Nonlinear observability and an invariance principle for switched systems*

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Abstract

This paper proposes several definitions of observability for nonlinear systems and explores relationships between them. These observability properties involve the existence of a bound on the norm of the state in terms of the norm of the output on a small time interval. As an application, we prove a LaSalle-like stability theorem for switched nonlinear systems.

1 Introduction

For linear time-invariant systems with outputs, there are several equivalent ways to define observability. A standard approach is through distinguishability, which is the property that different initial conditions produce different outputs. This is equivalent to 0-distinguishability, which says that nonzero initial conditions produce nonzero outputs. The state of an observable linear system can be explicitly reconstructed from the output measurements on a time interval of arbitrary length by inverting the observability Gramian.

In the nonlinear context, various definitions of observability are no longer equivalent, and in general nonlinear observability is not as completely understood. In particular, the distinguishability concept has a natural counterpart for nonlinear systems, but does not lend itself to a constructive state reconstruction procedure as readily as in the linear case. In fact, it is well known that recovering the state of a nonlinear system from its output, even asymptotically by means of a dynamic observer, is a difficult task. Instead of building an observer, however, it is sometimes sufficient for control purposes (although still far from being trivial) to obtain a bound on the state using the output; see [14] for a discussion and references.

Another concept which is related to observability is detectability. In [14], a variant of detectability for nonlinear systems (called "output-to-state stability") is defined as the property that the state is bounded in terms of the supremum norm of the past output, modulo a decaying term depending on initial conditions. This turns out to be a very useful and natural property, which is dual to input-to-state stability (ISS).

The present work is related to this line of research in that we are concerned with obtaining state bounds. In Section 2 we present several possible definitions of observability for nonlinear systems with no inputs, which involve a bound on the norm of the state in terms of the norm of the output on some (arbitrarily) small time interval. We establish implications and equivalences among these notions in Section 3. We demonstrate, among other things, that the length of the time interval can affect the existence of a state bound. Systems with inputs and other generalizations are discussed in Section 4.

Observability is a stronger property than detectability, and we explore and clarify this relationship below. In fact, one of our definitions is obtained directly from the notion of output-to-state stability by imposing one additional requirement which says, loosely speaking, that the term describing the effects of initial conditions can be chosen to decay arbitrarily fast. In the spirit of [14], we derive a Lyapunov-like sufficient condition for this property in Section 5.

A motivating application for this work is extending LaSalle's invariance principle to switched systems. As shown in [5], a switched linear system is globally asymptotically stable if each subsystem possesses a weak Lyapunov function nonincreasing along its solutions and is observable with respect to the derivative of this function, and if one imposes a suitable non-chattering assumption on the switching signal and a coupling assumption on the multiple Lyapunov functions. This can be viewed as an invariance-like principle for switched linear systems. We generalize this result to switched nonlinear systems in Section 6, using one of the observability definitions introduced in this paper.

2 Observability properties

Consider the system

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function with $f(0) = 0$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ is a continuous function with $h(0) = 0$. We assume that this system is both forward and backward complete (i.e., solutions are globally defined), so that the issue of existence of its solutions on time intervals under consideration does not arise. We will denote by $\|z\|_J$ the supremum norm of a signal z on an interval $J \subset [0, \infty)$. The standard Euclidean norm will be denoted by $|\cdot|$ and the corresponding induced matrix norm by $\|\cdot\|$.

Inequalities written below are understood to hold for all initial conditions. We will say that the system (1) has

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Property 1 if[1]

$$\forall \tau > 0\ \exists \gamma \in \mathcal{K}_\infty\colon |x(0)| \le \gamma(\|y\|_{[0,\tau]}). \tag{2}$$

By time invariance this can be equivalently expressed as

$$\forall \tau > 0\ \exists \gamma \in \mathcal{K}_\infty\colon |x(t)| \le \gamma(\|y\|_{[t,t+\tau]})\ \forall t \ge 0 \tag{3}$$

or, after taking the supremum over $t \in [t_1, t_2]$ for arbitrary $t_2 \ge t_1 \ge 0$, as

$$\forall \tau > 0\ \exists \gamma \in \mathcal{K}_\infty\colon \|x\|_{[t_1,t_2]} \le \gamma(\|y\|_{[t_1,t_2+\tau]})\ \forall t_2 \ge t_1 \ge 0. \tag{4}$$

