$\mathcal{L}_2$-induced Gain Analysis for a class of Switched Systems

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Abstract

This paper addressed the computation of the $\mathcal{L}_2$-induced gain for a class of switched systems. The main contribution of the paper is to completely characterize the induced gain of the switched system through a system of differential inequalities, one for each system being switched. The motivation for computing the induced gain of a switched system is the application of robust stability tools to the analysis of hybrid systems.

I. INTRODUCTION

Hybrid dynamical systems whose behavior can be described using a mixture of event-based logic and differential or difference equations have been attracting significant interest. This is motivated by the observation that a wide variety of artificial/man-made and physical systems/processes are naturally modeled in a hybrid dynamical framework. Switched systems typically arise in the context of hybrid dynamical systems when it is possible to describe the behavior in each mode through a differential or difference equation and the event-based transitions as discontinuous switchings.

The stability of switched system has been extensively studied and several key results can be found in the survey papers [1], [2], [3] and references therein. Among these result, the ones in [4], [5] are especially relevant for the present paper. In these papers, it was proved that the existence of a *common* solution to a set of Lyapunov inequalities, one for each linear system being switched, is *equivalent* to the uniform asymptotic stability of the switched system for arbitrary switching signals. In this context, *uniformity*

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refers to the fact that asymptotic stability is guaranteed over an entire set of switching signals, which in this case contains every switching signal with a finite number of discontinuities on any finite time interval. However, if one were to demand a common *quadratic* solution to the set of Lyapunov inequalities, this would lead to a conservative sufficient condition for uniform asymptotic stability. In fact, it was shown in [5] by example, that a non-conservative necessary and sufficient condition for uniform asymptotic stability requires a *non-quadratic* common Lyapunov function. Nonlinear generalizations of these results can be found in [5], [6].

The input-output properties of dynamical systems, especially $\mathcal{L}_2$-induced gains are fundamental tools for robust control theories in linear [7], [8], and nonlinear [9] settings, particularly in $\mathcal{H}^\infty$ control problems [10]. In spite of their important roles, the progress on the study of input-output properties for switched systems has been difficult [11]. It was proved that uniformly asymptotically stable switched systems admit a *finite input-output gain* with several combinations of input/output signal norms [12]. In [13], *multiple-quadratic* solutions to Riccati-type inequalities, one for each mode of the switched system, were utilized to estimate an upper bound of the induced gain. Although quadratic solutions are attractive from a numerical computations point of view, they generally provide conservative estimates on the induced gain. Non-conservative necessary and sufficient conditions that can be used to establish the value of the induced gain are only available for special switched systems. A separation property between all the stabilizing and all the anti-stabilizing solutions to a set of algebraic Riccati equations of the systems being switched provides a complete solution to the induced gain analysis in the case of *slow-switching* signals, where slow-switching refers to the limit as the intervals between consecutive switchings grows infinity [14]. The variational approaches in [15] also provide a complete and non-conservative characterization of the induced gain for single-input single-output first-order systems. Other studies on input-output properties for switched systems include dissipativity and passivity type conditions [16], [17].

The main result of this paper is a necessary and sufficient condition for the $\mathcal{L}_2$-induced gain of a linear switched system to lie below a prescribed value, in terms of the existence of a *common* solution to a set of Hamilton-Jacobi inequalities, one for each system being switched. The key challenge in proving such result is the necessity of the condition, as the sufficiency is relatively straightforward. A necessary and sufficient condition, such as this, allows one to use a bisection procedure to construct a conceptual algorithm to compute the value of the induced gain of a switched system up to any pre-specified level of precision.

The paper also investigates under what condition one can deduce stability of the switched system from the knowledge that the induced gain is finite. This requires one to consider appropriate observability notions, as well as restrictions on the allowable classes of switching signals, similar to what was done in [18], in the context of extending LaSalle's invariance principle to switched systems.

We also show that the induced gain that is obtained for the class of all switching signals, remains the same for every more restricted class of switching signals that is closed under concatenation, i.e., for classes of switching signals $\mathcal{S}'$ for which if we take two switching signals s_1 and s_2 in $\mathcal{S}'$ and construct a third switching signal s_3 by making it equal to s_1 up to some time $t > 0$ and equal to s_2 after t, then s_3 is still in the class $\mathcal{S}'$.

The remainder of the paper is organized as follows: Section II describes the problem formulation and presents the main results. The proofs of the main results are presented in Section III. Section IV discusses a set of properties for switching signal classes, and this allows us to state results that correspond to versions of the results in Section II for more general classes of switching signals. Section V contains some concluding remarks. A subset of the results in this manuscript is submitted to the 48th IEEE Conference on Decision and Control [19].

II. Problem Descriptions and Main Results

The following Section II-A formulates the problems that will be considered. Section II-B contains the main results.

A. Problem Descriptions

The switched systems under consideration are represented by equations of the form

$$\frac{dx}{dt} = A_s x + B_s u \tag{1a}$$

$$y = C_s x \tag{1b}$$

where $s \in \mathcal{S}$ denotes a piecewise constant switching signal that selects appropriate triples (A_p, B_p, C_p) from a parametrized family

$$\{(A_p, B_p, C_p)|\ p \in \mathcal{P}\} \tag{1c}$$

of n-dimensional, m-input, k-output state space realizations, where $\mathcal{P}$ denotes an index set. Throughout the paper, the set of matrices in (1c) is assumed to be compact. The set of all piecewise constant switching signals is denoted by

$$\mathcal{S} = \{s :\ [\,0, \infty\,) \to \mathcal{P}\} \tag{1d}$$

and by a piecewise constant signal s we mean that s must only have a finite number of discontinuities on each bounded time interval. By convention, each piecewise constant signal $s \in \mathcal{S}$ is assumed to be continuous from above[1]. A function $x:\ [\,0,\infty\,) \to \mathbb{R}^n$ is said a solution to (1) if it is continuous and piecewise continuously differentiable and there exist a switching signal $s \in \mathcal{S}$ and an input signal $u \in \mathcal{L}_2^m$ such that the time-varying differential equation[2]

$$\frac{dx(t)}{dt} = A_{s(t)}x(t) + B_{s(t)}u(t)$$

holds almost everywhere on $t \geq 0$.

Our primal interest is to determine the induced gain of the switched systems (1), but to proceed we need to introduce appropriate stability definitions.

Definition 1: Let $\mathcal{S}' \subset \mathcal{S}$. The system (1) is said to be *uniformly asymptotically stable over* $\mathcal{S}'$ if there exists a function β of class $\mathcal{KL}$ such that[3]

$$\|x(t)\| \leq \beta(\|x(0)\|, t)$$

for all $t \geq 0$, all $s \in \mathcal{S}'$ and all $x(0) \in \mathbb{R}^n$, where $x(t)$ denotes the solution to (1) obtained for the switching signal s, zero input signal u, and initial condition $x(0)$.

Definition 2: Let $\mathcal{S}' \subset \mathcal{S}$. The system (1) is said to be *(uniformly) exponentially stable over* $\mathcal{S}'$ if there exist constants $a > 0$ and $\lambda > 0$ such that the function β in Definition 1 can be chosen of the form $\beta(r,t) = ae^{-\lambda t}r$ for all $r \geq 0$ and all $t \geq 0$.

We now state the following definitions of induced gains.

Definition 3: Let $\mathcal{S}' \subset \mathcal{S}$ and $\gamma > 0$. The system (1) is said to have an *$\mathcal{L}_2$-induced gain smaller than or equal to γ uniformly over $\mathcal{S}'$* if

$$\int_0^t \|y(\tau)\|^2 d\tau \leq \gamma^2 \int_0^t \|u(\tau)\|^2 d\tau \tag{2}$$

for all $t \geq 0$, all $s \in \mathcal{S}'$ and all $u \in \mathcal{L}_2^m$, where $y(\tau)$ denotes the output of (1) obtained for the switching signal s, input signal u, and zero initial condition.

Definition 4: Let $\mathcal{S}' \subset \mathcal{S}$ and $\gamma > 0$. The system (1) is said to have an *$\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}'$* if the system has an $\mathcal{L}_2$-induced gain smaller than or equal to some $\gamma' < \gamma$ uniformly over $\mathcal{S}'$.

[1]For any $t \geq 0$, the limit from above of $s(\tau)$ as $\tau \downarrow t$ is equal to $s(t)$.

[2]We denote by $\mathcal{L}_2^m$ the set of square integrable functions with values on $\mathbb{R}^m$ defined on $[\,0,\infty\,)$.

