On the robust stability and stabilization of sampled-data systems: A hybrid system approach

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***Abstract*— We consider the stability analysis and state-feedback stabilization of LTI uncertain sampled-data systems. There are two sources of uncertainty: the plant's parameters can be uncertain and the sampling intervals can be unknown and variable. We model the sampled-data system as a hybrid system and we employ a Lyapunov function with discontinuities to establish the stability of the system. Our stability and stabilization results are presented as Linear Matrix Inequalities (LMIs). By solving these LMIs, one can find a positive constant which determines an upper bound on the sampling intervals such that the stability of the closed-loop is guaranteed. The control design LMIs also provide controller gains that can be used to stabilize a given process. To reduce the conservativeness we use slack matrices; however, we require a smaller number of slack matrices than in the previous results and we show that we have done it without making the results more conservative. As a special case we consider sampled-data systems with constant sampling intervals and provide results for this class of systems that are less conservative than the ones obtained for the general case of variable sampling time.**

I. Introduction

We consider the stability analysis and state-feedback stabilization of uncertain sampled-data systems. We consider an LTI plant connected to a digital state-feedback controller through sampling and hold. There are two sources of uncertainty: the values of the plant's parameters can be unknown while satisfying a polytopic condition and the sampling interval can be uncertain and variable. This framework is general enough to capture networked control systems (NCSs) with packet dropout [1].

Sampled-data systems have been studied extensively over the past decades and three main approaches have been used for robust sampled-data stabilization. The first one is based on *lifting* [2, 3], in which the problem is transformed into an equivalent finite-dimensional discrete-time problem while maintaining the inter-sampling information of the system. This approach does not work in the case of uncertain sampling interval and polotopic parametric uncertainty in the system matrices. The second approach is based on modeling the sampled-data system as a continuous-time system with a delayed control input [4, 5]. In this case, the closed-loop becomes an *infinite-dimensional* Delay Differential Equation (DDE) and the stability is established using Razumikin or Lyapunov-Krasovskii Theorems. The third approach is based on the hybrid modeling of sampled-data systems in which a *time-varying periodic* Lyapunov function is used [6, 7]. The downside of this approach is that the sampling interval of the plant's output must be constant and the sampling interval of the controller's output is ℓ-times ($\ell \in \mathbb{N}$) smaller than the plant's output sampling interval.

Our approach is based on a hybrid modeling of sampled-data systems but we use a *Lyapunov function with jumps*. However, this Lyapunov function is inspired by the Lyapunov functionals that appear in DDEs (c.f. [5]). The stability conditions are presented as Linear Matrix Inequalities (LMIs) which can be solved numerically using software packages such as MATLAB. By solving these LMIs, one can find a positive constant which determines the maximum sampling interval. In the NCSs literature this maximum sampling interval is often called τ_{MATI}, e.g., in [8].

We also consider the control design problem, in which a static controller gain becomes a parameter to be selected. When this gain is viewed as an unknown, the stability conditions mentioned above become Bilinear Matrix Inequalities (BMIs). For numerical efficiency, these BMIs are converted back into a LMI feasibility problem. These LMI provides not only the controller gain, but also the largest τ_{MATI} for which stability can be assured. We show that our conditions improve upon previous results in the sense that the closed-loop system remains stable for larger sampling intervals.

Sampled-data systems whose continuous state evolves according to an ordinary differential equation (ODE) are finite-dimensional because the future evolution of the system can be completely determined from the current value of the ODE's state and the last received sample (given that the sampling sequence is known). We use a *Lyapunov function* with discontinuities to establish the stability of such systems. In [5] these systems are modeled as infinite dimensional DDEs and a *Lyapunov functional* is used to prove their stability. We show that for *any system* our sufficient condition for stability of sampled-data system is less (or equally) conservative than the one in [5, 9]. From this perspective, considering an infinite dimensional DDE model and using a Lyapunov functional to prove its stability offers no advantage for this class of finite-dimensional systems.

Slack matrices introduce degrees of freedom that can be exploited to minimize conservativeness and they have been used extensively for analyzing the stability of sampled-data system and DDEs (c.f. [5, 9, 10]). We also use slack matrices; however, we use *a smaller number* of slack matrices, without

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making the results more conservative. Decreasing the number of slack matrices reduces the number of scalar variables in the LMIs and enables us to study larger dimensional systems. Developing matrix inequalities for the design problem also becomes simpler.