This last condition includes (2) as a special case (just let $t_1 = t_2 = 0$), and so it is easy to see that (2), (3), and (4) are equivalent. As we will show, they are actually also equivalent to

$$\forall \tau > 0 \quad \exists \gamma \in \mathcal{K}_\infty\colon \|x\|_{[t_1,t_2]} \le \gamma(\|y\|_{[t_1,t_2]})\ \forall t_2 \ge t_1 + \tau. \tag{5}$$

Rather than bounding the state at the beginning of an interval in terms of the future output on that interval, we can bound the state at the end of an interval in terms of the past output on that interval. Let us say that the system (1) has **Property 1′** if

$$\forall \tau > 0\ \exists \gamma \in \mathcal{K}_\infty\colon |x(\tau)| \le \gamma(\|y\|_{[0,\tau]}). \tag{6}$$

By time invariance, this is equivalent to

$$\forall \tau > 0\ \exists \gamma \in \mathcal{K}_\infty\colon |x(t)| \le \gamma(\|y\|_{[t-\tau,t]})\ \forall t \ge \tau. \tag{7}$$

Taking the supremum over $t \in [t_1, t_2]$ for arbitrary $t_2 \ge t_1 \ge \tau$, we arrive at

$$\forall \tau > 0\ \exists \gamma \in \mathcal{K}_\infty\colon \|x\|_{[t_1,t_2]} \le \gamma(\|y\|_{[t_1-\tau,t_2]})\ \forall t_2 \ge t_1 \ge \tau. \tag{8}$$

We now define a different set of observability properties, similar to the above, as follows. Let us say that the system (1) has **Property 2** if

$$\exists\, \tau > 0,\ \gamma \in \mathcal{K}_\infty\colon |x(0)| \le \gamma(\|y\|_{[0,\tau]}). \tag{9}$$

By time invariance, this is equivalent to

$$\exists\, \tau > 0,\ \gamma \in \mathcal{K}_\infty\colon |x(t)| \le \gamma(\|y\|_{[t,t+\tau]})\ \forall t \ge 0. \tag{10}$$

Taking the supremum over $t \in [t_1, t_2]$, we can further rewrite this as

$$\exists\, \tau > 0,\ \gamma \in \mathcal{K}_\infty\colon \|x\|_{[t_1,t_2]} \le \gamma(\|y\|_{[t_1,t_2+\tau]})\ \forall t_2 \ge t_1 \ge 0. \tag{11}$$

[1]Recall that a function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then it is said to be of class $\mathcal{K}_\infty$. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \ge 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \ge 0$. We will write $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$, etc.

The condition (9) is a special case of (11), and we easily see that (9), (10), and (11) are equivalent. It turns out that they are also equivalent to

$$\exists\, \tau > 0,\ \gamma \in \mathcal{K}_\infty\colon \|x\|_{[t_1,t_2]} \le \gamma(\|y\|_{[t_1,t_2]})\ \forall t_2 \ge t_1 + \tau. \tag{12}$$

Note that the only difference between Properties 1 and 2 is that in the former the length τ of the time interval can be arbitrary, while the latter requires the inequalities to hold for at least one positive τ (of course, they will then also hold for all larger values of τ). For linear systems these two properties are known to be equivalent, but for nonlinear systems this is in general not true, as we will see below.

As before, we can bound the state in terms of past output rather than future output. We will say that the system (1) has **Property 2′** if

$$\exists\, \tau > 0,\ \gamma \in \mathcal{K}_\infty\colon |x(\tau)| \le \gamma(\|y\|_{[0,\tau]}). \tag{13}$$

Again, by time invariance we can equivalently express this as

$$\exists\, \tau > 0,\ \gamma \in \mathcal{K}_\infty\colon |x(t)| \le \gamma(\|y\|_{[t-\tau,t]})\ \forall t \ge \tau \tag{14}$$

or, taking the supremum over $t \in [t_1, t_2]$, as

$$\exists\, \tau > 0,\ \gamma \in \mathcal{K}_\infty\colon \|x\|_{[t_1,t_2]} \le \gamma(\|y\|_{[t_1-\tau,t_2]})\ \forall t_2 \ge t_1 \ge \tau. \tag{15}$$

Let us say that the system (1) has **Property 3** if there exists a function $\gamma \in \mathcal{K}_\infty$ such that

$$\|x\|_{[0,\infty)} \le \gamma(\|y\|_{[0,\infty)}) \quad \forall\, x(0),\ t \ge 0. \tag{16}$$

This is the *strong observability* property, considered in [13] for the more general case of systems with inputs (cf. Section 4 below).

