Motivation

Largest sampling interval that system remains stable?

Less comm. ~ more users

Important for high cost comm. e.g.

Wireless comm. ~ longer battery life

Example:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
0 & -0.1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0.1
\end{bmatrix}
u, \quad u(t_k) = [3.75, 11.5] x(t_k)
\]

<table>
<thead>
<tr>
<th></th>
<th>Max. sampling interval</th>
<th># of plants in CAN based NCS</th>
<th>Battery life</th>
</tr>
</thead>
<tbody>
<tr>
<td>Walsh ACC 99</td>
<td>$2.7 \times 10^{-4}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Zhang Allerton 01</td>
<td>0.0593</td>
<td>12</td>
<td>1 month</td>
</tr>
<tr>
<td>Naghshtabrizi</td>
<td>1.6</td>
<td>400</td>
<td>30 month</td>
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Outline

- Sampled-data systems (SDSs) with variable sampling & delay
- Different Network Control Systems (NCSs) can be presented by SDSs
- SDSs/NCSs as impulsive systems
- Stability of impulsive systems
- NCSs protocols
- Conclusions and future work

SDSs with variable sampling & delay

\[ \dot{x}(t) = Ax + Bu, \]

\[ y(t) = Cx(t) \]

\[ u(t) = z(t) = x(s_k), \quad t_k \leq t < t_{k+1}, \]

\[ z(t_{k+1}^+) = x(s_{k+1}), \quad k \in \mathbb{N} \]

Variable sampling

Variable delay

Delay \( \tau_k \)

K-th sampling time
update time
\[ t_k := s_k + \tau_k \]

Missing samples:
If we only index samples that get to destination model can capture missing samples.
NCSs (Network Control Systems) V.S. SDSs (Sampled-Data Systems)

Network: Variable sampling, delays, packet dropouts.

Plant: \( \dot{x}_p = A_p x_p + B_p u_p \)
Cont.: \( y_c = K x_p \)
\[ A := A_p, \ B := B_p K \]

All states are measurable. Measurements/control command can be sent in a single packet.

Multi-Input, Multi-Output (MIMO) SDSs

\( K \)-th sampling time
update time \( t_k := s_k + \tau_k \)

\( z := \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix} \)
\( y := \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \)

\( k, j : \text{consecutive sampling time indexes in } K \), \( 1 \leq i \leq m \),
\( z_i(t) := y_i(s_k), \ t_k \leq t < t_j \),

a, c \in K_1
b, d \in K_2
1. One-channel NCS with dynamic feedback controller

Plant:
\[
\begin{align*}
\dot{x}_p &= A_p x_p + B_p u_p \\
y_p &= C_p x_p
\end{align*}
\]

Cont:
\[
\begin{align*}
\dot{x}_c &= A_c x_c + B_c u_c \\
y_c &= C_c x_c + D_c u_c
\end{align*}
\]

\(K\)-th sampling time \( t_k := \delta_k + \tau_k \)

\(y_i(t) := y_{pi}(t), \quad x := \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \quad A := \begin{bmatrix} A_p & B_p C_c \\ 0 & A_c \end{bmatrix}, \quad B := \begin{bmatrix} B_p D_c \\ 0 \end{bmatrix}, \quad C := \begin{bmatrix} C_p & 0 \end{bmatrix}.\)

2. Two-channel NCS with dynamic feedback controller (assume \( D_c = 0 \))

\(K\)-th sampling time \( t_k := \delta_k + \tau_k \)

\(x := \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \quad y_i := \begin{cases} y_{pi}, & 1 \leq i \leq m_p \\ y_{ci}, & m_p + 1 \leq i \leq m_p + m_c \end{cases}, \quad A := \begin{bmatrix} A_p & 0 \\ 0 & A_c \end{bmatrix}, \quad R := \begin{bmatrix} 0 & B_p \\ 0 & 0 \end{bmatrix}, \quad C := \begin{bmatrix} C_p & 0 \\ 0 & C_c \end{bmatrix}.\)
NCSs configurations Modeled by MIMO SDSs

3. Two-channel NCS with anticipative feedback controller

For simplicity, sampling intervals and delays are constant in control channel equal to $h, \tau$

