Discrete Approximations to Continuous Shortest-Path: Application to Minimum-Risk Path Planning for Groups of Uninhabited Air Vehicles (UAVs)

Jongrae Kim                 João Hespanha

University of California
Santa Barbara

Outline

1. Motivation—minimum-risk path planning
2. Discretization approach to shortest-path
3. Sampling methods
4. Back to minimum-risk path planning…
**Anisotropic shortest-path problem**

Compute path $\rho$ from initial position $x_i$ to final position $x_f$ minimizing the integral cost

$$J[\rho] := \int_0^T \ell(\rho(t), \dot{\rho}(t)) \, dt$$

as opposed to isotropic shortest path

$$J[\rho] := \int_0^T \ell(\rho(t), \|\dot{\rho}(t)\|) \, dt$$

**Motivating problems**

- Minimum-risk path planning for fixed-wing UAVs
- Path-planning for UAV surveillance in complex 3D environments

- Probability of radar acquisition depends on UAVs attitude
- “Safe-space” is very much nonconvex

- Region covered by sensor depends on UAVs attitude
- Free-space is very much nonconvex
Radar acquisition

Probability of UAV being acquired by radar on an elementary time interval \( dt \):

\[ \eta(x, \dot{x}, m) dt \]

\( \eta \) changes significantly with the attitude of the UAV with respect to the radar.

Survivability ("risk")

Probability of UAV being acquired by a radar on a path \( \rho \) with duration \( T \)

\[ p^{\text{acq}}[\rho] = 1 - e^{- \int_0^T \eta^{\text{acq}}(\rho(t), \dot{\rho}(t)) \, dt} \]

Probability of UAV being destroyed by a radar on a path \( \rho \) with duration \( T \)

\[ p^{\text{kill}}[\rho] = p^{\text{leth}} \left( 1 - e^{- \int_0^T \eta^{\text{acq}}(\rho(t), \dot{\rho}(t)) \, dt} \right) \]

Conditional probability of kill, given that UAV was acquired by radar.
Survivability ("risk")

$k$ SAM sites

probability of UAV being acquired by a radar on a path $\rho$ with duration $T$

$$p^{\text{acq}}[\rho] = 1 - e^{-\int_0^T \ell^{\text{acq}}(\rho(t), \dot{\rho}(t)) dt}$$

$\ell^{\text{acq}}(x, v) := \sum_{i=1}^k \eta_i(x, v, m_i)$

probability of UAV being destroyed by a radar on a path $\rho$ with duration $T$

$$p^{\text{kill}}[\rho] = p^{\text{letha}} \left( 1 - e^{-\int_0^T \ell^{\text{acq}}(\rho(t), \dot{\rho}(t)) dt} \right)$$

Anisotropic shortest-path problem

minimize $J[\rho] := \int_0^T \ell(\rho(t), \dot{\rho}(t)) dt$

over all paths $\rho$ with (non-fixed) duration $T > 0$ such that

$\rho(0) = x_i$ initial position

$\rho(T) = x_f$ final position

$||\dot{\rho}|| \leq$ maximum velocity of slowest UCAV
Anisotropic shortest-path problem

\[
\begin{align*}
\text{minimize} \quad J[\rho] := & \int_0^T \ell(\rho(t), \dot{\rho}(t)) dt \\
\text{over all paths } \rho \text{ with (non-fixed) duration } T > 0 \text{ such that} \\
\rho(0) &= x_i \quad \text{initial position} \\
\rho(T) &= x_f \quad \text{final position} \\
\|\dot{\rho}\| &\leq \text{maximum velocity of slowest UCAV}
\end{align*}
\]

We would obtain similar problem formulations if the previous setup was generalized to:

1. *Unknown radar positions* (need to take expected value with respect to the distribution of the radars, analytically or using randomized methods)
2. *Groups of UAVs* flying together
3. *Multi-criteria* optimization (e.g., considering constraints on path duration and fuel consumption)

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Variational approach
Assuming that there exists a solution to the Hamilton-Jacobi-Bellman PDE

\[
\min_{v \in V} H(x, v, \nabla_x V(x)) = 0 \quad V(x_f) = 0
\]

the optimal path is given by

\[
\rho(t) = x_i + \int_0^t \arg \min_{v \in V} H(\rho(\tau), v, \nabla_x V(\rho(\tau))) d\tau
\]
Anisotropic shortest-path problem

\[ \begin{align*}
\text{minimize} & \quad J[\rho] := \int_0^T \ell(\rho(t), \dot{\rho}(t))dt \\
\text{over all paths} & \quad \rho \quad \text{with (non-fixed) duration} \\
\rho(0) & \quad = x_i \quad \text{initial} \\
\rho(T) & \quad = x_f \quad \text{final} \\
\|\dot{\rho}\| & \quad \leq \text{maximum}
\end{align*} \]

**Variational approach**

Assuming that there exists a solution to the Hamilton-Jacobi-Bellman PDE

\[ \min_{v \in \mathcal{V}} H(x, v, \nabla_x V(x)) = 0 \quad V(x_f) = 0 \]

the optimal path is given by

\[ \rho(t) = x_i + \int_0^t \arg \min_{v \in \mathcal{V}} H(\rho(\tau), v, \nabla_x V(\rho(\tau)))d\tau \]

Numerical methods based on the maximum principle also run into difficulties because of local minima.