In [14], the authors define the property of output-to-state stability, which is a variant of detectability and is characterized by an inequality of the form

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(\|y\|_{[0,t]}) \quad \forall\, x(0),\ t \ge 0 \tag{17}$$

where $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$. Strengthening this notion, we say that the system (1) has **Property 4** if for every $\varepsilon > 0$ and every function $\nu \in \mathcal{K}$ there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that the inequality (17) holds and, moreover, we have

$$\beta(r, \varepsilon) \le \nu(r) \qquad \forall r \ge 0. \tag{18}$$

The condition (18) can be interpreted as saying that β can be chosen to decay arbitrarily fast, because ε can be arbitrarily small and ν can grow arbitrarily slowly. (Note that there are no additional conditions on the function γ, which may then have to be increased.)

In the same spirit as before, we introduce a variant of Property 4 by requiring that (18) hold for all $\nu \in \mathcal{K}$ and at least one positive ε (but not necessarily for all ε). Namely, we will say that the system (1) has **Property 5** if there exists an $\varepsilon > 0$ such that for every function $\nu \in \mathcal{K}$ there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ for which the conditions (17) and (18) are satisfied.

3 Implications and equivalences

The following technical lemma is a straightforward consequence of forward completeness, continuous dependence of solutions on initial conditions, and the presence of an equilibrium at the origin.

Lemma 1 *For every $\tau > 0$ there exists a function $\nu_f \in \mathcal{K}_\infty$ such that along all solutions of the system* (1) *we have $|x(t_2)| \le \nu_f(|x(t_1)|)$ for each pair of times t_1, t_2 satisfying $0 \le t_1 \le t_2 \le t_1 + \tau$.*

We also need the backward in time version.

Lemma 2 *For every $\tau > 0$ there exists a function $\nu_b \in \mathcal{K}_\infty$ such that along all solutions of the system* (1) *we have $|x(t_1)| \le \nu_b(|x(t_2)|)$ for each pair of times t_1, t_2 satisfying $0 \le t_1 \le t_2 \le t_1 + \tau$.*

These results allow us to conclude, in particular, that Properties 1 and 1′ defined in the previous section are equivalent. Indeed, (2) implies (6) in view of Lemma 1, and the converse follows from Lemma 2. The equivalence between (4) and (5) is deduced with the help of Lemma 1. Thus the properties expressed by conditions (2)–(8) are all equivalent. The same arguments (for a given τ) demonstrate that the properties expressed by conditions (9)–(15) are also equivalent.

The following theorem explains the relationship between the above properties (we refer to the observability properties by their numbers, so that for example $1 \Rightarrow 2$ means that Property 1 implies Property 2).

Theorem 3 *The only implications that hold among the properties introduced in Section 2 are:*

$$\boxed{1 \Leftrightarrow 1' \Leftrightarrow 4 \Rightarrow 2 \Leftrightarrow 2' \Leftrightarrow 5 \Rightarrow 3}$$

Remark 1 It is easy to see that each of the above properties implies the standard 0-distinguishability notion: the only invariant set in $\ker h$ is $\{0\}$. Note that the converse does not hold. As an example, consider the scalar system $\dot{x} = x$, $y = \arctan x$. It is clearly 0-distinguishable (in fact, distinguishable: the output map is invertible), but x blows up while y stays bounded.

Remark 2 It is interesting to compare the above findings with the results reported in [12] for discrete-time systems. For example, the counterparts of Properties 1 and 1′, or 2 and 2′, in discrete time (i.e., *initial-state* vs. *final-state* observability) are no longer equivalent. To see why, it is enough to consider a system whose output map is zero and whose state becomes zero after one step.

In view of the equivalences established in Theorem 3, we can now give one name to Properties 1, 1′ and 4 and also give one name to Properties 2, 2′ and 5. Prompted by terminology used in the controllability literature [4], let us call the system (1) *small-time norm-observable* if it satisfies Properties 1, 1′ and 4, and *large-time norm-observable* if it satisfies Properties 2, 2′ and 5. In the latter case, we will refer to every τ provided by Properties 2 and 2′ as a *large-time norm-observability constant* of (1).