When it is known that the sampling intervals are constant, the work reported in [5, 9, 10] is unable to take advantage of this information. It therefore arrives at the same value for τ_{MATI} that would be obtained for variable (but upper-bounded) sampling intervals. ([9, 10] allow the delay in the control loop; however, we assume the delay is zero). The conditions in our paper distinguish between the cases of constant vs. variable sampling intervals and provide less conservative results when the sampling interval is fixed. Although the classical time domain, frequency domain, and lifting approaches provide necessary and sufficient conditions for stability (and stabilization) of sampled-data systems with constant sampling intervals, they are not readily applicable when there is a polytopic uncertainty in the parameters of the continuous-time plant model.

In section II we provide the sufficient conditions for the stability of sampled-data systems in terms of LMIs. In section III we consider the problem of finding a stabilizing state-feedback based on the results in section II. The last section is dedicated to conclusions and future work.

Notation: We denote the transpose of a matrix A by A' and we write $P>0$ (or $P<0$) when P is a symmetric positive (or negative) definite matrix. We write a symmetric matrix $\begin{bmatrix} A & B \\ B' & C \end{bmatrix}$ as $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$.

II. Stability of sampled-data systems

Consider a sampled-data system consisting of an LTI plant and a state feedback controller with constant gain K connected through sample and hold blocks. The LTI plant has a state space model of the form

$$\dot{x}(t) = Ax(t) + B_u u(t), \quad y = x,$$

where x,u,y are the state, input and output of the plant respectively. At the sampling time t_k, $k \in \mathbb{N}$ the plant's state, $x(t_k)$, is sent to the controller and the control command $Kx(t_k)$ is sent back to the plant to be used as soon as it arrives until the next control command update. The resulting closed-loop system can be written as a hybrid system of the form

$$\dot{\xi}(t) = F\xi(t), \qquad t_k \le t < t_{k+1}, \tag{1a}$$

$$\xi(t_{k+1}) = \begin{bmatrix} x(t_{k+1}) \\ x(t_{k+1}) \end{bmatrix}, \quad \forall k \in \mathbb{N}, \tag{1b}$$

where

$$F := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \qquad \xi(t) := \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \qquad B := B_u K,$$
$$z(t) := x(t_k), \qquad t_k \le t < t_{k+1}.$$

The maximum interval between samples will be denoted by

$$\tau_{MATI} := \sup_{k\in\mathbb{N}} (t_{k+1} - t_k).$$

We define $\rho(t) := t - t_k$, $t_k \le t < t_{k+1}$, and therefore we have $\tau_{MATI} = \sup_{t\ge 0} \rho(t)$. We assume that there exists a $t_D > 0$ such that $t_D \le t_{k+1} - t_k$.

A. Stability of sampled-data systems with variable sampling

We now construct a finite-dimensional Lyapunov function for the sampled-data system (1) with variable sampling. Consider the candidate Lyapunov function

$$V(\zeta) := V_1(x) + V_2(\zeta) + V_3(\zeta), \tag{2}$$

where

$$V_1(x) := x'Px,$$
$$V_2(\zeta) := \xi'\Big(\int_{-\rho}^{0} (s+\tau_{MATI})(F\exp(Fs))'\tilde{R}F\exp(Fs)ds\Big)\xi,$$
$$V_3(\zeta) := (\tau_{MATI} - \rho)(x-z)'X(x-z),$$

and $\zeta := \begin{bmatrix} \xi \\ \rho \end{bmatrix}$, $\tilde{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}$ with R,P,X symmetric positive definite matrices. It is clear that the Lyapunov function is positive (for any x and z not both equal to zero). Along jumps this Lyapunov function does not increase since $V_2(\zeta)$ and $V_3(\zeta)$ are non-negative before the jumps and they become zero right after the jumps.

Remark 1: Note that $V_2(\zeta)$ can be written as

$$V_2(\zeta) = \int_{-\rho}^{0}\int_{t+\theta}^{t} \dot{x}'(s)R\dot{x}(s)dsd\theta. \tag{3}$$

This type of functional (or closely related forms) appeared in the DDE and NCSs literature extensively, e.g., in [5, 11] but with the ρ replaced by τ_{MATI} in (3) (and without $V_3(\zeta)$).

The next theorem provides a sufficient condition for the decrease of the Lyapunov function (2) along the solution to the system (1) between the jumps. Decreasing the Lyapunov function between and at the jumps provides a sufficient condition for the exponential stability of the system (1) [12].