[3]A function $\alpha:\ [\,0,\infty\,) \to [\,0,\infty\,)$ is said to be of class $\mathcal{K}$, and we write $\alpha \in \mathcal{K}$, if α is continuous, strictly increasing and $\alpha(0) = 0$. A function $\beta:\ [\,0,\infty\,) \times [\,0,\infty\,) \to [\,0,\infty\,)$ is said to be of class $\mathcal{KL}$, and we write $\beta \in \mathcal{KL}$, if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r,t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

Our goal is to provide non-conservative conditions on the realizations in (1c) that guarantee that the system has an $\mathcal{L}_2$-induced gain smaller than (or equal) to a given constant γ.

B. Main Results

This section summarizes the main results of this paper. Their proofs are provided in the next Section III.

Theorem 1: Suppose that the system (1) is uniformly asymptotically stable over $\mathcal{S}$ and let $\gamma > 0$. The following statements are equivalent:

1) The system (1) has an $\mathcal{L}_2$-induced gain smaller than or equal to γ uniformly over $\mathcal{S}$.
2) There exits a function $V : \mathbb{R}^n \to \mathbb{R}$ that is convex, zero at zero, and homogeneous of degree two, which satisfies

$$\frac{\partial V}{\partial x}(x) A_p x + x^{\mathrm{T}} C_p^{\mathrm{T}} C_p x + \frac{1}{4}\gamma^{-2} \frac{\partial V}{\partial x}(x) B_p B_p^{\mathrm{T}} \frac{\partial V^{\mathrm{T}}}{\partial x}(x) \leq 0 \qquad \text{for all } p \in \mathcal{P} \tag{3}$$

almost everywhere in x.

We recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous of degree q if $f(kx) = k^q f(x)$ for all $x \in \mathbb{R}^n$ and all $k \in \mathbb{R}$. Because of convexity, the function V is differentiable almost everywhere in x, but not necessarily everywhere (see, e.g., [20], [21]). Therefore, the quantification over x in (3) should be interpreted as for every $x \in \mathbb{R}^n$ except the zero-measure set of points at which $x \mapsto V(x)$ is not differentiable.

Remark 1: The statement 2) in Theorem 1 requires the existence of a *common* solution V that satisfies the inequalities in (3), one for each p in the index set $\mathcal{P}$. The function V can be regarded as a *common storage function* for all the systems being switched. It is probably not surprising that the existence of a common storage function suffices to guarantee an induced gain smaller than γ. It is perhaps more unexpected that this is actually a necessary condition.

The following Theorem 2 provides necessary and sufficient conditions for a switched system to admit an induced gain strictly smaller than $\gamma > 0$ uniformly over $\mathcal{S}$.

Theorem 2: Suppose that the system (1) is uniformly asymptotically stable over $\mathcal{S}$ and let $\gamma > 0$. The following statements are equivalent:

1) The system (1) has an $\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}$.
2) There exist a constant $\epsilon > 0$ and a function $V : \mathbb{R}^n \to \mathbb{R}$ that is convex, zero at zero, and homogeneous of degree two, which satisfies

$$\frac{\partial V}{\partial x}(x) A_p x + x^{\mathrm{T}} C_p^{\mathrm{T}} C_p x + \frac{1}{4}\gamma^{-2} \frac{\partial V}{\partial x}(x) B_p B_p^{\mathrm{T}} \frac{\partial V^{\mathrm{T}}}{\partial x}(x) \leq -\epsilon \|x\|^2 \qquad \text{for all } p \in \mathcal{P} \tag{4}$$

almost everywhere in x.

Remark 2: Theorems 1 and 2 assume that the system is uniformly asymptotically stable over $\mathcal{S}$. The inequalities in (3) or (4) do not necessarily imply the stability of the switched systems (1), hence this is a necessary assumption. One might expect that an observability assumption could remove the need for the stability assumption. However, even when all pair (C_p, A_p) are observable, stability does not necessarily follow for every class of allowable switching signals (cf. [18] for details). This topic is further explained in Corollaries 1 and 2.

Remark 3: From (4), we have

$$\frac{\partial V}{\partial x}(x)A_p x \leq -\epsilon \|x\|^2 \qquad \text{for all } p \in \mathcal{P} \tag{5}$$

almost everywhere in x. Therefore, the function V could be a candidate *common* Lyapunov function, provided that it is positive definite. The following Corollary 2 unifies stability analysis and Theorem 2.

The following Corollary 1 shows how the combination of an observability condition with the inequalities in (3) suffices to establish both stability and a finite induced gain, provided that the class of switching signals satisfies appropriate regularity conditions.

To state Corollary 1, we need to introduce the following subset of $\mathcal{S}$: the set of persistent dwell-time switching signals $\mathcal{S}_{\text{p-dwell}}[\,\tau_D, \infty\,]$ for some $\tau_D > 0$ and $T \in [\,0, \infty\,]$ consists of those switching signals s with an infinite number of disjoint intervals on which s is constant of length no smaller than the *persistent dwell-time* τ_D, and consecutive intervals with this property are separated by no more than the *period of persistence* T. For a more detailed description of this and related classes of switching signals, the reader is referred to, e.g., [12], [18].

Corollary 1: Suppose that each (C_p, A_p), $p \in \mathcal{P}$ is an observable pair and let $\tau_D > 0$, $T < \infty$, and $\gamma > 0$. The following statement 1) implies statement 2):

1) There exits a function $V: \ \mathbb{R}^n \to \mathbb{R}$ that is convex, zero at zero, homogeneous of degree two, which satisfies

$$\frac{\partial V}{\partial x}(x)A_p x + x^{\mathrm{T}} C_p^T C_p x + \frac{1}{4}\gamma^{-2}\frac{\partial V}{\partial x}(x) B_p B_p^T \frac{\partial V}{\partial x}^{\mathrm{T}}(x) \leq 0 \qquad \text{for all } p \in \mathcal{P} \tag{6}$$

almost everywhere in x.

2) The system (1) is (uniformly) exponentially stable over $\mathcal{S}_{\text{p-dwell}}[\,\tau_D, T\,]$ and has an $\mathcal{L}_2$-induced gain smaller than or equal to γ uniformly over $\mathcal{S}_{\text{p-dwell}}[\,\tau_D, T\,]$.

Remark 4: Suppose that $\gamma_{\mathcal{S}}$ and $\gamma_{\mathcal{S}_{\text{p-dwell}}[\,\tau_D, T\,]}$ denote the $\mathcal{L}_2$-induced gains of the switched systems (1) for switching signals in $\mathcal{S}$ and $\mathcal{S}_{\text{p-dwell}}[\,\tau_D, T\,]$, respectively (see (12) later for more precise descriptions of this notation). Since $\mathcal{S}_{\text{p-dwell}}[\,\tau_D, T\,] \subset \mathcal{S}$, we necessarily have that $\gamma_{\mathcal{S}_{\text{p-dwell}}[\,\tau_D,T\,]} \leq \gamma_{\mathcal{S}}$ and, in fact, one should expect the inequality to be strict in general. However, Corollary 1 provides the same estimate for the induced gain $\gamma_{\mathcal{S}_{\text{p-dwell}}[\,\tau_D,T\,]}$ as Theorem 1 provides for $\gamma_{\mathcal{S}}$. This indicates that the sufficient condition in Corollary 1 is conservative, in contrast to the conditions in Theorem 1 which is not. In Section IV, we investigate for which classes of switching signals the conditions (3) or (4) provides a non-conservative estimate for the induced gain.

The following Corollary 2 shows how the combination of an observability condition with the inequalities in (4) establishes both stability and a non-conservative induced gain estimate for the switching signal class $\mathcal{S}$, provided that the function V is *strictly* convex.

Corollary 2: Suppose that there exits a system index $q \in \mathcal{P}$ such that (C_q, A_q) is an observable pair and let $\gamma > 0$. The following statements are equivalent:

1) The system (1) is (uniformly) exponentially stable over $\mathcal{S}$ and has an $\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}$.
2) There exist a constant $\epsilon > 0$ and a function $V:\ \mathbb{R}^n \to \mathbb{R}$ that is strictly convex, zero at zero, and homogeneous of degree two, which satisfies

$$\frac{\partial V}{\partial x}(x)A_p x + x^{\mathrm{T}} C_p^{\mathrm{T}} C_p x + \frac{1}{4}\gamma^{-2}\frac{\partial V}{\partial x}(x) B_p B_p^{\mathrm{T}} \frac{\partial V}{\partial x}^{\mathrm{T}}(x) \leq -\epsilon \|x\|^2 \qquad \text{for all } p \in \mathcal{P} \tag{7}$$

almost everywhere in x.