For linear systems, all of the above properties are equivalent to the usual observability. For Properties 1–3 this can be easily shown using the observability Gramian. Property 4 is less obvious, and can be viewed as a generalization of the *squashing lemma* from [11] whose proof relies on the well-known result about arbitrary pole placement by output injection. This lemma says that if (C, A) is an observable pair, then for every $\varepsilon > 0$ and every $\delta > 0$ there exist a $\lambda > 0$ and an output injection matrix K such that we have $\|e^{(A+KC)t}\| \le \delta e^{-\lambda(t-\varepsilon)}$, which implies $\|e^{(A+KC)\varepsilon}\| \le \delta$. Therefore, in the linear case the function β in (17) can be chosen to satisfy $\beta(r,\varepsilon) \le \delta r$ with δ arbitrarily small. Property 4 can be deduced from this if the function ν is restricted to be bounded from below by a linear function, but otherwise Property 4 expresses a more general fact—even for linear systems.

4 Extensions

In this section we consider, instead of (1), the system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{19}$$

where u is a measurable locally essentially bounded disturbance or control input taking values in a set $\mathcal{U} \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuously differentiable function with $f(0,0) = 0$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is a continuous function with $h(0) = 0$. Forward and backward completeness of this system mean that solutions are globally defined for all inputs. We want to investigate how the definitions of Section 2 and the results of Section 3 can be extended to this case.

First, let us assume that $\mathcal{U}$ is a compact set and $f(0,u) = 0$ for all $u \in \mathcal{U}$. Then we can define observability properties for the system (19) in the same way as in Section 2, simply adding the quantification "for all $u \in \mathcal{U}$". In other words, we now require that Properties 1 through 5 hold uniformly over all inputs. It follows from the results of [10, Section 5] that Lemmas 1 and 2 still hold, where solutions of the system are now parameterized by all initial conditions and all inputs. Therefore, the results of Section 3 are still true for these modified properties.

Now, let us drop the assumptions that $\mathcal{U}$ is compact and $f(0,u) \equiv 0$. In this more general situation, imposing uniformity over inputs is too restrictive. More meaningful observability properties result if we add the term $\chi(\|u\|_J)$ to the right-hand sides of the inequalities (2)–(17), where χ is a class $\mathcal{K}_\infty$ function and for J one must substitute the interval over which the norm of y is taken. This is equivalent to replacing y by the vector $\binom{u}{y}$ in the corresponding formulas. In particular, the inequality (16) which describes Property 3 transforms precisely into the strong observability property [13], while the inequality (17), which is one of the two conditions describing

Property 4 and which corresponds to output-to-state stability, transforms into the input-output-to-state stability property [14]. The results of [10, Section 5] imply that Lemma 1 is valid if the inequality $|x(t_2)| \le \nu_f(|x(t_1)|)$ is replaced by $|x(t_2)| \le \nu_f(|x(t_1)|) + \chi_f(\|u\|_{[t_1,t_2]})$ for some $\chi_f \in \mathcal{K}_\infty$, and similarly for Lemma 2. Therefore, it is not hard to check that all arguments still go through and the results still hold for the modified properties.

Another way to generalize our earlier developments is to replace the forward and backward completeness assumption by the weaker *unboundedness observability* property, which means that the output becomes unbounded whenever the state becomes unbounded. This can be done for the original system (1) as well as for the system with inputs (19). The results of [1, Section 2] extend the aforementioned results of [10] and show that in this case, the estimates of Lemmas 1 and 2 (or the corresponding results in the presence of inputs described above) need to be modified by adding a term of the form $\gamma(\|y\|_{[t_1,t_2]})$, $\gamma \in \mathcal{K}_\infty$ to the right-hand side. It is not difficult to see that this does not affect the results of Section 3. The definitions of Properties 1–5 should now be restricted to intervals on which solutions exist (although even when the solutions are not defined, the inequalities are formally true in the sense that $\infty \le \infty$).

5 Lyapunov functions

An attractive feature of Properties 4 and 5 is that they can be characterized in terms of Lyapunov-like inequalities, as we now show. We present the result for Property 4, the case of Property 5 being analogous.