Theorem 1: The system (1) is exponentially stable for any sampling interval satisfying $t_D \le t_{k+1} - t_k \le \tau_{MATI}, \forall k \in \mathbb{N}$, provided that there exist symmetric positive definite matrices P,R,X and a (not necessarily symmetric) matrix N that satisfy the following LMIs:

$$\Psi + \tau_{MATI}(M_1 + M_2) < 0, \tag{4a}$$

$$\begin{bmatrix} \Psi + \tau_{MATI}M_1 & \tau_{MATI}N \\ * & -\tau_{MATI}R \end{bmatrix} < 0, \tag{4b}$$

where

$$\bar{F} := \begin{bmatrix} A & B \end{bmatrix},$$
$$\Psi := \begin{bmatrix} P \\ 0 \end{bmatrix}\bar{F} + \bar{F}'\begin{bmatrix} P & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} X \begin{bmatrix} I & -I \end{bmatrix}$$
$$- N\begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} N',$$
$$M_1 := \bar{F}'R\bar{F},$$
$$M_2 := \begin{bmatrix} I \\ -I \end{bmatrix} X\bar{F} + \bar{F}'X\begin{bmatrix} I & -I \end{bmatrix}. \tag{5}$$

See the appendix for the proof of Theorem 1.

Remark 2: Suppose there exist matrices $P_{1f} > 0$, P_{2f}, P_{3f}, Z_f and $R_f > 0$ satisfying the following stability conditions from Lemma 1 of [5]:

$$\Psi_{1f} < 0, \qquad -Z_f + P_f' \begin{bmatrix} 0 \\ B \end{bmatrix} R_f^{-1} \begin{bmatrix} 0 \\ B \end{bmatrix}' P_f < 0,$$

where

$$P_f := \begin{bmatrix} P_{1f} & 0 \\ P_{2f} & P_{3f} \end{bmatrix},$$
$$\Psi_{1f} := \Psi_{0f} + \tau_{MATI} Z_f + \tau_{MATI} \begin{bmatrix} 0 & 0 \\ 0 & R_f \end{bmatrix},$$
$$\Psi_{0f} := P' \begin{bmatrix} 0 & I \\ A+B & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A+B & -I \end{bmatrix}' P.$$

Then we have

$$\Psi_{0f} + \tau_{MATI} \begin{bmatrix} 0 & 0 \\ 0 & R_f \end{bmatrix} + \tau_{MATI} P_f' \begin{bmatrix} 0 \\ B \end{bmatrix} R_f^{-1} \begin{bmatrix} 0 \\ B \end{bmatrix}' P_f < 0. \quad (6)$$

Multiplying (6) from the right and left by $\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}$ and its transpose we obtain (21) with

$$\rho = \tau_{MATI} = h_f, \qquad P = P_{1f},$$
$$R = R_f, \qquad N' = -\begin{bmatrix} B'P_{2f} + B'P_{3f}A & B'P_{3f}B \end{bmatrix}.$$

This means that if there are matrix variables satisfying the conditions of Lemma 1 in [5] then the conditions in Theorem 1 will necessarily also be satisfied. It is also possible to show that when the stability condition in [9] holds (given by equation (12) in [9]), then the condition in Theorem 1 must also necessarily hold with

$$\rho = \tau_{MATI} = \eta_y \qquad R = T_y,$$
$$P = P_y, \qquad N = \begin{bmatrix} N_{1y}' + N_{3y}'A & N_{2y}' + N_{3y}'B \end{bmatrix}.$$

Hence our Lyapunov function leads to conditions in Theorem 1 that are less conservative than the stability conditions in [5, 9] using a Lyapunov functional. From this perspective, considering an infinite dimensional DDE model and using a Lyapunov functional to prove its stability offers no advantage for this class of finite dimensional systems.

Remark 3: Suppose that the system matrices $\Omega := \begin{bmatrix} A & B \end{bmatrix}$ are not exactly known and instead they are specified through the following polytopic condition:

$$\Omega \in \Big\{ \sum_{j=1}^{\kappa} f_j \Omega_j, \quad 0 \le f_j \le 1, \quad \sum_{j=1}^{\kappa} f_j = 1 \Big\},$$

where the κ vertices of the polytope are described by $\Omega_j := \begin{bmatrix} A^j & B^j \end{bmatrix}$. Stability of the system can be checked by solving the LMIs in Theorem 1 for each of the individual vertices with the same matrix variables P, X, R, N.