Remark 5: Since $s^q(\tau) = q(=\text{ constant})$, $\tau \geq 0$ is a possible switching signal, uniform asymptotic stability in the statement 1) implies A_q is Hurwitz. Other note on this corollary is that strict convexity, $V(0) = 0$ and homogeneity with degree two implies that the function V is positive definite, $V(x) > 0$, $x \neq 0$[4].

Remark 6: Since Theorems 1, 2 and Corollary 2 provide non-conservative conditions for the induced gain to lie below a given constant γ, one can use a bisection procedure to establish a conceptual algorithm to compute the value of the induced gain of a switched system up to any pre-specified level of precision.

III. Proofs of Main Results

In this section, we prove the main results stated above. Before proceeding, we note that, because of causality of the system dynamics [22], having an induced gain smaller than or equal to $\gamma > 0$ uniformly

[4] Let $\theta \in (0,1)$ and $x \neq 0$. We have $\theta^2 V(x) = V(\theta x) = V(\theta x + (1-\theta)0) < \theta V(x) + (1-\theta)V(0) = \theta V(x)$, thus $\theta(1-\theta)V(x) > 0$. Since $\theta \in (0,1)$, this implies that $V(x) > 0$.

over $\mathcal{S}'$ is equivalent to

$$\sup_{s\in\mathcal{S}'}\sup_{u\in\mathcal{L}_2^m}\lim_{t\to\infty}\int_0^t \|y(\tau)\|^2-\gamma^2\|u(\tau)\|^2 d\tau \le 0 \tag{8}$$

where $y(\tau)$, $\tau \ge 0$ denotes the output of (1) obtained for the switching signal s, input signal u, and zero initial condition (see also Appendix B).

A. *Proof of Theorem 1*

In Theorems 1 and 2, the system (1) is assumed to be uniformly asymptotically stable over $\mathcal{S}$ (see Definition 1). For switched systems in the form (1), uniform asymptotic stability is equivalent to exponential stability (see Definition 2) [18], and exponential stability guarantees that the switched system (1) admits a finite induced gain [18], [12]. Actually, let $b = \sup_{p\in\mathcal{P}}\|B_p\|$ and $c = \sup_{p\in\mathcal{P}}\|C_p\|$, then we can have

$$(\int_0^t \|y(\tau)\|^2 d\tau)^{\frac{1}{2}} \le \frac{ca}{\sqrt{2\lambda}}\|x\| + \frac{cab}{\lambda}(\int_0^t \|u(\tau)\|^2 d\tau)^{\frac{1}{2}} \tag{9}$$

for all $t \ge 0$, all $x \in \mathbb{R}^n$, all $s \in \mathcal{S}$ and all $u \in \mathcal{L}_2^m$, where $y(\tau)$, $t \ge \tau \ge 0$ denotes the output of (1) obtained for the switching signal s, input signal u, and initial condition $x(0) = x$; see e.g., [12] for details.

The following Section III-A1 proves that 2) $\Rightarrow$ 1), whereas Section III-A3 proves that 1) $\Rightarrow$ 2) via a series of lemmas stated in Section III-A2.

1) Theorem 1, 2) $\Rightarrow$ 1): Convexity of the function V implies that this function is Lipschitz, continuous and differentiable almost everywhere on $\mathbb{R}^n$; see, e.g., [20], [21]. Since $x \mapsto V(x)$ is not necessarily differentiable everywhere in x, the following lemma will be needed [23, Lemma 1], [24, Section 2].

Lemma 1: Let $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\alpha: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be continuous, and $w: \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Suppose that

$$\frac{\partial w}{\partial x}(x)f(x,u) \le \alpha(x,u) \qquad \text{for all } x \in \mathbb{R}^n \setminus \Omega \text{ and all } u \in \mathbb{R}^m$$

where Ω has Lebesgue measure zero and contains all points at which $x \mapsto w(x)$ is not differentiable. For every absolutely continuous solution $x: [t_0, t_1) \to \mathbb{R}^n$ to $\dot{x}(t) = f(x(t), u(t))$ with locally bounded and Lebesgue measurable $u(t) \in \mathbb{R}^m$, we have that $t \mapsto w(x(t))$ is absolutely continuous and

$$\frac{dw(x(t))}{dt} \le \alpha(x(t), u(t)) \qquad \text{for all } t \in [t_0, t_1) \setminus \mathcal{T}$$

for some $\mathcal{T}$ which has Lebesgue measure zero and contains all points at which $t \mapsto w(x(t))$ is not differentiable.

Let $x \in \mathbb{R}^n$ be a point at which $x \mapsto V(x)$ is differentiable. From (3) and using a square completion, we have

$$\begin{aligned}\frac{\partial V}{\partial x}(x)(A_p x + B_p u) &\leq -x^{\mathrm{T}} C_p^{\mathrm{T}} C_p x + \gamma^2 u^{\mathrm{T}} u \\ &\qquad -\gamma^2 \left(u^{\mathrm{T}} u + \frac{1}{4}\gamma^{-4} \frac{\partial V}{\partial x}(x) B_p B_p^{\mathrm{T}} \frac{\partial V^{\mathrm{T}}}{\partial x}(x) - \gamma^{-2} \frac{\partial V}{\partial x}(x) B_p u \right) \\ &= -\|C_p x\|^2 + \gamma^2 \|u\|^2 - \gamma^2 \|u - \frac{1}{2}\gamma^{-2} B_p^{\mathrm{T}} \frac{\partial V^{\mathrm{T}}}{\partial x}(x)\|^2 \\ &\leq -\|C_p x\|^2 + \gamma^2 \|u\|^2\end{aligned}$$

This concludes that

$$\frac{\partial V}{\partial x}(x)(A_p x + B_p u) \leq -\|C_p x\|^2 + \gamma^2 \|u\|^2 \qquad \text{for all } p \in \mathcal{P} \text{ and all } u \in \mathbb{R}^m \tag{10}$$

almost everywhere in x (except at the points where $\frac{\partial V}{\partial x}(x)$ does not exist).

Let $x(t)$, $t \geq 0$ be a solution to (1) obtained for some $s \in \mathcal{S}$, $u \in \mathcal{L}_2^m$, and zero initial condition. Since the function V is locally Lipschitz, we conclude from (10) and Lemma 1 that, between any two consecutive switching times t_{k-1} and t_k of s, the function $t \mapsto V(x(t))$ is absolutely continuous and

$$\frac{dV(x(t))}{dt} \leq -\|C_{s(t)} x(t)\|^2 + \gamma^2 \|u(t)\|^2 \tag{11}$$

almost everywhere in $t \in (t_{k-1}, t_k)$.

Since both of $x \mapsto V(x)$ and $t \mapsto x(t)$ are continuous, even at switching instances $t \mapsto V(x(t))$ is continuous. By integrating both sides of (11), we obtain,

$$\int_0^t \|y(\tau)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau \leq -V(x(t)) + V(x(0)) = -V(x(t))$$

where we used the fact that convexity, $V(0) = 0$ and homogeneity with degree two imply that the function V is positive semi-definite, $V(x) \geq 0$ for all x[5]. By letting $t \to \infty$, from the uniform asymptotic stability and $u \in \mathcal{L}_2^m$, we conclude that $x(t) \to 0$, and thus $V(x(t)) \to 0$. Because s and u are arbitrary, we conclude that (8) holds.

2) Preliminaries: We start by introducing the following notation for the *exact* $\mathcal{L}_2$-induced gain of the switched systems (1):

$$\gamma_{\mathcal{S}'} = \inf\{\gamma > 0 \mid \text{(2) holds for all } t \geq 0, \text{ all } s \in \mathcal{S}' \text{ and all } u \in \mathcal{L}_2^m\} \tag{12a}$$

$$= \inf\{\gamma > 0 \mid \text{(8) holds}\} \tag{12b}$$

[5]This can be proved by replacing strict convexity condition with non-strict one in the footnote of Remark 5.

This and the following section assume that the switched system (1) has an induced gain smaller than or equal to some $\gamma > \gamma_{\mathcal{S}} \geq 0$ uniformly over $\mathcal{S}$ [Theorem 1, 1)].

Let us define

$$v(x) = \sup_{s \in \mathcal{S}} v(x,s) \qquad x \in \mathbb{R}^n \tag{13a}$$

where

$$v(x,s) = \sup_{u \in \mathcal{L}_2^m} \lim_{t \to \infty} \int_0^t \|y(\tau)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau \qquad x \in \mathbb{R}^n \text{ and } s \in \mathcal{S} \tag{13b}$$

where $y(\tau)$, $\tau \geq 0$ denotes the output of (1) obtained for the switching signal s, input signal u, and initial condition $x(0) = x$.