Proposition 4 *Consider the system* (1). *Suppose that for every $\varepsilon > 0$ and every $\nu \in \mathcal{K}$ there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions α_1, α_2 and ρ, and a positive definite locally Lipschitz function $\alpha_3 : [0,\infty) \to [0,\infty)$ such that we have*

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$$

and

$$|x| \ge \rho(|y|) \Rightarrow \frac{\partial V}{\partial x} f(x) \le -\alpha_3(V(x)) \tag{20}$$

and moreover

$$\eta^{-1}(\eta(r)+\varepsilon) \le \alpha_1 \circ \nu \circ \alpha_2^{-1}(r) \qquad \forall r \ge 0 \tag{21}$$

where η is defined by[2]

$$\eta(r) := -\int_1^r \frac{ds}{\alpha_3(s)}.$$

Then Property 4 holds.

The proof, not given due to space constraints, follows the arguments of [13, 14]. An informal interpretation of Proposition 4 is that Property 4 holds if there exists a positive definite radially unbounded function V which decays along solutions whenever $|x|$ is sufficiently large compared to $|y|$ and, moreover, this decay rate—described by the function α_3—can be made arbitrarily fast by a proper choice of V. (The "gain margin" function ρ, on the other hand, may have to be increased in order to achieve this; note that the extra condition (21) does not involve ρ.) To better understand the role of α_3, note that if α_3 grows rapidly, then the graph of η is "flat", and consequently the function $\eta^{-1}(\eta(\cdot)+\varepsilon)$ is small. In fact, this function is approximated, up to the first-order term in ε, by $r - \alpha_3(r)\varepsilon$. To illustrate with the linear case, suppose that $\alpha_1(r) = c_1 r^2$, $\alpha_2(r) = c_2 r^2$, and $\alpha_3(r) = kr$ so that $\eta^{-1}(\eta(r)+\varepsilon) = e^{-k\varepsilon} r$. We see that by choosing a sufficiently large k we can satisfy the condition (21) if and only if ν is bounded from below by a linear function. Thus working with quadratic V and linear α_3 is in general not sufficient, even for linear systems.

It is straightforward to extend the above result to the system (19). Property 4 then needs to be interpreted as explained in Section 4. In the case when the inputs do not take values in a compact set and uniformity with respect to inputs is not required, one needs to replace y by $\binom{u}{y}$ in (20).

6 Invariance principle

Consider the system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. One version (in fact, a special case) of the well-known LaSalle's invariance principle can be stated as follows. If there exists a positive definite, radially unbounded, continuously differentiable (C^1) function $V : \mathbb{R}^n \to \mathbb{R}$ whose derivative along solutions satisfies $\dot{V}(x) := \frac{\partial V}{\partial x} f(x) \le 0$, and if moreover the largest invariant set contained in the set $\{x : \dot{V}(x) = 0\}$ is equal to $\{0\}$, then the system is globally asymptotically stable. The second condition can be regarded as observability (0-distinguishability) with respect to the auxiliary output $y := -\dot{V}(x)$. Here the negative sign is used for convenience, so that $y \ge 0$.

In this section we derive an extension of the above result to switched systems. This generalizes the earlier work on switched linear systems reported in [5]. Some remarks on relationships to other LaSalle-like theorems available in the literature are provided at the end of the section.

Consider a family of systems

$$\dot{x} = f_p(x), \qquad p \in \mathcal{P}$$

where $\mathcal{P}$ is a finite index set and $f_p : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function for each $p \in \mathcal{P}$. We make the following two assumptions regarding these systems, which parallel the assumptions for the traditional LaSalle's theorem stated above. The first assumption is the existence of a weak (i.e., nonstrictly decreasing) Lyapunov function for each system, and the second one is observability with respect to the derivative of this function playing the role of an auxiliary output (however, instead of 0-distinguishability we require the stronger small-time norm-observability property; cf. Remark 1 in Section 3).

[2] We use the conventions $\eta(0) = \infty$ and $\eta^{-1}(\infty) = 0$, which are consistent with continuity.

