucial that the number of LMIs grows linearly too. The next theorem presents another stability test which is more conservative; however, the stability test is based on the feasibility of a single LMI.

*Theorem 3:* The system (10) is exponentially stable for any delay and sampling interval satisfying $0 \leq \tau_k \leq \tau_{i\max}$ and $s_j + \tau_j - s_k \leq \rho_{i\max}$, where $k$ and $j$ are the consecutive sampling indexes in $\mathscr{K}_i$, $i = 1, 2$ provided that there exist symmetric positive definite matrices $P, R_{1i}, R_{2i}$ and (not necessarily symmetric) matrices $N_{1i}, N_{2i}$ that satisfy the following LMIs:

$$\begin{bmatrix} \bar{M}_1 + \rho_{1\max} \bar{M}_{21} + \rho_{2\max} \bar{M}_{22} & \tau_{1\max} N_{11} & \tau_{2\max} N_{12} & G_{11} & G_{12} \\ * & -\tau_{1\max} R_{11} & 0_{q_1 q_2} & 0_{q_1 q_1} & 0_{q_1 q_2} \\ * & * & -\tau_{2\max} R_{12} & 0_{q_2 q_1} & 0_{q_2 q_2} \\ * & * & * & G_{21} & 0_{q_1 q_2} \\ * & * & * & * & G_{22} \end{bmatrix} < 0, \tag{16}$$

where $\bar{F} := \begin{bmatrix} A & B & 0_{nq} \end{bmatrix}$ and

$$\begin{aligned}
\bar{M}_1 :=& \bar{F}' \begin{bmatrix} P & 0_{nq} & 0_{nq} \end{bmatrix} + \begin{bmatrix} P \\ 0_{qn} \\ 0_{qn} \end{bmatrix} \bar{F} - N_{11} T_3 - T_3' N_{11}' - N_{21} T_1 \\
&- T_1' N_{21}' - N_{12} T_4 - T_4' N_{12}' - N_{22} T_2 - T_2' N_{22}', \\
\bar{M}_{2i} :=& \bar{F}' C_i' (R_{1i} + R_{2i}) C_i \bar{F}, \\
G_{1i} :=& \rho_{i\max}(N_{1i} + N_{2i}), \qquad G_{2i} := \rho_{i\max}(R_{1i} + R_{2i}).
\end{aligned}$$

with the matrix variables defined in (14).

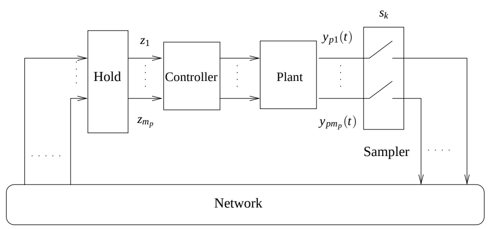


Fig. 5. One channel NCSs with the plant (17) and the controller (18).

*Proof of Theorem 3.* After taking the time derivative of the Lypunov functional and following the steps in [6], we conclude that the derivative is negative if (15) holds. Equation (15) with $\rho_i = \rho_{i\max}$ is a sufficient condition for (15). So we obtain

$$M_1 + \sum_{i=1}^{2} \rho_{i\max}(M_{2i} + M_{4i}) < 0,$$

and by applying Schur Lemma we get (16).

### A. NCSs modeled by MIMO sampled-data systems

The impulsive system (10) can be used to represent the distributed sensors/actuators configurations shown in Figs. 5, and 6. The LTI plant has the state space of the form

$$\dot{x}_p(t) = A_p x_p(t) + B_p u_p(t), \qquad y_p(t) = C_p x_p(t), \qquad (17)$$

where $x_p \in \mathbb{R}^{n_p}, u_p := \left[u'_{p1} \cdots u'_{pm_c}\right]' \in \mathbb{R}^{m_c}$, and $y_p := \left[y'_{p1} \cdots y'_{pm_p}\right] \in \mathbb{R}^{m_p}$ are the state, input and output of the plant. At time $t_k, k \in \mathscr{K}_i, 1 \le i \le m_p$, sensor $i$ sends $y_{pi}(t_k)$ to the controller, which arrives at the destination at time $t_k := s_k + \tau_k$. When a new measurement of the sensor $i$ arrives at the controller side, the corresponding value at the hold block, $z_i$, is updated and held constant until another measurement of the sensor $i$ arrives (all other measurement values remain unchanged when the value of the sensor $i$ is updated). Hence $u_{ci}(t) = z_i(t) = y_{pi}(t_k), t_k \le t < t_j$ where $k$ and $j$ are consecutive sampling indexes in $\mathscr{K}_i, 1 \le i \le m$ . An output feedback controller (or a state feedback controller) uses the measurements to construct the control signal. The controller has the state space of the form