Two-channel and One-channel NCSs

For analysis purposes:

Two-ch NCS with anticipative controller  One-ch NCS with dynamic feedback
SDSs (with delay) as infinite-dim. impulsive systems

\[ z(t) = Az(t) + Bu, \quad x(t) = z(s_k), \quad t_k \leq t < t_{k+1}, \]
\[ x(s_{k+1}) = x(s_k + \tau_k), \quad k \in \mathbb{N} \]
\[ \zeta := \begin{bmatrix} x \\ z \end{bmatrix}, \quad F := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \]

Flow:
\[ \dot{\zeta}(t) = F\zeta(t), \quad t_k \leq t < t_{k+1}, \]

Jumps or impulses:
\[ \zeta(t_{k+1}^+) = \begin{bmatrix} x(t_{k+1}) \\ x(s_{k+1} + \tau_{k+1}) \end{bmatrix}, \quad k \in \mathbb{N} \]

MIMO case

\[ z(t) := \begin{bmatrix} z_1(t) \\ \vdots \\ z_m(t) \end{bmatrix} \]
\[ u := \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad y := \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \]
\[ \zeta := \begin{bmatrix} x \\ z \end{bmatrix}, \quad F := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \]

Flow:
\[ \dot{\zeta}(t) = F\zeta(t), \quad t_k \leq t < t_{k+1}, \]

Jumps or impulses:
\[ \zeta(t_{k+1}^+) = \begin{bmatrix} x(t_{k+1}) \\ y(s_{k+1}) \\ x(s_{k+1} + \tau_{k+1}) \\ \vdots \\ z_m(t_{k+1}) \end{bmatrix}, \quad k + 1 \in \mathcal{K}_i, 1 \leq i \leq m_i \]
Stability of (finite dimensional) impulsive systems

Consider impulsive system (finite dimensional)
\[
\begin{align*}
\dot{x} &= f_k(x, t), & a_k \leq t < a_{k+1}, \\
x(t_k) &= g_k(x(t_k)), & k \in \mathbb{N},
\end{align*}
\]
and a class $\mathcal{S}$ of impulse sequences. System is GUES over the class $\mathcal{S}$ if for every impulse sequence in $\mathcal{S}$, there exists $\exists V : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ s.t.

\begin{itemize}
  \item[(a)] $c_1|x|^b \leq V(x, t) \leq c_2|x|^b$ \hspace{1cm} $\forall x \in \mathbb{R}^n$
  \item[(b)] $\frac{dV(x, t)}{dt} \leq -c_3|x(t)|^b$ \hspace{1cm} $\forall a_k \leq t < a_{k+1}, \ k \in \mathbb{N}$
  \item[(c)] $V(x, t_k) \leq \lim\limits_{t \uparrow t_k} V(x, t)$ \hspace{1cm} $k \in \mathbb{N}$
\end{itemize}

for $c_1, c_2, c_3, b > 0$.

Adopted from Decarlo-Branicky ITAC 00, Liberzon (book) 03

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Stability of infinite-dimensional impulsive systems

Consider delay impulsive system
\[
\begin{align*}
\dot{x} &= f(x, t), & t_k \leq t < t_{k+1}, \\
x(t_{k+1}) &= g(x(t_k), t_{k+1} - \tau_k)) & k \in \mathbb{N},
\end{align*}
\]
and a class $\mathcal{S}$ of impulse-delay sequences. System is GUES over the class $\mathcal{S}$ if for every sequence in $\mathcal{S}$, there exists $\exists V : C([-r, 0], \mathbb{R}^n) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ s.t.

\begin{itemize}
  \item[(a)] $d_1|\phi(0)|^b \leq V(\phi, t) \leq d_2|\phi(0)|^b + \tilde{d}_2 \int_{t-r}^t |\phi(s)|^b ds \hspace{1cm} \forall \phi \in C([-r, 0]), t \in \mathbb{R}^+$
  \item[(b)] $\frac{dV(x_k, t)}{dt} \leq -d_3|x(t)|^b$ \hspace{1cm} $\forall t_k \leq t < t_{k+1}, \ k \in \mathbb{N}$
  \item[(c)] $V(x_{t_k}, t_k) \leq \lim\limits_{t \uparrow t_k} V(x, t)$ \hspace{1cm} $k \in \mathbb{N}$
\end{itemize}

for $d_1, d_2, \tilde{d}_2, d_3, b > 0$.