Remark 4: When the sampling intervals approach zero (guarantee that $\tau_{MATI} \to 0$) the conditions (4a) and (4b) reduce to

$$\begin{bmatrix} P \\ 0 \end{bmatrix} \bar{F} + \bar{F}' \begin{bmatrix} P & 0 \end{bmatrix} - N \begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} N' < 0 \qquad (7)$$

(since M_2 does not appear and the only remaining term in equation (4) that contains X is negative semi-definite, one may simply take $X = 0$). A sufficient condition for (7) to be satisfied is

$$(A+B)'P + P(A+B) < 0, \qquad P = P' > 0, \qquad (8)$$

because if (8) holds, then (7) holds with the choice $N = \left[\begin{smallmatrix} -PB+I \\ -I \end{smallmatrix}\right]$. The condition (8) is the necessary and sufficient condition for the stability of the closed-loop system $\dot{x} = (A+B)x$. Hence the sampled-data system is stable for small enough sampling intervals if the corresponding closed-loop continuous system is stable. By the Matrix Elimination Lemma it turns out that (8) is also a necessary condition for (7). Therefore as the sampling intervals approach zero, Theorem 1 recovers exactly the continuous-time stability condition. This does not happen for the conditions that appeared in [6]. Moreover using a Lyapunov function instead of a Lyapunov functional facilitates proving the exponential stability (instead of just asymptotic stability) of the system (1).

B. Stability of sampled-data systems with constant sampling

Now we consider the case where the sampling intervals are constant. This case may appear uninteresting since there are classical results giving necessary and sufficient conditions for stability and stabilization of such sampled-data systems. However the LMI conditions presented here can be used for more interesting problems such as studying the stability and stabilization of sampled-data system with input saturation for which one wants to maximize the region of attraction. These results are also extendable to the study of multi-rate sampled-data system for which the ratio of the different sampling rates is not a rational number. For constant sampling instead of $V_3(\xi)$ in (2) we use

$$W(\zeta) := V_1(x) + V_2(\zeta) + W_3(\zeta) \qquad (9)$$

where

$$W_3(\zeta) := (\tau_{MATI} - \rho)\xi' \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}' \bar{X} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \xi, \qquad (10)$$

$$\bar{X} := \begin{bmatrix} 0 & X_1 \\ X_1' & X_2 \end{bmatrix}. \qquad (11)$$

Note that

$$W_3(\zeta) = (\tau_{MATI} - \rho)\big((x-z)'X_2(x-z) + 2(x-z)X_1 z\big),$$

and consequently $W(\zeta)$ is not necessarily a positive function. However, right before and after every sampling time $W_3(\zeta)$ is zero (because right before $\tau_{MATI} - \rho = 0$ and right after $x - z = 0$), therefore $W(\zeta(t_k)) > 0, \forall k \in \mathbb{N}$. If between sampling times we have that $\frac{dW(\zeta(t))}{dt} < 0$, for $t_k \le t < t_{k+1}$, then (a) $W(\zeta(t_k))$ is smaller right after t_k than it was just before t_k, but still positive and; (b) between sampling times $W(\zeta(t))$ must always be positive. Item (b) can be justified as follows: By contradiction suppose that at some point $W(\zeta(t))$ becomes negative. Then, since this function is monotonically decreasing, at the next sampling time it

Th. 1	Th. 2	[5]	[9]	[10]
$3.5n^2+1.5n$	$5n^2+n$	$5n^2+2n$	$7n^2+n$	$16n^2+3n$
1.1137	1.3277	0.8696	0.8696	0.8871

TABLE I
THE SECOND ROW SHOWS THE NUMBER OF VARIABLES IN THE LMIS USED TO TEST STABILITY AND THE THIRD ROW SHOWS THE VALUE OF τ_{MATI} FOR EXAMPLE 1.

must still be negative, which contradicts item (a). The next theorem provides sufficient conditions that guarantee that $\frac{\mathrm{d}W(\zeta(t))}{\mathrm{d}t} < 0$, for $t_k \le t < t_{k+1}$. The decrease of the Lyapunov-like function (9) along the solution to the system (1) both between and at the sampling times provides a sufficient conditions that guarantee that for the exponential stability of the system (1) when the sampling interval is fixed.