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t), \qquad y_c(t) = C_c x_c(t) + D_c u_c(t), \qquad (18)$$

where $x_c \in \mathbb{R}^{n_c \times 1}, u_c := \left[u'_{c1} \cdots u'_{cm_p}\right]' \in \mathbb{R}^{m_p \times 1}, y_c := \left[y'_{c1} \cdots y'_{cm_c}\right] \in \mathbb{R}^{m_p \times 1} \in \mathbb{R}^{m_c \times 1}$ are the state, input and output of the plant and matrices $A_c, B_c, C_c, D_c$ have the proper dimensions. The main difference between the different NCS configurations in this section is the control signal construction.

*1) One-channel NCS with dynamic feedback controller:* Fig. 5 shows a one-channel NCS consisting of a plant with the state-space (17) in feedback with a dynamic output controller with state-space (18). Note that in one-channel NCS the controller is directly connected to the plant and only the measurements of the plant are sent through the network. Hence in this case $y_i(t) := y_{pi}(t), m := m_p$, and $x := \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}\right]$. The closed-loop system can be written as the impulsive system (10) where

$$A := \begin{bmatrix} A_p & B_p C_c \\ 0 & A_c \end{bmatrix}, \quad B := \begin{bmatrix} B_p D_c \\ B_c \end{bmatrix}, \quad C := \begin{bmatrix} C_p & 0 \end{bmatrix}. \quad (19)$$

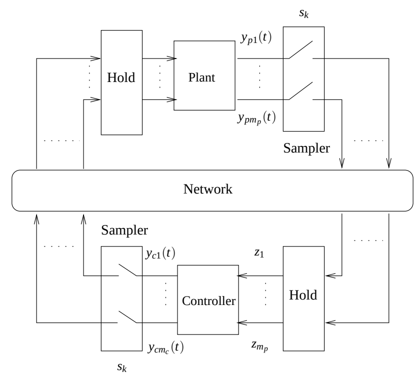


Fig. 6. Two channel NCSs with the plant (17) and the anticipative or non-anticipative controller (18).

*2) Two-channel NCS with non-anticipative controller:* Fig. 6 shows a two-channel NCS consisting of a plant with the state-space (17) in feedback with a *non-anticipative* controller with state-space (18) where $D_c = 0$. Non-anticipative controllers are simply output-feedback controllers which a single value control command is calculated. Now the controller is located away from the actuators and the control commands should be sent through the network. The control signal for the actuator $i$, $y_{ci}(t)$, is sampled at $s_k, k \in \mathscr{K}_i, m_p + 1 \le i \le m_p + m_c$, and samples get to the actuator $i$ at $t_k := s_k + \tau_k$. Note that a non-anticipative control unit sends a single-value control command to be applied to the actuator $i$ of the plant and held until the next control update of the actuator $i$ arrives (all other actuator values remain unchanged while the value of actuator $i$ is updated). Hence $u_{pi}(t) = z_i(t) = y_{ci}(t_k), t_k \le t < t_j$, where $k$ and $j$ are the consecutive sampling indexes in $\mathscr{K}_i, m_p + 1 \le i \le m_p + m_c$. So in this case

$$y_i := \begin{cases} y_{pi}, & 1 \le i \le m_p \\ y_{ci}, & m_p + 1 \le i \le m_p + m_c \end{cases},$$

$m := m_p + m_c$ and $x := \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix}\right]$. The closed-loop system with state can be written as the impulsive system (10) where

$$A := \begin{bmatrix} A_p & 0 \\ 0 & A_c \end{bmatrix}, \quad B := \begin{bmatrix} 0 & B_p \\ B_c & 0 \end{bmatrix}, C := \begin{bmatrix} C_p & 0 \\ 0 & C_c \end{bmatrix}. \quad (20)$$

*3) Two-channel NCS with anticipative controller:* Fig.6 can represent a two-channel NCS with a plant with the state-space (17) in feedback with a *anticipative* controller with state-space (18). Anticipative controller attempts to compensate the sampling and delay introduced by the actuation channels. For simplicity, we assume that the actuation channels are sampled with constant sampling intervals $h = t_k - t_j$, for any consecutive sampling indexes $k, j \in \mathscr{K}_i, m_p + 1 \le i \le m_p + m_c$, and that the delay in the actuation channels is constant and equal to $\tau = \tau_k$, $\forall k \in \mathscr{K}_i, m_p + 1 \le i \le m_p +$

$m_c$. At each sampling time $s_k$, $k \in \mathcal{K}_i, m_p+1 \leq i \leq m_p+m_c$ the controller sends a time-varying control signal $y_{ci}(\cdot)$ to the actuator $i$. This control signal should be used from the time $s_k+\tau$ at which it arrives until the time $s_k+h+\tau$ at which the next control update of the actuator $i$ will arrive. This leads to

$$u_{pi}(t) = y_{ci}(t), \quad \forall t \in [s_k+\tau, s_k+h+\tau),\ k_i \in \mathcal{K}_i,$$

where $m_p+1 \leq i \leq m_p+m_c$. However, the prediction of control signal $y_{ci}(t)$ needed in the interval $[s_k+\tau, s_k+h+\tau)$ must be available at the transmission time $s_k$, which requires the control unit to calculate the control signal up to $h+\tau$ time units into the future.