- Extended version of L-K Theorem for infinite dimensional (delay) systems with jumps.
- Results by Liu ITAC 01, Sun-Michel ITAC 05, didn’t lead to LMI cond. for linear case.
Stability of NCSs with delay

Assume that
\[ s_{k+1} + \tau_{k+1} - s_k \leq \tau_{MATI}, \]
\[ \tau_{\min} \leq \tau_k \leq \tau_{\max}, \forall k \in \mathbb{N}, \]
The system is exponentially stable if \( \exists P, X, Z, R_i > 0, N_i, 1 \leq i \leq 4 \) s.t.
\[
\begin{bmatrix}
M_1 + \beta_{max}(M_2 + M_3) & \tau_{max}N_1 & \tau_{min}N_1 \\
\tau_{max}N_2 & -\tau_{max}R_1 & -\tau_{min}R_1 \\
-\tau_{max}N_3 & -\tau_{min}R_2 & -\tau_{max}N_4
\end{bmatrix} < 0,
\]
where \( \beta_{max} = \tau_{MATI} - \tau_{\min} \) \( M_1 := \cdots \)

\[ V := x^TPx + \int_{t_{k-2}}^{t_{k-1}} (\sigma_{2, k_{max}} - t - s)z'(s)z(s)ds + \int_{t_{k-1}}^{t_{k}} (\sigma_{2, k_{max}} - t - s)z'(s)z(s)ds \]
\[ \rho_1(t) := t - s, \rho_2(t) := t - t_k, t_k \leq t < t_k+1 \]
\[ \sigma_{2, k_{max}} := \sup_{t \geq t_k} \rho_1(t), \sigma_{2, k_{max}} := \sup_{t \geq t_k} \rho_2(t) \ldots \]

Benchmark problem

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u, \quad u(t_k) = \begin{bmatrix} 3.75 \\ 11.5 \end{bmatrix} x(t_k)
\]

Variable sampling:
\[ s_{k+1} - s_k \leq \tau_{MATI}, \forall k \in \mathbb{N} \]

Variable sampling+delay:
\[ s_{k+1} + \tau_{k+1} - s_k \leq \tau_{MATI}, \]
\[ \tau_{\min} \leq \tau_k \leq \tau_{\max}, \forall k \in \mathbb{N}, \]

- This approach constant delay \( \tau_{max} = \tau_{\min} \)
- This approach largest delay upper bound \( \tau_{max} = \tau_{MATI} \)
- Yue et al. Automatica 05
- Naghshtabrizi et al. CDC 05
Stability of NCSs, distributed sensors/actuators ($\tau_{MATI} \text{ VS } \rho_{\text{max}}$)

- Based on previous slide we can extend the results to MIMO case.
- For simplicity, no delay. By solving stability LMIS one gets constants $\rho_{\text{max}}^1, \ldots, \rho_{\text{max}}^n$.

\[ \forall i \quad \text{Interval between consecutive sampling of } y_i \leq \rho_{\text{max}}^i \quad \text{Exp. Stability} \]

- Previous results:

\[ \text{interval between any consecutive samplings } \leq \tau_{MATI} \quad \text{Exp. Stability} \]

\[ y_1 \quad y_2 \quad y_1 \quad y_2 \quad a, c \in K_1 \]
\[ t_a \quad t_b \quad t_c \quad t_d \quad b, d \in K_2 \]

\[ \tau_{MATI} : \quad t_b - t_a, t_c - t_b, t_d - t_c, \ldots \leq \tau_{MATI} \]
\[ \rho^i : \quad t_c - t_a < \rho_{\text{max}}^i, t_d - t_b < \rho_{\text{max}}^i. \]

Benchmark problem: batch reactor

Linearized model of a batch reactor controlled by a PI controller through one-ch. NCS

\[
\begin{align*}
\dot{x}_p &= A_p x_p + B_p u, \\
\dot{y}_p &= C_p x_p
\end{align*}
\]

\[
\begin{align*}
\dot{x}_c &= A_c x_c + B_c u_c, \\
y_c &= C_c x_c + D_c u_c
\end{align*}
\]