**Discretization approach**

- extract a finite set \( \mathcal{X} \) of points
- restrict the search to piecewise linear paths between points in \( \mathcal{X} \)

\[ \rho(t) = x_{k-1} + \frac{x_k - x_{k-1}}{\|x_k - x_{k-1}\|} (t - t_{k-1}) \]

\[ \forall t \in [t_{k-1}, t_k], \ k \in \{1, 2, \ldots, m\} \]

with

\[ 0 = t_0 < t_1 < \cdots < t_m = T \]

\[ x_k \in \mathcal{X} \quad \forall k \in \{0, 1, \ldots, m\} \]

\( \odot \) original continuous problem was converted into an optimization on a finite graph

\( \odot \) at the expense of a higher cost
**Discretization approach**

- Extract a finite set $X$ of points
- Restrict the search to piecewise linear paths between points in $X$

$$\rho(t) = x_{k-1} + \frac{x_k - x_{k-1}}{||x_k - x_{k-1}||} (t - t_{k-1})$$

$\forall t \in [t_{k-1}, t_k], \ k \in \{1, 2, \ldots, m\}$

$$0 = t_0 < t_1 < \cdots < t_m = T$$

$$x_k \in X \ \forall k \in \{0, 1, \ldots, m\}$$

- Original continuous problem was converted into an optimization on a finite graph
- At the expense of a higher cost
- The cost penalty can be made arbitrarily small

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**Suboptimality**

**Theorem:** For every $\epsilon$, $A > 0$ there exists a finite set $X$ of $N_\epsilon$ points such that

$$\min_{\rho \in P_\epsilon} J[\rho] - \min_{\rho \in P_{\|\rho\| \leq A}} J[\rho] \leq \epsilon$$

Optimal cost over piecewise linear paths

Optimal cost over all paths with acceleration bounded by $A$

But less cost penalty requires more points
**Optimized sampling**

**Theorem:** For every $\epsilon$, $A > 0$ there exists a finite set $X$ of $N$ points such that

$$\min_{\rho \in \mathcal{F}_X} J[\rho] - \min_{\rho \in \mathcal{F}_{\|\rho\| \leq A}} J[\rho] \leq \epsilon$$

One can show that the cost penalty $\epsilon$ is proportional to

$$\frac{g_{x,k} \epsilon_x + g_{v,k} \epsilon_v + \left(\frac{g_{x,k} + g_{v,k} A}{2}\right) \|x_k - x_{k-1}\|}{\text{norm of the cost gradient around the points}} \quad \text{distance between consecutive points}$$

$$g_{x,k} := \sup_{x \in \mathcal{R}, \|v\| \leq \epsilon} \|\nabla_x \ell(x, v)\|$$

$$g_{v,k} := \sup_{x \in \mathcal{R}, \|v\| \leq \epsilon} \|\nabla_v \ell(x, v)\|$$

one can under-sample where gradient is low and over-sample where gradient is high

**Sampling methods**

**Uniform sampling**

extract points randomly over the whole region, with uniform probability

**Gradient sampling**

extract points randomly over the whole region, with probability proportional to the norm of the gradient

sampling over a regular grid is much worse for anisotropic costs and will not be reported here (not enough directions!)
**Optimized sampling**

Honeycomb sampling

1. extract points randomly over the whole region with probability proportional to the norm of the gradient

\[ \frac{g_{r,k} s_x + g_{r,k} s_y + \frac{g_{r,k} + g_{r,k-1}}{2} \|x_k - x_{k-1}\|}{\text{gradient-norm}} \]

2. compute the Voronoi diagram for these points

3. uniformly sample the edges of the Voronoi diagram

**Sampling methods – spots configurations**

- **Gradient-norm**
- **Cost & optimal path**
- **Uniform sampling**
- **Gradient-proportional sampling**
- **Honeycomb sampling**
Since the optimal path can avoid danger, gradient-based methods does not help with respect to uniform sampling.
Sampling methods – X configurations

probability of radar acquisition is minimized through azimuth/elevation angles management

Gradient-based methods improve upon uniform sampling

honeycomb sampling is fast and provides good performance over a wide range of configurations
Sampling methods – complex config.

Uniform sampling

Honeycomb sampling

Gradient-proportional sampling

Honeycomb sampling is fast and provides good performance over a wide range of configurations.
Conclusions

• Proposed a method to solve anisotropic shortest path problems based on discretization of the continuous state-space
• Proposed a “honeycomb” sampling algorithm to minimize the penalty introduced
• Illustrated the potentialities of the algorithm for minimum-risk path planning for groups of UAVs—“honeycomb” consistently produces low-risk paths with less computation time!

Future work
• Apply this type of algorithm to other path planning problems (e.g., UAV surveillance in complex 3D environments)
• Support incremental computation to obtain any-time optimization algorithms (for real-time implementation)