*Remark 3:* Anticipative controllers send actuation signals to be used during time intervals of duration $h$, therefore the sample and hold blocks in Fig. 6 should be understood in a broad sense. In practice, the sample block would send over the network some parametric form of the control signal $u_{ci}(\cdot)$ (e.g., the coefficients of a polynomial approximation to this signal).

*Remark 4:* Anticipative controllers are similar to predictive controllers in the sense that both calculate the future control actions. However in predictive controllers only the most recent control prediction is applied until the new control commands arrive. Anticipative controllers are predictive controllers that send a control prediction for a certain duration. In the expense of sending more information to the actuators in each packet, one expect fewer packets to be transmitted to stabilize the system.

An estimate $\hat{x}_c(t)$ of $x_c(t+h+\tau)$ is constructed as follows:

$$\dot{\hat{x}}_c(t) = A_c\hat{x}_c(t) + B_c u_c(t), \tag{21}$$

where

$$u_{ci}(t) = y_{pi}(t_k), \forall t \in [t_k+h+\tau, t_j+h+\tau), \tag{22}$$

where $k, j$ are two consecutive sample indexes in $\mathcal{K}_i, 1 \leq i \leq m_p$. To compensate for the time varying delay and sampling intervals in the actuation channels, $\hat{x}_c$ would have to be calculated further into the future. Hence the assumptions of constant delay and sampling interval for actuation channel can be relaxed by predicting $x_c$ further into the future. With such a controller state prediction available, the signal $y_{ci}(t)$ sent at times $s_k, k \in \mathcal{K}_i, m_p+1 \leq i \leq m_p+m_c$, to be used in $\left[s_k+\tau, s_k+h+\tau\right)$, is then given by

$$y_{ci}(t) = C_{ci}\hat{x}_c(t-h-\tau) + D_c u_{ci}, \forall t \in [s_k+\tau, s_k+h+\tau), \tag{23}$$

which only requires the knowledge of $\hat{x}_c(.)$ in the interval $t \in \left[(s_k-h, s_k\right)$, and therefore is available at the transmission times $s_k$. The closed-loop system can be written as

$$\begin{bmatrix}\dot{\hat{x}}_p(t)\\ \dot{\hat{x}}_c(t)\end{bmatrix} = \begin{bmatrix}A_p & B_pC_c\\ 0 & A_c\end{bmatrix}\begin{bmatrix}\hat{x}_p(t)\\ \hat{x}_c(t)\end{bmatrix} + \begin{bmatrix}B_pD_c\\ B_c\end{bmatrix}u_c(t), \tag{24}$$

where $\hat{x}_p(t) := x_p(t+h+\tau)$ and the elements of $u_c(t)$ are defined by (22). Hence in this case $y_i(t) := y_{pi}(t), \quad m := m_p$ and the closed-loop system with state $x := \left[\begin{smallmatrix}\hat{x}_p\\ \hat{x}_c\end{smallmatrix}\right]$ can be written as the impulsive system (10) where

$$A := \begin{bmatrix}A_p & B_pC_c\\ 0 & A_c\end{bmatrix}, \quad B := \begin{bmatrix}B_pD_c\\ B_c\end{bmatrix}, \quad C := \begin{bmatrix}C_p & 0\end{bmatrix}. \tag{25}$$

*Remark 5:* Equation (19) that represents one-channel NCS with dynamic output feedback is similar to equation (25) that represents two-cannel NCS with anticipative controller. Consequently for analysis perpose one can model a two-cannel NCS with anticipative controller as a one-channel NCS with *'fictitious'* delays equal to the sum of the delay in the sensor to actuator channels, the delay in the actuator to sensor channels, and the sampling of the actuator channel.