The following Lemmas 2, 3 and 4 assure that the function v has a quadratic growth rate bound and is positive semi-definite. Moreover, $v(x,s)$ is actually quadratic on x for each fixed s.

Lemma 2: Suppose that the system (1) is uniformly asymptotically stable over $\mathcal{S}$. There exists a constant $\alpha_2 > 0$ such that $v(x) \leq \alpha_2 \|x\|^2$ for all $x \in \mathbb{R}^n$, and consequently $v(x,s) \leq \alpha_2 \|x\|^2$ for all $x \in \mathbb{R}^n$ and all $s \in \mathcal{S}$.

Proof: See Appendix A. ■

Lemma 3: For all $x \in \mathbb{R}^n$ and all $s \in \mathcal{S}$, $v(x,s) \geq 0$, and consequently $v(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Proof: See Appendix A. ■

Lemma 4: Suppose that the system (1) is uniformly asymptotically stable over $\mathcal{S}$. For each $s \in S$, $v(x,s)$ is a quadratic form in x.

Proof: See Appendix A ■

From Lemma 4, for each $s \in \mathcal{S}$, there exists a positive semi-definite $Q(s) \in \mathbb{R}^{n \times n}$ such as $v(x,s) = x^{\mathrm{T}} Q(s) x$. We note that, since $v(x,s) \leq \alpha_2 \|x\|^2$, $Q(s)$ is bounded. Similarly to what was done in [5], let us define

$$W = \{Q(s) \in \mathbb{R}^{n \times n} | \; s \in \mathcal{S}\} \tag{14}$$

and K to be closure of W, i.e., $K = \mathrm{cl} W$. Since $Q(s)$ is bounded, W is also bounded, and hence K is compact. These definitions allow us to write $v(x) = \sup_{s \in \mathcal{S}} v(x,s) = \sup_{s \in \mathcal{S}} x^{\mathrm{T}} Q(s) x = \max_{Q \in K} x^{\mathrm{T}} Q x$.

The next lemma summarizes some important properties of $v(x)$.

Lemma 5: Suppose that the system (1) is uniformly asymptotically stable over $\mathcal{S}$, then $v(x)$ satisfies:

1) $v(0) = 0$

2) v is homogeneous of degree two.

3) v is convex.

4) v is continuously differentiable almost everywhere on $\mathbb{R}^n$.

Proof: See Appendix A. ■

We close this subsection with stating the next lemma which describes a useful property of the switching signal class $\mathcal{S}$.

Lemma 6: Let $\gamma > 0$ and $x \in \mathbb{R}^n$. For any $t > 0$

$$v(x) = \sup_{s\in\mathcal{S}[\,0,t\,)} \sup_{u\in\mathcal{L}_2^m[\,0,t\,)} \left[\int_0^t \|C_{s(\tau)}x(\tau)\|^2 - \gamma^2\|u(\tau)^2\|d\tau + v(x(t))\right] \tag{15}$$

holds where $x(\tau)$, $t \geq \tau \geq 0$ denotes the solution of (1) obtained for the switching signal s, input signal u, and initial condition $x(0) = x$.

In the above lemma, the set $\mathcal{S}[\,0,t\,)$ consists of restrictions of all $s \in \mathcal{S}$ to the interval $[\,0,t\,)$, and the set $\mathcal{L}_2^m[\,0,t\,)$ is defined in a similar manner (see Section IV for details).

Remark 7: Lemma 6 basically says that, for every time $t > 0$, evaluating $v(x)$ can be accomplished by separating the time horizon into two parts: one from 0 to t and the other from t onwards. The equality in (15) holds for the switching signal class $\mathcal{S}$. However, if one would replace $\mathcal{S}$ by some subset $\mathcal{S}'$ of $\mathcal{S}$, this property would generally not hold. This crucial property of $\mathcal{S}$ will be proved and further discussed in Section IV.

3) Theorem 1, 1) ⇒ 2): We have already seen that the function $v(x)$ is continuously differentiable almost everywhere on $\mathbb{R}^n$ [Lemma 5, 4)]. Let $x \in \mathbb{R}^n$ be a point at which $x \mapsto v(x)$ is differentiable. We show next that the function $v(x)$ satisfies (3).

Let $t > 0$ and pick some $p \in \mathcal{P}$. We consider a constant switching signal $s^p(\tau) = p$ in the interval $[\,0,t\,)$. Since a particular switching signal s^p is selected, from Lemma 6, the inequality

$$v(x) \geq \sup_{u\in\mathcal{L}_2^m[\,0,t\,)} \left[\int_0^t \|C_p x(\tau)\|^2 - \gamma^2\|u(\tau)\|^2 d\tau + v(x(t))\right]$$

follows for every $p \in \mathcal{P}$ and every $t > 0$. We note here that $x(\tau)$, $t \geq \tau \geq 0$ represents the solution of the time-invariant system (1) obtained for the switching signal $s^p(\tau) = p$, input signal u, and initial condition $x(0) = x$.

By the mean value theorem, there exists a $\theta \in [\,0,1\,]$ such that

$$v(x) \geq \sup_{u\in\mathcal{L}_2^m[\,0,t\,)} \left[(\|C_p x(\theta t)\|^2 - \gamma^2\|u(\theta t)\|^2)t + v(x(t))\right]$$

Applying Taylor's theorem to expand $v(x(t))$, we obtain[6]

$$v(x) \geq \sup_{u\in\mathcal{L}_2^m[\,0,t\,)} \left[(\|C_p x(\theta t)\|^2 - \gamma^2\|u(\theta t)\|^2)t + v(x) + \frac{\partial v}{\partial x}(x)\,[A_p x + B_p u(0)]\,t + O(t^2)\right]$$

[6] $O(\delta)$ represents any functions such that $|f(\delta)| \leq k|\delta|$ for some constant k.

and

$$0 \geq \sup_{u \in \mathcal{L}_2^m[\,0,t\,)} \left[(\|C_p x(\theta t)\|^2 - \gamma^2 \|u(\theta t)\|^2) + \frac{\partial v}{\partial x}(x)[A_p x + B_p u(0)] + O(|t|) \right]$$

Letting $t \to 0$, to conclude that

$$0 \geq \sup_{u \in \mathbb{R}^m} \left[(\|C_p x\|^2 - \gamma^2 \|u\|^2) + \frac{\partial v}{\partial x}(x)(A_p x + B_p u) \right]$$

By a square completion argument, we have that

$$\begin{aligned}
0 &\geq \frac{\partial v}{\partial x}(x) A_p x + \|C_p x\|^2 + \sup_{u \in \mathbb{R}^m} \left[-\gamma^2 \|u\|^2 + \frac{\partial v}{\partial x}(x) B_p u \right] \\
&= \frac{\partial v}{\partial x}(x) A_p x + \|C_p x\|^2 + \sup_{u \in \mathbb{R}^m} \left[-\gamma^2 \left(\|u - \frac{1}{2}\gamma^{-2} B_p^{\mathrm{T}} \frac{\partial v^{\mathrm{T}}}{\partial x}(x)\|^2 - \|\frac{1}{2}\gamma^{-2} B_p^{\mathrm{T}} \frac{\partial v^{\mathrm{T}}}{\partial x}(x)\|^2 \right) \right] \\
&= \frac{\partial v}{\partial x}(x) A_p x + x^{\mathrm{T}} C_p^{\mathrm{T}} C_p x + \frac{1}{4}\gamma^{-2} \frac{\partial v}{\partial x}(x) B_p B_p^{\mathrm{T}} \frac{\partial v^{\mathrm{T}}}{\partial x}(x)
\end{aligned}$$

This conclude that $v(x)$ satisfies (3) almost everywhere in x (except the points at which $x \mapsto v(x)$ is not differentiable).