Theorem 2: The system (1) is exponentially stable for any non-zero *constant sampling* interval smaller than τ_{MATI} provided that there exist symmetric positive definite matrices P, R and (not necessarily symmetric) matrices N, X_1, X_2 that satisfy the following LMIs:

$$\bar{\Psi} + \tau_{MATI}(M_1 + \bar{M}_2) < 0,$$
$$\begin{bmatrix} \bar{\Psi} + \tau_{MATI} M_1 & \tau_{MATI} N \\ * & -\tau_{MATI} R \end{bmatrix} < 0,$$

where

$$\begin{aligned}\bar{\Psi} :=& \begin{bmatrix} P \\ 0 \end{bmatrix} \bar{F} + \bar{F}' \begin{bmatrix} P & 0 \end{bmatrix} - \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}' \bar{X} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \\ & - N \begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} N', \\ \bar{M}_2 :=& \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}' \bar{X} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} F + F' \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}' \bar{X} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix},\end{aligned}$$

and $M_1, \bar{X}$ are defined in (5) and (11).

See the appendix for the proof of Theorem 2.

Example 1: Consider the plant model from [13]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u, \qquad (13)$$

with the state feedback gain $K = -\begin{bmatrix} 3.75 & 11.5 \end{bmatrix}$. In our notation, this corresponds to

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \qquad B = -\begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \times \begin{bmatrix} 3.75 & 11.5 \end{bmatrix}.$$

constant sampling: Using standard techniques from digital control one can show that the maximum constant sampling interval for which the closed-loop system remains stable is $1.7s$. The maximum constant sampling interval given by Theorem 2 is 1.3277.

variable sampling: The stability results in [5, 9, 1] provide an upper bound on the sampling interval, for which stability is guaranteed equal to 0.8696. This bound is improved to 0.8871 in [10]. Theorem 1 gives the upper bound equal to 1.1137. When compared to the constant-sampling bound given by Theorem 2, we now obtain a more conservative value. This is reasonable because we are now guaranteeing stability for every sequence of sampling times, with consecutive samples separated by no more than 1.1137, but potentially with different sampling intervals from one sample to the next.

Remark 5: In table I we compare the number of scalar unknowns that appear in the LMIs of the different papers assuming that the dimension of the plant is n. Notice that for an $n \times n$ symmetric matrix variable $\frac{n(n+1)}{2}$ scalar variables are needed whereas for an $m \times n$ matrix variable mn scalar variables are required. We can see that our results use fewer variables, but this is not at the expense of degrading the value obtained for τ_{MATI}. It is not fair to compare the number of variables in [10] to the others in Table I because this paper considered the sampled-data system with delays (although we presented the number of variables in [10] in Table I).

III. STABILIZATION OF SAMPLED-DATA SYSTEMS

The LMIs presented in the previous section become Bilinear Matrix Inequalities (BMIs) when the controller gain K in $B = KB_u$ is unknown, because there are cross terms between B and P. This situation arises in the design problem, when we want to find a feedback gain K that stabilizes the closed-loop system. The next theorem provides an LMI conditions that enables us to find a stabilizing feedback gain for variable sampling intervals. Following the same steps, one can find the state feedback for the constant sampling case.

Theorem 3: There exists a state feedback gain K that exponentially stabilizes the sampled-data system (1) for any sampling interval satisfying $t_D \le t_{k+1} - t_k \le \tau_{MATI}, \forall k \in \mathbb{N}$, provided that there exists a symmetric positive definite matrix Q, (not necessarily symmetric) matrices N_d, Y, and positive scalars $\varepsilon_1, \varepsilon_2$ that satisfy the following LMIs:

$$\begin{bmatrix} \Psi_d + \tau_{MATI} M_d & \tau_{MATI} F_d' \\ * & -\tau_{MATI} \varepsilon_1^{-1} Q \end{bmatrix} < 0,$$
$$\begin{bmatrix} \Psi_d & \tau_{MATI} F_d' & \tau_{MATI} N_d \\ * & -\tau_{MATI} \varepsilon_1^{-1} Q & 0 \\ * & * & -\tau_{MATI} \varepsilon_1 Q \end{bmatrix} < 0,$$

where

$$\begin{aligned} F_d :=& \begin{bmatrix} AQ & BY \end{bmatrix}, \\ \Psi_d :=& \begin{bmatrix} I \\ 0 \end{bmatrix} F_d + F_d' \begin{bmatrix} I & 0 \end{bmatrix} - \varepsilon_2 \begin{bmatrix} I \\ -I \end{bmatrix} Q \begin{bmatrix} I & -I \end{bmatrix} \\ & - N_d \begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} N_d', \\ M_d :=& \varepsilon_2 \begin{bmatrix} I \\ -I \end{bmatrix} F_d + \varepsilon_2 F_d' \begin{bmatrix} I & -I \end{bmatrix}. \end{aligned}$$

When these LMIs are feasible, the stabilizing state feedback gain is given by $K = YQ^{-1}$.