### *B. Stability of NCSs modeled by MIMO sampled-data systems*

Exponential stability of one-channel NCS, two-channel NCS with non-anticipative controller and two-channel NCS with anticipative controller can be concluded from Theorem 2 or 3 (for $m = 2$ or generalization of these theorems for arbitrary $m$) with the choice of (19),(20) and (25) respectively. By solving the LMIs one can find positive constants $\rho_{i\max}, 1 \leq i \leq m$ which determines the upper bound between the sampling time $s_k$ and the next update time $s_j+\tau_j$ of the output $i$ for $1 \leq i \leq m$, where $k, j$ are two consecutive sampling instances of output $i$, for which the stability of the closed-loop system is guaranteed for a given lower and upper bound on the total delay in the loop $i$.

However most of the work in the literature has been devoted to finding $\tau_{MATI}$ ([13, 14, 10, 11, 4] and references therein) which determines the upper bound between the sampling time $s_k$ and the next input update time $t_{k+1} = s_{k+1}+\tau_{k+1}$ for *any consecutive sampling instances* which the stability of the closed-loop system is guaranteed for a given lower and upper bound on the total delay in the loop. One can expect that having $m$ constants $\rho_{i\max}, 1 \leq i \leq m$ instead of one single constant $\tau_{MATI}$, reveals more information about the system as we explore in the following example.

*Example 2:* This example appeared in [14, 10, 5, 12] and considers a linearized model of the form (17) for a two-input, two-output batch reactor where

$$A_p := \begin{bmatrix}1.38 & -0.2077 & 6.715 & -5.676\\ -0.5814 & -4.29 & 0 & 0.675\\ 1.067 & 4.273 & -6.654 & 5.893\\ 0.048 & 4.273 & 1.343 & -2.104\end{bmatrix}, \quad B_p := \begin{bmatrix}0 & 0\\ 5.679 & 0\\ 1.136 & -3.146\\ 1.136 & 0\end{bmatrix},$$
$$C_p := \begin{bmatrix}1 & 0 & 1 & -1\\ 0 & 1 & 0 & 0\end{bmatrix}.$$

This system is controlled by a PI controller of the form (18) where

$$A_c := \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \qquad B_c := \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$
$$C_c := \begin{bmatrix}-2 & 0\\ 0 & 8\end{bmatrix}, \qquad D_C := \begin{bmatrix}0 & -2\\ 5 & 0\end{bmatrix}.$$

Following the assumptions of [14, 10, 5, 12], we assume that only the outputs of the plant are transmitted over the network, there is no dropouts and the outputs are sent in a round-robin fashion and consecutively. We compare $\tau_{MATI}$ of this example given by the stability results in [14, 10, 5, 12],

| | no drop |
|---|---|
| Maximum deterministic time interval between samples from [12] | 0.0123 |
| Maximum stochastic arbitrary inter-sampling time distribution from [5] | 0.0279 |
| Maximum stochastic uniform inter-sampling time distribution from [5] | 0.0517 |
| Maximum deterministic time interval between samples from Theorem 2 | 0.0405 |

TABLE I

COMPARISON OF $\tau_{MATI}$ FOR EXAMPLE 2 WHEN THERE IS NO DELAY.

where in all the references the effect of the delay is ignored. From Theorem 2 we compute $\rho_{1\max} = 0.081, \rho_{2\max} = 0.113$ when there is no delay. We can show that if the upper bound between any sonsecutive sampling, $\tau_{MATI}$, is smaller that $\frac{1}{2}\min(\rho_{1\max}, \rho_{2\max})$, then the upper bound between the samples of $y_{p1}$ or $y_{p2}$ are smaller than $\min(\rho_{1\max}, \rho_{2\max})$ and the system is stable, so we get $\tau_{MATI} = \frac{1}{2}\min(\rho_{1\max}, \rho_{2\max}) = 0.0405$ Table I shows the less conservative results in the literature and our $\tau_{MATI}$ for comparison. Only $\tau_{MATI}$ for a (stochastic) uniform inter-sampling time distribution give by [5] is less conservative than $\tau_{MATI}$ given by (2). However, for fair comparison our result should be compared to stochastic arbitrary inter-sampling time distribution give by [5]. When the maximum of delay is 0.05 then $\rho_{1\max} = 0.097, \rho_{2\max} = 0.126$.

## IV. CONCLUSIONS AND FUTURE WORK

We considered a large class of Networked Control Systems (NCSs) and we show that the resulting NCSs can be viewed as *abstract* MIMO sampled-data systems with variable sampling intervals and delay. We modeled the linear delay sampled-data system as a linear infinite-dimensional impulsive systems and we provided conditions for the stability of the closed-loop expressed in terms of LMIs. By solving these LMIs, one can determine positive constants related to each variable sent through the network that determines an upper bound between the sampling time and the next update time at the destination of that variable.

Although in this paper we focused on the stability problem, it is possible to derive LMI conditions that lead to $H_\infty$ designs. Another direction for future research is the medium access scheduling problem and the co-design of feedback controllers and medium-access protocols.

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