B. Proof of Theorem 2

1) Theorem 2, 2) ⇒ 1): Since $\epsilon > 0$ in (4), there exists a constant $\rho > 1$ such that

$$C_p^{\mathrm{T}} C_p + \epsilon I \geq (\rho C_p)^{\mathrm{T}} (\rho C_p) \geq C_p^{\mathrm{T}} C_p \qquad \text{for all } p \in \mathcal{P}$$

We set $\delta > 0$ as

$$\rho = \frac{\gamma}{\gamma - \delta} > 1$$

From (4), we have

$$\begin{aligned}
0 &\geq \frac{\partial V}{\partial x}(x) A_p x + x^{\mathrm{T}} C_p^T C_p x + \epsilon \|x\|^2 + \frac{1}{4}\gamma^{-2} \frac{\partial V}{\partial x}(x) B_p B_p^T \frac{\partial V^{\mathrm{T}}}{\partial x}(x) \\
&\geq \frac{\partial V}{\partial x}(x) A_p x + x^{\mathrm{T}} (\frac{\gamma}{\gamma - \delta} C_p)^T (\frac{\gamma}{\gamma - \delta}) C_p x + \frac{1}{4}\gamma^{-2} \frac{\partial V}{\partial x}(x) B_p B_p^T \frac{\partial V^{\mathrm{T}}}{\partial x}(x) \qquad \text{for all } p \in \mathcal{P}
\end{aligned}$$

almost everywhere in x. From Theorem 1 we then conclude that

$$\int_0^t \|\frac{\gamma}{\gamma - \delta} C_{s(\tau)} x(\tau)\|^2 d\tau \leq \gamma^2 \int_0^t \|u(\tau)\|^2 d\tau$$

or equivalently

$$\int_0^t \|y(\tau)\|^2 d\tau \leq (\gamma - \delta)^2 \int_0^t \|u(\tau)\|^2 d\tau$$

Hence, the system has an induced gain smaller than or equal to $\gamma' := \gamma - \delta < \gamma$ with some $\delta > 0$.

2) Theorem 2, 1) $\Rightarrow$ 2): The system has an induced gain strictly smaller than $\gamma > 0$ uniformly over $\mathcal{S}$, thus there exists a constant $\delta > 0$ such that

$$\int_0^t \|y(\tau)\|^2 d\tau \le (\gamma-\delta)^2 \int_0^t \|u(\tau)\|^2 d\tau \tag{16}$$

for all $t \ge 0$, all $s \in \mathcal{S}$ and all $u \in \mathcal{L}_2^m$, where $y(\tau)$ denotes the output of (1) obtained for the switching signal s, input signal u, and zero initial condition.

Let $r = \mathrm{rank} C_p \le n$ and define $C_p^\perp \in \mathbb{R}^{(n-r)\times n}$ such as $\mathrm{rank}[\, C_p^{\mathrm{T}} \quad (C_p^\perp)^{\mathrm{T}} \,]^{\mathrm{T}} = n$. Consider $z = [\, C_p^{\mathrm{T}} \quad (C_p^\perp)^{\mathrm{T}} \,]^{\mathrm{T}} x$, from (16), we have

$$\begin{aligned}\int_0^t \|z(\tau)\|^2 d\tau &= \int_0^t \|y(\tau)\|^2 d\tau + \int_0^t \|C_{s(\tau)}^\perp x(t)\|^2 d\tau \\ &\le (\gamma-\delta)^2 \int_0^t \|u(\tau)\|^2 d\tau + \int_0^t \|C_{s(\tau)}^\perp x(t)\|^2 d\tau\end{aligned}$$

From (9), we further have

$$\begin{aligned}\int_0^t \|z(\tau)\|^2 d\tau &\le (\gamma-\delta)^2 \int_0^t \|u(\tau)\|^2 d\tau + \|C_p^\perp\|^2 \int_0^t \|x(\tau)\|^2 d\tau \\ &\le (\gamma-\delta)^2 \int_0^t \|u(\tau)\|^2 d\tau + \|C_p^\perp\|^2 \frac{a^2 b^2}{\lambda^2} \int_0^t \|u(\tau)\|^2 d\tau\end{aligned}$$

where second inequality comes from by setting $x(0) = 0$ and $C_p = I$, $p \in \mathcal{P}$ in (9). We can make the norm $\|C_p^\perp\|$ arbitrarily small and conclude that the system with output equation $z = [\, C_p^{\mathrm{T}} \quad (C_p^\perp)^{\mathrm{T}} \,]^{\mathrm{T}} x$ still has an induced gain smaller than or equal to $\gamma - \bar{\delta} > 0$ for some constant $\bar{\delta} > 0$, i.e.,

$$\int_0^t \|z(\tau)\|^2 d\tau \le (\gamma-\bar{\delta})^2 \int_0^t \|u(\tau)\|^2 d\tau$$

or equivalently

$$\frac{\gamma^2}{(\gamma-\bar{\delta})^2} \int_0^t \|z(\tau)\|^2 d\tau \le \gamma^2 \int_0^t \|u(\tau)\|^2 d\tau$$

Therefore, from Theorem 1, we conclude that

$$\frac{\partial V}{\partial x}(x) A_p x + x^{\mathrm{T}} \Big(\frac{\gamma}{\gamma-\bar{\delta}} \begin{bmatrix} C_p \\ C_p^\perp \end{bmatrix}\Big)^{\mathrm{T}} \Big(\frac{\gamma}{\gamma-\bar{\delta}} \begin{bmatrix} C_p \\ C_p^\perp \end{bmatrix}\Big) x + \frac{1}{4}\gamma^{-2} \frac{\partial V}{\partial x}(x) B_p B_p^T \frac{\partial V}{\partial x}^{\mathrm{T}}(x) \le 0 \tag{17}$$

for all $p \in \mathcal{P}$ and almost everywhere in x.

Since $\gamma/(\gamma-\bar{\delta}) > 1$ and $\mathrm{rank}[\, C_p^{\mathrm{T}} \quad (C_p^\perp)^{\mathrm{T}} \,]^{\mathrm{T}} = n$, there exists an $\epsilon > 0$ such that

$$\begin{aligned}\Big(\frac{\gamma}{\gamma-\bar{\delta}} \begin{bmatrix} C_p \\ C_p^\perp \end{bmatrix}\Big)^{\mathrm{T}} \Big(\frac{\gamma}{\gamma-\bar{\delta}} \begin{bmatrix} C_p \\ C_p^\perp \end{bmatrix}\Big) &\ge \begin{bmatrix} C_p \\ C_p^\perp \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} C_p \\ C_p^\perp \end{bmatrix} + \epsilon I \\ &= C_p^{\mathrm{T}} C_p + (C_p^\perp)^{\mathrm{T}} C_p^\perp + \epsilon I \\ &\ge C_p^{\mathrm{T}} C_p + \epsilon I \qquad \text{for all } p \in \mathcal{P}\end{aligned}$$

From (17), we have

$$0 \geq \frac{\partial V}{\partial x}(x)A_p x + x^{\mathrm{T}}(\frac{\gamma}{\gamma-\bar{\delta}}\begin{bmatrix} C_p \\ C_p^{\perp}\end{bmatrix})^{\mathrm{T}}(\frac{\gamma}{\gamma-\bar{\delta}}\begin{bmatrix} C_p \\ C_p^{\perp}\end{bmatrix})x + \frac{1}{4}\gamma^{-2}\frac{\partial V}{\partial x}(x)B_p B_p^T\frac{\partial V}{\partial x}^{\mathrm{T}}(x)$$

$$\geq \frac{\partial V}{\partial x}(x)A_p x + x^{\mathrm{T}}C_p^T C_p x + \epsilon\|x\|^2 + \frac{1}{4}\gamma^{-2}\frac{\partial V}{\partial x}(x)B_p B_p^T\frac{\partial V}{\partial x}^{\mathrm{T}}(x)$$

This concludes that (4) holds.

C. Proof of Corollary 1

The following lemma [18, Lemma 5] will be needed to prove exponential stability of the system (1) over $\mathcal{S}_{\tau_D}[\,\tau_D, T\,]$.

Lemma 7: Suppose that finite $\tau_D, T, \lambda > 0$ are given and each (C_p, A_p), $p \in \mathcal{P}$ is an observable pair. Then, there exit constants $a, \kappa > 0$ such that

$$\|\Phi_s(t,\tau)\| \leq a e^{\lambda(t-\tau)} \qquad \text{for all } s \in \mathcal{S}_{\text{p-dwell}}[\,\tau_D, T\,] \text{ and all } t \geq \tau \geq 0$$

where $\Phi_s(t,\tau)$ denotes the state transition matrix of the time-varying system

$$\frac{dz(t)}{dt} = (A_{s(t)} + K(t)C_{s(t)})z(t) \qquad t \geq 0$$

for some appropriately chosen time-varying output injection matrix K satisfying $\|K(t)\| \leq \kappa$ for all $t \geq 0$.