Proof. Assume $Q = P^{-1}$. Multiplying (4a) by $\bar{Q} := \operatorname{diag}(Q,Q)$ and (4b) by $\operatorname{diag}(Q,Q,Q)$ from the right and the left. By using Schur lemma and defining $N_d := \bar{Q}NQ, Y := KQ,$ and choosing $X = \varepsilon_2 Q^{-1}$, we get

$$\begin{bmatrix} \Psi_d + \tau_{MATI} M_d & \tau_{MATI} F_d' \\ * & -\tau_{MATI} R^{-1} \end{bmatrix} < 0, \tag{15}$$

$$\begin{bmatrix} \Psi_d & \tau_{MATI} F_d' & \tau_{MATI} N_d \\ * & -\tau_{MATI} R^{-1} & 0 \\ * & * & -\tau_{MATI} QRQ \end{bmatrix} < 0. \tag{16}$$

The matrix inequalities in (15) and (16) are not LMIs. We make them linear by choosing $R = \varepsilon_1 Q^{-1}$, which results in the LMIs in Theorem 3.

Remark 6: By choosing $R = \varepsilon_1 Q^{-1}$ we transformed the BMIs into LMIs that are numerically tractable, but this was achieved at the expense of over design. This conservativeness could be reduced by using the cone complementarity algorithm [14]. Consider a matrix variable Z such that $Z < QRQ$. Now the matrix inequalities to be considered are

$$\begin{bmatrix} \Psi_d + \tau_{MATI} M_d & \tau_{MATI} F_d' \\ * & -\tau_{MATI} R^{-1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \Psi_d & \tau_{MATI} F_d' & \tau_{MATI} N_d \\ * & -\tau_{MATI} R^{-1} & 0 \\ * & * & -\tau_{MATI} Z \end{bmatrix} < 0,$$

$$\begin{bmatrix} R & Q^{-1} \\ Q^{-1} & R \end{bmatrix} > 0.$$

Clearly these matrix inequalities are not LMIs because the inverse of the matrix variables appear. However, the cone complementarity algorithm transforms this problem into a linear optimization subject to a set of LMIs that can be solved numerically. The improvement of this type of approach that uses the cone complementarity algorithm has been investigated in [15].

Example 2: Now we consider the state feedback controller design for the plant (13). We would like to find a feedback gain K that maximizes the upper bound of the variable sampling intervals. Yue et al. [9] found a stabilizing controller that guarantees stability up to a sampling interval equal to 402. Our results provide the controller $K = \begin{bmatrix} 5\times 10^{-5} & -0.0348 \end{bmatrix}$, which improves this upper bound to 820. This upper bound on the variable sampling intervals is very large because the plant (13) is marginally stable and little control action is needed to exponentially stabilize the plant. In fact by choosing $u = -\alpha(0.1x_1 + 10x_2)$ with a small α, we can get very large τ_{MATI}. For example when $\alpha = 0.001$ we get $\tau_{MATI} = 10^6$. The reason that this input leads to very large τ_{MATI} is that we can define $z := 0.1x_1 + x_2$ then $\dot{z} = 0.1u$. This system with the input $u = -\alpha z$ is stable for large sampling intervals with small enough α. Although our LMIs improve upon the previous results, most likely because of numerical errors they are unable to give the controller that previous observation suggests.

IV. CONCLUSION AND FUTURE WORK

We considered the stability analysis and state-feedback stabilization of uncertain sampled-data systems. We introduced a Lyapunov function with jumps to study this class of systems. As special cases we considered sampled-data systems with fixed sampling intervals. Our stability and stabilization results were presented as Linear Matrix Inequalities (LMIs). To reduce the conservativeness we used slack matrices; however, we used fewer slack matrices than the previous results in the literature. We showed that we have done it without making the results more conservative.

In the future we will extend our results to multi-rate sampled-data system. This setting does not require a rational ratio between sampling intervals of different variables. Our setting is also amenable to the analysis of linear systems with input saturation [5] or nonlinear uncertainty [16]. We will also consider the H_∞ design for uncertain sampled-data system. We are also interested in studying the sampled-data systems with delays in the control loop.