1. For each $p \in \mathcal{P}$ there exists a positive definite radially unbounded C^1 function $V_p : \mathbb{R}^n \to \mathbb{R}$ which satisfies

$$\frac{\partial V_p}{\partial x} f_p(x) \le 0 \qquad \forall x.$$

2. For each $p \in \mathcal{P}$ the system

$$\begin{aligned} \dot{x} &= f_p(x) \\ y &= -\frac{\partial V_p}{\partial x} f_p(x) \end{aligned} \tag{22}$$

is small-time norm-observable as defined at the end of Section 3 (i.e., has the equivalent Properties 1, 1′ and 4 introduced in Section 2).

We now consider the *switched system*

$$\dot{x} = f_\sigma(x) \tag{23}$$

where $\sigma : [0, \infty) \to \mathcal{P}$ is a piecewise constant *switching signal*, continuous from the right. We denote by t_i, $i = 1, 2, \dots$ the consecutive discontinuities of σ (the *switching times*). Two more assumptions are needed, with regard to this switched system. The first one is a rather mild non-chattering requirement on σ (which will be further discussed below), and the second is a typical condition on the evolution of the functions V_p, $p \in \mathcal{P}$ encountered in results using multiple Lyapunov functions (see [3, 9, 6]).

3. If there are infinitely many switching times, there exists a $\tau > 0$ such that for every $T \ge 0$ we can find a positive integer i for which $t_{i+1} - \tau \ge t_i \ge T$. In other words, we persistently encounter intervals of length at least τ between switching times.

4. For each $p \in \mathcal{P}$ and every pair of consecutive intervals $[t_i, t_{i+1})$, $[t_j, t_{j+1})$ on which $\sigma = p$ we have $V_p(x(t_j)) \le V_p(x(t_{i+1}))$. In other words, the value of V_p at the beginning of each interval on which $\sigma = p$ does not exceed the value of V_p at the end of the previous such interval (if one exists).

Theorem 5 *Under assumptions 1–4 the switched system* (23) *is globally asymptotically stable.*

Proof. Stability of the origin in the sense of Lyapunov follows from assumptions 1 and 4 and the finiteness of $\mathcal{P}$ as in the proof of [2, Theorem 2.3]. Now, take an arbitrary solution of (23). Our goal is to prove that it converges to 0. We are assuming that there are infinitely many switching times, for otherwise the result immediately follows from Remark 1 and the standard LaSalle's theorem cited earlier. In light of assumption 3 and the fact that $\mathcal{P}$ is finite, we can pick an infinite subsequence of switching times $t_{i_1}, t_{i_2}, \dots$ such that the corresponding intervals $[t_{i_j}, t_{i_j+1})$, $j = 1, 2, \dots$ have length no smaller than some fixed $\tau > 0$ and the value of σ on all these intervals is the same, say, $q \in \mathcal{P}$. Let us denote the union of these intervals by $\mathcal{Q}$ and consider the auxiliary function

$$y_{\mathcal{Q}}(t) := \begin{cases} y(t) & \text{if } \ t \in \mathcal{Q} \\ 0 & \text{otherwise} \end{cases}$$

In view of assumption 4, for every $t \ge 0$ we have

$$\int_0^t y_{\mathcal{Q}}(t) \le V_q(x(t_{i_1})) - V_q(x(t)) \le V_q(x(t_{i_1})).$$

Since $y_{\mathcal{Q}}$ is nonnegative by assumption 1, we see that $\int_0^\infty y_{\mathcal{Q}}(t)dt < \infty$, i.e., $y_{\mathcal{Q}} \in \mathcal{L}_1$.

We proceed to prove that $y_{\mathcal{Q}}(t) \to 0$ as $t \to \infty$. Suppose that this is not true. Then there exist an $\varepsilon > 0$ and an infinite sequence of times $s_1, s_2, \dots$ such that the values $y_{\mathcal{Q}}(s_1)$, $y_{\mathcal{Q}}(s_2)$, $\dots$ are bounded away from zero by at least ε. It follows from the definition of $y_{\mathcal{Q}}$ that the times $s_1, s_2, \dots$ necessarily belong to $\mathcal{Q}$. Now, assumption 1 guarantees that x remains bounded, hence $\dot{x}$ is also bounded and so $y_{\mathcal{Q}}$ is uniformly continuous on $\mathcal{Q}$. Therefore, we can find a $\delta > 0$ such that each s_i is contained in some interval of length δ on which $y_{\mathcal{Q}}(t) \ge \varepsilon/2$. This contradicts the assertion proved earlier that $y_{\mathcal{Q}} \in \mathcal{L}_1$, thus indeed $y_{\mathcal{Q}}(t) \to 0$.