If the exponential stability over $\mathcal{S}_{\text{p-dwell}}[\,\tau_D, T\,]$ follows for all $\tau_D > 0$ and all $T < \infty$, then Theorem 1 concludes the corollary. To this effect, we investigate stability of

$$\frac{dx}{dt} = A_s x \qquad y = C_s x \tag{18}$$

At this moment, we do not introduce restrictions on switching signals, hence we suppose that $s \in \mathcal{S}$.

Let us define

$$w(x) = \sup_{s \in \mathcal{S}[\,\tau,\infty\,)} w(x,s) \qquad \tau \geq 0 \text{ and } x \in \mathbb{R}^n$$

where

$$w(x,s) = \lim_{t\to\infty}\int_\tau^t \|y(h)\|^2 dh \qquad \tau \geq 0, x \in \mathbb{R}^n \text{ and } s \in \mathcal{S}[\,\tau,\infty\,)$$

where $y(h)$, $h \geq \tau \geq 0$ denotes the output of (18) obtained for the switching signal s, and initial condition $x(\tau) = x$. We denote by $\mathcal{S}[\,\tau,\infty\,)$ as the set of restrictions of all switching signals $s \in \mathcal{S}$ to the interval $[\,\tau,\infty\,)$ (see Section IV for details).

The following Lemmas 8 and 9 assure that the function w has quadratic shrink and growth rate bounds.

Lemma 8: Suppose that there exits a system index $q \in \mathcal{P}$ such that (C_q, A_q) is an observable pair. There exits an $\alpha_1 > 0$ such that $w(x) \geq \alpha_1 \|x\|^2$ for all $x \in \mathbb{R}^n$.

Proof: See Appendix A ∎

Lemma 9: Suppose that the statement 1) in Corollary 1 holds. There exists an $\alpha_2 \geq 0$ such that $w(x) \leq \alpha_2 \|x\|^2$ for all $x \in \mathbb{R}^n$.

Proof: See Appendix A ∎

Combining Lemmas 8 and 9, we conclude that there exist $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \|x\|^2 \leq w(x) \leq \alpha_2 \|x\|^2 \qquad \text{for all } x \in \mathbb{R}^n \tag{20}$$

Suppose now that switching signals are restricted as $s \in \mathcal{S}_{\text{p-dwell}}[\,\tau_D, T\,]$, and we rewrite (18) as

$$\frac{dx}{dt} = (A_s + KC_s)x - Ky \tag{21}$$

Applying Lemma 7 and according similar procedures to what was done in [18, Proof of Theorem 4 (ii)], we can have that

$$\|x(t)\| \leq ae^{-\lambda(t-\tau)}\|x(\tau)\| + \frac{a\kappa}{\sqrt{2\lambda}}\left(\int_\tau^t \|y(h)\|^2 dh\right)^{\frac{1}{2}} \qquad \text{for all } t \geq \tau \geq 0 \tag{22}$$

where $x(t)$ and $y(t)$, $t \geq \tau \geq 0$ denote the solution and output of (21) obtained for the switching signal s, and initial condition $x(\tau)$, respectively.

Let us denote $w(t) = w(x(t))$. From (20) and (22), for any $t \geq \tau \geq 0$, we have

$$\begin{aligned}
w(t) \leq \alpha_2 \|x(t)\|^2 &\leq \alpha_2 \left(ae^{\lambda(t-\tau)}\|x(\tau)\| + \frac{a\kappa}{\sqrt{2\lambda}} \int_\tau^t \|y(h)\|^2 dh \right)^2 \\
&\leq 2\alpha_2 a^2 e^{-2\lambda(t-\tau)} \|x(\tau)\|^2 + \frac{\alpha_2 a^2 \kappa^2}{\lambda} \int_\tau^t \|y(h)\|^2 dh \\
&\leq \frac{2\alpha_2 a^2}{\alpha_1} e^{-2\lambda(t-\tau)} w(\tau) + \frac{\alpha_2 a^2 \kappa^2}{\lambda} \int_\tau^t \|y(h)\|^2 dh \\
&= \frac{2\alpha_2 a^2}{\alpha_1} e^{-2\lambda(t-\tau)} w(\tau) + \frac{\alpha_2 a^2 \kappa^2}{\lambda} \left(w(\tau) - w(t) \right)
\end{aligned}$$

This concludes that

$$w(t) \leq \frac{c_1 e^{-2\lambda(t-\tau)} + c_2}{1 + c_2} w(\tau) \qquad \text{for all } t \geq \tau \geq 0$$

where we set $c_1 = \frac{2\alpha_2 a^2}{\alpha_1}$ and $c_2 = \frac{\alpha_2 a^2 \kappa^2}{\lambda}$.

Suppose now that we pick a constant $L > 0$ such that

$$\frac{c_1 e^{-2\lambda L} + c_2}{1 + c_2} = \rho < 1$$

Let N be an integer satisfying $(N-1)L+\tau \leq t \leq NL+\tau$. Since $w((k+1)L+\tau) \leq \rho w(kL+\tau)$ for any positive integer k, we have

$$\begin{aligned} w(t) &\leq \frac{c_1 e^{-2\lambda[t-((N-1)L+\tau)]}+c_2}{1+c_2} w((N-1)L+\tau) \\ &\leq \frac{c_1+c_2}{1+c_2} w((N-1)L+\tau) \\ &\leq \frac{c_1+c_2}{1+c_2} \rho^{N-1} w(\tau) = \frac{c_1+c_2}{(1+c_2)\rho} \rho^{N} w(\tau) \\ &\leq \frac{c_1+c_2}{(1+c_2)\rho} \rho^{\frac{t-\tau}{L}} w(\tau) \end{aligned}$$

Together with (20), this leads to

$$\begin{aligned} \|x(t)\|^2 \leq \frac{w(t)}{\alpha_1} &\leq \frac{c_1+c_2}{\alpha_1(1+c_2)\rho} \rho^{\frac{t-\tau}{L}} w(\tau) \\ &\leq \frac{\alpha_2(c_1+c_2)}{\alpha_1(1+c_2)\rho} \rho^{\frac{t-\tau}{L}} \|(x(\tau)\|^2 \qquad \text{for all } t \geq \tau \geq 0 \end{aligned}$$

Thus, $x(\cdot)$ converges to zero exponentially fast. The above bound is applicable for any $s \in \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$, this conclude that (18) is exponentially stable over $\mathcal{S}_{\text{p-dwell}}[\,\tau_D, T\,]$.

D. Proof of Corollary 2

1) Corollary 2, 2) $\Rightarrow$ 1): We show that a strictly convex function V satisfying (7) implies exponential stability of the system (1) over $\mathcal{S}$.

Let us consider the compact set $\{\xi \in \mathbb{R}^n | \ \|\xi\| = 1\}$. Since the function V is positive definite[7] and continuous, there exist $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 = \min_{\substack{\xi\in\mathbb{R}^n \\ \|\xi\|=1}} V(\xi) \qquad \alpha_2 = \max_{\substack{\xi\in\mathbb{R}^n \\ \|\xi\|=1}} V(\xi)$$

Then we have

$$\alpha_1 \leq V(\xi) \leq \alpha_2 \qquad \text{for all } \xi \in \mathbb{R}^n \text{ such as } \|\xi\| = 1$$

Homogeneity of the function V also leads to

$$\alpha_1\|x\|^2 \leq V(x) \leq \alpha_2\|x\|^2 \qquad \text{for all } x \in \mathbb{R}^n \tag{23}$$

Thus, the function V has quadratic shrink and growth rate bounds.

[7]see the footnote in Remark 5

Let $x(t)$, $t \geq 0$ be a solution to (1) obtained for some $s \in \mathcal{S}$, initial condition $x(0)$, and zero input signal u. Since the function V is locally Lipschitz, we conclude from (5) and Lemma 1 that, between any consecutive switching times t_{k-1} and t_k of s, the function $t \mapsto V(x(t))$ is absolutely continuous and

$$\frac{dV(x(t))}{dt} \leq -\epsilon \|x(t)\|^2 \tag{24}$$

almost everywhere in $t \in (\ t_{k-1}, t_k\ )$.

Since both of $x \mapsto V(x)$ and $t \mapsto x(t)$ are continuous, even at switching instances $t \mapsto V(x(t))$ is continuous. Therefore $t \mapsto V(x(t))$ is monotonically strictly decreasing, absolutely continuous on $t \geq 0$, and (24) holds almost everywhere on $t \geq 0$.