No delay, Policy: output1, output2 periodically

\[ \tau_{MATI} \]

<table>
<thead>
<tr>
<th>Authors</th>
<th>Year</th>
<th>Method</th>
<th>$\tau_{MATI}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Walsh et al.</td>
<td>2002</td>
<td>ITAC 02</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Nesic et al.</td>
<td>2004</td>
<td>ITAC 04</td>
<td>0.0082</td>
</tr>
<tr>
<td>Tabbara et al.</td>
<td>2005</td>
<td>CDC05</td>
<td>0.0123</td>
</tr>
<tr>
<td>Hespanha et al.</td>
<td>2006</td>
<td>MTNS 06</td>
<td>0.0279</td>
</tr>
<tr>
<td>Naghshtabrizi et al.</td>
<td>2004</td>
<td></td>
<td>0.0405</td>
</tr>
</tbody>
</table>

\[ \rho_{\text{max}}^1 = 0.081, \quad \rho_{\text{max}}^2 = 0.113 \]

\[ \tau_{MATI} = \frac{1}{2} \min (\rho_{\text{max}}^1, \rho_{\text{max}}^2) \]

Max of delay 0.05, $\rho_{\text{max}}^1 - 0.047, \rho_{\text{max}}^2 - 0.076$
**Round-Robin (RR) protocol**

- Protocol determines how the access to network is granted.

\[ \text{plant2} \rightarrow \text{controller1} \rightarrow \text{plant1} \rightarrow \text{controller2} \]

E.g. plant 2, controller 1, controller 2, plant 1, …

- RR is a static protocol, i.e., assigns access to network in a predetermined and in a cyclic manner.
- Simple, implementable by Token passing based network
- Sufficient condition for stability can be found in Literature. (\( \psi_{max} \) analysis)
- Our analysis also provides a sufficient condition for stability.
- Not robust

Ref: Lian-Talibury IEEE Cont. Sys. Magazine, Walsh ITAC 02, Nesic-Teel ITAC 04 …

**TOD (try once discard) protocol**

- TOD is a dynamic protocol, i.e., assigns access to network based on the current error in the network.

\[ e := \begin{bmatrix} y_1 - z_1 \\ \vdots \\ y_m - z_m \end{bmatrix} \]

The node \( i \in \{1, \ldots, m\} \) with the largest error will be granted the access to network.

- TOD is efficient (based on the current situation of network).
- TOD is robust.
- Sufficient condition for stability can be found in Literature. (\( \sigma_{max} \) analysis)
- TOD is not distributed: there is a central entity that has access to & compares all errors. (relaxed in Tabbara et al. cdc06 by introducing hybrid TOD)

Ref: Walsh ITAC 02, Nesic-Teel ITAC 04 …
Priority based protocol

- Each node has a sending priority based on a monotonically decreasing function of \( r_{\text{max}} - (t - T_i) \)
  \( T_i \): Last time that node i send a packet
  \( t \): Current time
  \( \hat{r}_{\text{max}} \): Deadline

- Inspired by Earliest Deadline First algorithm (Liu and Layland 73):

  \[
  \text{Given } r^{\text{max}}_i \text{ and } \hat{r}_{\text{max}} \text{ such that the stability LMIs are feasible and } \\
  \sum_{i=1}^{n} \frac{r^{\text{max}}_i}{\hat{r}_{\text{max}}} \leq 1 \]

  Then the algorithm is able to generate sampling sequence \( \{s_k\} \) for which system is exponentially stable.

- Robust
- Distributed/scalable
- Implementable on CAN based networks
  - CAN is designed for short (8 byte), time critical messages.
  - 11 bit identifier (version 2.0A) is used to prioritization

Conclusions and future work

Conclusions:
- Sampled-data systems (SDSs) with variable sampling and delay
- We show different Network Control Systems can be presented by SDSs
- Stability of SDSs/impulsive systems
- We introduce priority based protocol

Future work
- Sensor failure
- Ethernet or wireless networks, higher probability of transmission is assigned if deadline is close.
- Controller design.