To show that $x(t)$ converges to 0, we invoke assumption 2. Applying the condition (3) with $t = t_{i_j}$, $j = 1, 2, \dots$ and using the above analysis, we conclude that $x(t_{i_j}) \to 0$ as $j \to \infty$. It then follows from stability of the origin in the sense of Lyapunov that $x(t) \to 0$ as needed. □

One way to satisfy assumption 3 is to demand that consecutive switching times be separated by some positive *dwell time* τ_D. A less severe condition is provided by the following concept, introduced in [7]. The switching signal σ is said to have *average dwell time* $\tau_{AD} > 0$ if for some $N_0 > 0$ the number of its discontinuities on an arbitrary interval (t_1, t_2), denoted by $N_\sigma(t_2, t_1)$, satisfies

$$N_\sigma(t_2, t_1) \le N_0 + \frac{t_2 - t_1}{\tau_{AD}}.$$

Lemma 6 *If σ has average dwell time τ_{AD}, then the assumption 3 holds. As a desired τ, one can take an arbitrary number in the interval $(0, \tau_{AD})$.*

Note that the average dwell time τ_{AD} in the above result can be arbitrarily small, as long as it exists. If τ_{AD} is known, then we can relax assumption 2 by requiring only that the system (22) be large-time norm-observable as described by Property 2 with $\tau < \tau_{AD}$. Accordingly, if this system is known to be large-time norm-observable but not small-time norm-observable, then a variant of Theorem 5 can be established under a suitable slow switching condition. We thus introduce the following modified versions of assumptions 2 and 3.

2′. For each $p \in \mathcal{P}$ the system (22) is large-time norm-observable.

3′. If there are infinitely many switching times, for every $T \ge 0$ we can find a positive integer i for which $t_{i+1} - \tau \ge t_i \ge T$, where τ is a large-time norm-observability constant of the system (22).

The following result is proved by the same arguments as Theorem 5.

Theorem 7 *Under assumptions 1, 2′, 3′, and 4 the switched system* (23) *is globally asymptotically stable.*

The usefulness of Theorems 5 and 7 stems in part from the fact that it is sometimes easier to find weak Lyapunov functions nonincreasing along solutions and satisfying assumption 4 (or even a common weak Lyapunov function for a given family of systems) than to find Lyapunov functions strictly decreasing along solutions and satisfying assumption 4 (or, in particular, to find a common Lyapunov function). Under various slow switching conditions such as the one needed for Theorem 7, it is possible to deduce global asymptotic stability of a switched system from global asymptotic stability of individual subsystems (see [7, 9]). However, in the nonlinear context these results require additional, often restrictive assumptions, and the resulting bounds on the switching rate may be more conservative than the one obtained from Theorem 7.

A different version of LaSalle's invariance principle for systems with switching events has appeared in [15, Theorem 1]. That result states that if for a given hybrid system with a finite number of discrete states one can find a function of both the continuous and the discrete state which is nonincreasing along solutions, then all bounded solutions approach the largest invariant set inside the set of states where the instantaneous change of this function is zero. The proof is a relatively straightforward adaptation of the standard argument for continuous time-invariant systems. See also [8] for a similar result.

Theorems 5 and 7 apply to a different class of systems than the hybrid systems studied in [15], because in the present setting the switching is not assumed to be state-dependent. However, one way in which a switched system of the type considered here may arise is from a hybrid system by means of an abstraction procedure. In this case, our observability assumptions would serve as sufficient conditions for the largest invariant set mentioned earlier to be the origin, since they guarantee that along a nonzero solution the output cannot remain identically zero on any interval between switching times. Note, however, that we do not require the existence of a single function nonincreasing along solutions, and instead work with multiple weak Lyapunov functions satisfying the less restrictive assumption 4. This aspect of the results presented above—namely, that they rely to a large extent on separate conditions regarding the individual systems being switched—also sets them apart from LaSalle-like theorems available in the literature for certain classes of time-varying and other systems. (On the other hand, the conclusions provided by results such as Theorem 1 of [15] are stronger and closer in spirit to those of the classical LaSalle's theorem.)

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