The remaining of the proof follows standard Lyapunov arguments [25]. Combining (23) and (24) leads to

$$\dot{V}(x(t)) \leq -\epsilon \|x(t)\|^2 \leq -\frac{\epsilon}{\alpha_1} V(x(t))$$

Let $\eta(t)$, $t \geq 0$ be a solution to the differential equation

$$\dot{\eta}(t) = -\frac{\epsilon}{\alpha_1}\eta(t) \qquad \eta(0) = V(x(0)) \quad t \geq 0$$

Then, we have

$$V(x(t)) \leq \eta(t) = V(x(0)) e^{-\frac{\epsilon}{\alpha_1}t} \qquad \text{for all } t \geq 0$$

Together with (23), we have

$$\|x(t)\|^2 \leq \frac{1}{\alpha_1} V(x(t)) \leq \frac{1}{\alpha_1} V(x(0)) e^{-\frac{\epsilon}{\alpha_1}t} \leq \frac{\alpha_2}{\alpha_1} \|x(0)\|^2 e^{-\frac{\epsilon}{\alpha_1}t} \qquad \text{for all } t \geq 0$$

This concludes that $x(\cdot)$ converges to zero exponentially fast uniformly over $\mathcal{S}$.

From Theorem 2, the system (1) has an $\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}$.

2) Corollary 2, 1) $\Rightarrow$ 2): For switched systems in the form (1), uniform asymptotic stability is equivalent to exponential stability [18], thus Theorem 2, 1) $\Rightarrow$ 2) holds. We show that if we had at least a single mode q such as the pair (C_q, A_q) is observable, then the function $v(x)$ in (13a) is strictly convex [cf. Lemma 5, 3)]. To this effect, we first show that the function $v(x)$ is positive definite [cf. Lemma 3].

We note that $s^q(\tau) = q(=\text{constant})$, $\tau \geq 0$ is a possible switching signal. Hence, uniform asymptotic stability implies that A_q is Hurwitz, and therefore there exists an unique $Q_q > 0$ satisfying $A_q^{\mathrm{T}} Q_q + Q_q A_q = -C_q^{\mathrm{T}} C_q$. Since [cf. proof of Lemma 3 in Appendix A]

$$v(x) = \sup_{s \in \mathcal{S}} v(x, s) \geq v(x, s^q) \geq \lim_{t \to \infty} \int_0^t \|C_q x(\tau)\|^2 d\tau = x^{\mathrm{T}} Q_q x > 0 \qquad \text{for all } x \neq 0$$

we conclude that $v(x)$ is positive definite.

Since the system (1) is exponentially stable over $\mathcal{S}$, Lemmas 2 and 4 are applicable. We can redefine W in (14) as

$$W = \{Q(s) \in \mathbb{R}^{n\times n}|\ s \in \mathcal{S} \quad Q(s) \geq Q_q > 0\}$$

and, this leads to $v(x) = \max_{Q\in K} x^{\mathrm{T}} Qx$ where $K = \mathrm{cl}W$. We conclude that $v(x)$ is strictly convex, since it is a quadratic form with a positive definite matrix $Q \in K$.

IV. Non-conservative Induced Gain Estimates

This section contains the proof of Lemma 6 and shows that the equality in (15) holds for the switching signal class $\mathcal{S}$, but may not hold for subsets of $\mathcal{S}$. The results in this section make possible to state corollaries in Section IV-B that correspond to versions of Theorems 1, 2 and Corollary 2 for more general classes of switching signals.

We start with introducing some notations. For a given subset $\mathcal{S}'$ of $\mathcal{S}$ and for constants $b > a \geq 0$, possibly with $b = \infty$, we denote by $\mathcal{S}'[\,a, b\,)$ the set of restrictions of all switching signals $s \in \mathcal{S}'$ to the interval $[\,a, b\,)$. Similarly, $\mathcal{L}_2^m[\,a, b\,)$ denotes a restriction of $\mathcal{L}_2^m$.

Suppose that $c > b > a \geq 0$, and $\mathcal{S}' \subset \mathcal{S}$ is given. Let $\sigma \in \mathcal{S}'[\,a, b\,)$, $\mu \in \mathcal{S}'[\,b, c\,)$ and define $\sigma \oplus \mu$ by

$$(\sigma \oplus \mu)(\tau) = \begin{cases} \sigma(\tau) & a \leq \tau < b \\ \mu(\tau) & b \leq \tau < c \end{cases}$$

In addition, we also define $\mathcal{S}'[\,a, b\,) \bigoplus \mathcal{S}'[\,b, c\,) = \{\sigma \oplus \mu|\ \sigma \in \mathcal{S}'[\,a, b\,),\ \mu \in \mathcal{S}'[\,b, c\,)\}$.

Suppose that $t > 0$, and $\mathcal{S}' \subset \mathcal{S}$ is given. We define a set of switching signals defined on the interval $[\,0, \infty\,)$ by

$${}^{t}\mathcal{S}'[\,t, \infty\,) = \{\sigma|\ \sigma(\rho) = \mu(\rho + t),\ \mu \in \mathcal{S}'[\,t, \infty\,),\ \rho \geq 0\}$$

We have the following inclusion relations.

Lemma 10: Let $\mathcal{S}' \subset \mathcal{S}$. For any $t > 0$

1) $\mathcal{S}' \subset \mathcal{S}'[\,0, t\,) \bigoplus \mathcal{S}'[\,t, \infty\,)$
2) $\mathcal{S}' \subset {}^{t}\mathcal{S}'[\,t, \infty\,)$

hold, respectively.

Proof: See Appendix A. ■

We now state the following definitions.

Definition 5: Let $\mathcal{S}' \subset \mathcal{S}$. The switching signal class $\mathcal{S}'$ is said to be closed under concatenation if $\mathcal{S}' = \mathcal{S}'[\,0, t\,) \bigoplus \mathcal{S}'[\,t, \infty\,)$ for all $t > 0$.

Definition 6: Let $\mathcal{S}' \subset \mathcal{S}$. The switching signal class $\mathcal{S}'$ is said to be shift-invariant if $\mathcal{S}' = {}^t\mathcal{S}'[\,t, \infty\,)$ for all $t > 0$.

Actually, every switching signal class $\mathcal{S}' \subset \mathcal{S}$ is not necessarily closed under concatenation or shift-invariant. The following lemmas says that the former is a more strong property than the later.

Lemma 11: Let $\mathcal{S}' \subset \mathcal{S}$. The switching signal class $\mathcal{S}'$ is shift-invariant if it is closed under concatenation.

Proof: See Appendix A. ■

A. Proof of Lemma 6

Lemma 6 can be proved as a direct consequence of the following lemma.

Lemma 12: The switching signal class $\mathcal{S}$ is closed under concatenation, and consequently it is shift-invariant.

Proof: See Appendix A. ■

We now can complete the proof of Lemma 6. Let $x \in \mathbb{R}^n$ and $t > 0$, we have

$$\begin{aligned} v(x) &= \sup_{s \in \mathcal{S}[\,0,t\,)} \sup_{u \in \mathcal{L}_2^m[\,0,t\,)} \bigg[\int_0^t \|C_{s(\tau)}\phi(\tau; x, 0, s, u)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau \\ &\qquad + \sup_{\mu \in \mathcal{S}[\,t,\infty\,)} \sup_{u_2 \in \mathcal{L}_2^m[\,t,\infty\,)} \int_t^\infty \|C_{\mu(\tau)}\phi(\tau; \phi(t; x, 0, s, u), t, \mu, u_2)\|^2 - \gamma^2 \|u_2(\tau)\|^2 d\tau \bigg] \qquad \text{(25a)} \\ &= \sup_{s \in \mathcal{S}[\,0,t\,)} \sup_{u \in \mathcal{L}_2^m[\,0,t\,)} \bigg[\int_0^t \|C_{s(\tau)}\phi(\tau; x, 0, s, u)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau \\ &\qquad + \sup_{\mu \in \mathcal{S}} \sup_{u_2 \in \mathcal{L}_2^m} \int_0^\infty \|C_{\mu(\tau)}\phi(\tau; \phi(t; x, 0, s, u), 0, \mu, u_2)\|^2 - \gamma^2 \|u_2(\tau)\|^2 d\tau \bigg] \qquad \text{(25b)} \\ &= \sup_{s \in \mathcal{S}[\,0,t\,)} \sup_{u \in \mathcal{L}_2^m[\,0,t\,)} \bigg[\int_0^t \|C_{s(\tau)}\phi(\tau; x, 0, s, u)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau + v(\phi(t; x, 0, s, u)) \bigg] \end{aligned}$$

where $\phi(\tau; x, t_0, s, u)$, $\tau \geq t_0 \geq 0$ denotes the solution of (1) obtained for the switching signal s, input signal u, and initial condition x. The equalities in (25a) and (25b) follow from the fact that the switching signal class $\mathcal{S}$ is closed under concatenation and shift-invariant, respectively, thus we have the equality in (15).