APPENDIX

Proof of Theorem 1. Along the trajectory of the system (1) we have

$$\frac{\mathrm{d}V_1(\zeta)}{\mathrm{d}t} = \xi' \Big(\begin{bmatrix} P \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} P & 0 \end{bmatrix} \Big) \xi, \tag{17}$$

$$\frac{\mathrm{d}V_3(\zeta)}{\mathrm{d}t} = -(x-z)'X(x-z) + 2(\tau_{MATI} - \rho)(x-z)'X(Ax+Bz), \tag{18}$$

$$\begin{aligned}
\frac{\mathrm{d}V_2(\zeta)}{\mathrm{d}t} &= (\tau_{MATI} - \rho)\xi' \big(F\exp(-F\rho)'\tilde{R}F\exp(-F\rho)\big)\xi \\
&+ \xi' \Big(\int_{-\rho}^{0} (s + \tau_{MATI}) \frac{d}{ds} \big[F\exp(-Fs)'\tilde{R}F\exp(-Fs)ds\big] \Big)\xi = \\
&\xi'\big((\tau_{MATI} - \rho)F\exp(-F\rho)'\tilde{R}F\exp(-F\rho) + \tau_{MATI}F'\bar{R}F\big)\xi \\
&- (\tau_{MATI} - \rho)\xi'\big(\big(F\exp(-F\rho)'\tilde{R}F\exp(-F\rho)\big)\xi \\
&- \xi'\Big(\int_{-\rho}^{0} (F\exp(Fs))'\tilde{R}F\exp(Fs)ds\Big)\xi = \\
&\tau_{MATI}\xi'F'RF\xi - \xi'\Big(\int_{-\rho}^{0}(F\exp(Fs))'\tilde{R}F\exp(Fs)ds\Big)\xi.
\end{aligned} \tag{19}$$

Since $x(t) - z(t) = x(t) - x(t-\rho)$, for any matrix N, we have

$$\begin{aligned}
&2\xi'N\begin{bmatrix} I & -I \end{bmatrix}\xi = 2\xi'N\Big(\int_{-\rho}^{0} \begin{bmatrix} I & 0 \end{bmatrix} F\exp(Fs)ds\Big)\xi \\
&\le \xi'\Big(\int_{-\rho}^{0} NR^{-1}N' + (F\exp(Fs))'\tilde{R}F\exp(Fs)ds\Big)\xi \\
&\le \rho\xi'NR^{-1}N'\xi + \xi'\Big(\int_{-\rho}^{0}(F\exp(Fs))'\tilde{R}F\exp(Fs)ds\Big)\xi,
\end{aligned} \tag{20}$$

which relies on the fact that if we flow backward in time for ρ seconds and look at the x component, we get the z component. The matrix variable N represents a degree of freedom that can be exploited to minimize conservativeness and we call it a slack matrix. Combining (17),(18), (19) and (20) we get

$$\frac{\mathrm{d}V_2(\zeta)}{\mathrm{d}t} \leq \xi\left(\tau_{MATI}\bar{F}'R\bar{F} + \rho NR^{-1}N' - 2N\begin{bmatrix} I & -I \end{bmatrix}\right)\xi.$$

Hence $\frac{\mathrm{d}V(\zeta)}{\mathrm{d}t} < 0$ if

$$\Psi + \tau_{MATI}(M_1 + M_2) + \rho(M_3 - M_2) < 0, \qquad (21)$$

where Ψ, M_1, M_2 are defined in (5) and $M_3 := NR^{-1}N'$.
A necessary and sufficient condition to satisfy (21) is

$$\Psi + \tau_{MATI}(M_1 + M_2) < 0, \qquad (22)$$
$$\Psi + \tau_{MATI}(M_1 + M_3) < 0. \qquad (23)$$

To see that these matrix inequalities are sufficient, consider $\alpha \in [0,1]$ and note that

$$\alpha\big(\Psi + \tau_{MATI}(M_1 + M_3)\big) + (1-\alpha)\big(\Psi + \tau_{MATI}(M_1 + M_2)\big) = \Psi + \tau_{MATI}M_1 + \alpha\tau_{MATI}M_3 + (1-\alpha)\tau_{MATI}M_2.$$

Setting $\alpha = \rho/\tau_{MATI}$ we conclude that (21) holds. Now suppose that (21) holds for every $\rho \leq \tau_{MATI}$. Hence it should hold when $\rho = 0$ and $\rho = \tau_{MATI}$ which gives (22) and (23) respectively. By Schur complement, the matrix inequalities in (22) and (23) can be written as the LMIs given in Theorem 1.

Finally, when the LMIs in Theorem 1 are feasible, there exists a constant $c_1 > 0$ such that $\frac{\mathrm{d}V(\zeta)}{\mathrm{d}t} \leq -c_1|\xi|^2$. It is also easy to show that there exists a constant $c_2 > 0$ such that $V(\zeta) \leq c_2|\xi|^2$. Then $\frac{\mathrm{d}V(\zeta)}{\mathrm{d}t} \leq -\frac{c_1}{c_2}V(\zeta)$ and consequently $V(\zeta)$ and $V_1(\zeta)$ both decrease to zero exponentially fast which in tern means that $x(t) \to 0$ exponentially fast.

Proof of Theorem 2. Along the flow

$$\frac{\mathrm{d}W_3(\xi)}{\mathrm{d}t} = \xi'\Big(-\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}'\bar{X}\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} + 2(\tau_{MATI} - \rho)\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}'\bar{X}\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}F\Big)\xi$$

and rest of the proof goes exactly as in proof of the Theorem 1.

References

[1] P. Naghshtabrizi and J. Hespanha, "Designing observer-based controller for network control system," in *Proc. of the 44th Conf. on Decision and Contr.*, vol. 4, June 2005, pp. 2876–2880.

[2] B. Bamieh, J. Pearson, B. Francis, and A. Tannenbaum, "A lifting technique for linear periodic systems," *Syst. & Contr. Lett.*, vol. 17, no. 2, pp. 79–88, Aug 1991.

[3] T. Chen and B. A. Francis, *Optimal sampled-data control systems*, 2nd ed. Springer, 1996.

[4] K. Astrom and B. Wittenmark, *Adaptive control*. Addison-Wesley, 1989.

[5] E. Fridman, A. Seuret, and J. P. Richard, "Robust sampled-data stabilization of linear systems: an input delay approach," *Automatica*, vol. 40, no. 8, pp. 1441–1446, Aug. 2004.

[6] L. Hu, Y. Cao, and H. Shao, "Constrained robust sampled-data control of nonlinear uncertain systems," *Int. J. of Robust and Nonlinear Contr.*, vol. 12, pp. 447–464, 2002.

[7] L. Hu, J. Lam, Y. Cao, and H. Shao, "A LMI approach to robust H_2 sampled-data control for linear uncertain systems," *IEEE Trans. Syst. and Man Cybernetics*, vol. 33, no. 1, pp. 149–155, Feb 2003.

[8] G. C. Walsh, H. Ye, and L. Bushnell, "Stability analysis of networked control systems," *IEEE Trans. on Contr. Syst. Tech.*, vol. 10, no. 3, pp. 438–446, May 2002.

[9] D. Yue, Q.-L. Han, and C. Peng, "State feedback controller design for networked control systems," *IEEE Trans. on Automat. Contr.*, vol. 51, no. 11, pp. 640–644, Nov. 2004.

[10] D. Yue, Q. L. Han, and J. Lam, "Network-based robust H_∞ control of systems with uncertainty," *Automatica*, vol. 41, no. 6, pp. 640–644, 2005.

[11] J. P. Richard, "Time-delay systems: an overview of some recent advances and open problems," *Automatica*, vol. 39, pp. 1667–1694, 2003.

[12] M. S. Branicky, "Stability of hybrid systems: State of the art," in *Proc. of the 36th Conf. on Decision and Contr.*, vol. 1, Dec. 1997, pp. 120–125.

[13] M. Branicky, S. M. Philips, and W. Zhang, "Stability of networked control systems: explicit analysis of delay," in *Proc. of the 2000 Amer. Contr. Conf.*, June 2000, pp. 2352–2357.

[14] L. E. Ghaoui, F. Oustry, and M. AitRami, "A cone complementarity linearization algorithm for static output-feedback and related problems," *IEEE Trans. on Automat. Contr.*, vol. 42, no. 8, pp. 1171–1176, Aug 1997.

[15] H. Gao and C. Wang, "Comments and further results on "a descriptor system approach to H_∞ control of linear time-delay system"," *IEEE Trans. on Automat. Contr.*, vol. 48, Mar. 2003.

[16] M. Yu, L. Wang, and T. Chu, "Sampled-data stabilisation of networked control systems with nonlinearity," in *IEEE Proc. of Cont. and applications*, vol. 152, no. 6, 2005, pp. 609 – 614.