Let $\mathcal{S}' \subset \mathcal{S}$ and consider the function v in (13) for the switching signal class $\mathcal{S}'$. Since the switching signal class $\mathcal{S}'$ is not necessary closed under concatenation, we have that the quantity of the right hand side in (15) is larger than or equals to the left hand side, and one should expect that the inequality to be strict in general.

B. Non-conservative Induced Gain Estimate Conditions

We have a complete characterization of the induced gain in Theorem 1. The proof of this theorem utilized the result in Lemma 6, and the derivation of the equality in (15) depends on the fact that the class $\mathcal{S}$ of all piecewise constant switching signals is closed under concatenation. Non-conservative induced gain estimate conditions for more general switching signal classes can be summarized as in the following corollaries.

Corollary 3: Let $\mathcal{S}' \subset \mathcal{S}$ and $\gamma > 0$. Suppose that $\mathcal{S}'$ is closed under concatenation, and the system (1) is uniformly asymptotically stable over $\mathcal{S}'$. The following statements are equivalent:

1) The system (1) has an $\mathcal{L}_2$-induced gain smaller than or equal to γ uniformly over $\mathcal{S}'$.
2) There exits a function $V : \mathbb{R}^n \to \mathbb{R}$ that is convex, zero at zero, and homogeneous of degree two, which satisfies
$$\frac{\partial V}{\partial x}(x)A_p x + x^{\mathrm{T}} C_p^{\mathrm{T}} C_p x + \frac{1}{4}\gamma^{-2}\frac{\partial V}{\partial x}(x)B_p B_p^{\mathrm{T}}\frac{\partial V^{\mathrm{T}}}{\partial x}(x) \le 0 \qquad \text{for all } p \in \mathcal{P}$$
almost everywhere in x.

Corollary 4: Let $\mathcal{S}' \subset \mathcal{S}$ and $\gamma > 0$. Suppose that $\mathcal{S}'$ is closed under concatenation, and the system (1) is uniformly asymptotically stable over $\mathcal{S}'$. The following statements are equivalent:

1) The system (1) has an $\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}'$.
2) There exist a constant $\epsilon > 0$ and a function $V : \mathbb{R}^n \to \mathbb{R}$ that is convex, zero at zero, and homogeneous of degree two, which satisfies
$$\frac{\partial V}{\partial x}(x)A_p x + x^{\mathrm{T}} C_p^{\mathrm{T}} C_p x + \frac{1}{4}\gamma^{-2}\frac{\partial V}{\partial x}(x)B_p B_p^{\mathrm{T}}\frac{\partial V^{\mathrm{T}}}{\partial x}(x) \le -\epsilon\|x\|^2 \qquad \text{for all } p \in \mathcal{P}$$
almost everywhere in x.

Corollary 5: Let $\mathcal{S}' \subset \mathcal{S}$ and $\gamma > 0$. Suppose that $\mathcal{S}'$ is closed under concatenation, and there exits a system index $q \in \mathcal{P}$ such that (C_q, A_q) is an observable pair. The following statements are equivalent:

1) The system (1) is (uniformly) exponentially stable over $\mathcal{S}'$ and has an $\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}'$.
2) There exist a constant $\epsilon > 0$ and a function $V : \mathbb{R}^n \to \mathbb{R}$ that is strictly convex, zero at zero, and homogeneous of degree two, which satisfies
$$\frac{\partial V}{\partial x}(x)A_p x + x^{\mathrm{T}} C_p^{\mathrm{T}} C_p x + \frac{1}{4}\gamma^{-2}\frac{\partial V}{\partial x}(x)B_p B_p^{\mathrm{T}}\frac{\partial V^{\mathrm{T}}}{\partial x}(x) \le -\epsilon\|x\|^2 \qquad \text{for all } p \in \mathcal{P}$$
almost everywhere in x.

These results essentially show that any class of switching signals that is closed under concatenation will exhibit the same induced gain as the class $\mathcal{S}$ of every piecewise constant switching signal.

V. Conclusion

This paper provided non-conservative conditions to compute the induced gain of a switched system in terms of a common solution to a system of Hamilton-Jacobi inequalities. We also showed that for observable systems, stability follows from a finite induced gain, provided that the class of switching signals satisfies appropriate conditions. We have also shown that the induced gain that is obtained for the class $\mathcal{S}$ of all piecewise constant switching signals, remains the same for more restricted classes of switching signals that are closed under concatenation. Future work is needed. In particular how to solve (perhaps approximately) the Hamilton-Jacobi inequalities.

Appendix A
Proofs of Lemmas

Proof: [Lemma 2] We will start with for having some further considerations on the inequality in (9). For notational convenience, for a give switching signal $s \in \mathcal{S}$, define T_s^0 to be the dynamics of the system (1) from initial state $x(0)$ to the output y with zero inputs. Similarly, define T_s^1 to be the dynamics from u to y with zero initial state. Both of T_s^0, T_s^1 are linear and, if one assumed that the system (1) is uniformly asymptotically stable over $\mathcal{S}$, bounded. Actually, from (9), we have

$$\|T_s^0\| \leq \frac{ca}{\sqrt{2\lambda}} \qquad \|T_s^1\| = \gamma_{\mathcal{S}} \leq \frac{cab}{\lambda}$$

where $\|T_s^1\| = \gamma_{\mathcal{S}}$ is clear from definitions.

Adopting the above temporary notations, we rewrite the left hand side of (9) as

$$(\lim_{t\to\infty}\int_0^t \|y(\tau)\|^2 d\tau)^{\frac{1}{2}} = (\lim_{t\to\infty}\int_0^t \|(T_s^0 x)(\tau) + (T_s^1 u)(\tau)\|^2 d\tau)^{\frac{1}{2}} \tag{26}$$

Since we have both of

$$\lim_{t\to\infty}\int_0^t \|(T_s^0 x)(\tau)\|^2 d\tau \leq \|T_s^0\|^2\|x\|^2 \leq \frac{c^2a^2}{2\lambda}\|x\|^2 < +\infty$$

and

$$\lim_{t\to\infty}\int_0^t \|(T_s^1 u)(\tau)\|^2 d\tau \leq \|T_s^1\|^2 \lim_{t\to\infty}\int_0^t \|u(\tau)\|^2 d\tau \leq \gamma_{\mathcal{S}}^2 \lim_{t\to\infty}\int_0^t \|u(\tau)\|^2 d\tau < +\infty$$

we can apply the Minkowski inequality to the above (26). Thus, from (26), we have

$$\begin{aligned}(\lim_{t\to\infty}\int_0^t \|y(\tau)\|^2 d\tau)^{\frac{1}{2}} &= (\lim_{t\to\infty}\int_0^t \|(T_s^0 x)(\tau) + (T_s^1 u)(\tau)\|^2 d\tau)^{\frac{1}{2}} \\ &\leq (\lim_{t\to\infty}\int_0^t \|(T_s^0 x)(\tau)\|^2 d\tau)^{\frac{1}{2}} + (\lim_{t\to\infty}\int_0^t \|(T_s^1 u)(\tau)\|^2 d\tau)^{\frac{1}{2}} \\ &\leq \frac{ca}{\sqrt{2\lambda}}\|x\| + \gamma_{\mathcal{S}}(\lim_{t\to\infty}\int_0^t \|u(\tau)\|^2 d\tau)^{\frac{1}{2}}\end{aligned} \tag{27}$$

From hereafter, we further adopt the following additional temporary notations

$$\|u\| = (\lim_{t\to\infty}\int_0^t \|u(\tau)\|^2 d\tau)^{\frac{1}{2}}$$

and

$$v(x,s,u) = \lim_{t\to\infty}\int_0^t \|y(\tau)\|^2 - \gamma^2\|u(\tau)\|^2 d\tau$$

where $y(\tau)$, $\tau \geq 0$ denotes the output of (1) obtained for the switching signal s, input signal u, and initial condition $x(0) = x$.

Let us define

$$\begin{aligned} w(\|x\|,\|u\|) &= \left(\frac{ca}{\sqrt{2\lambda}}\|x\| + \gamma_{\mathcal{S}}(\lim_{t\to\infty}\int_0^t \|u(\tau)\|^2 d\tau)^{\frac{1}{2}}\right)^2 - \gamma^2 \lim_{t\to\infty}\int_0^t \|u(\tau)\|^2 d\tau \\ &= (\eta\|x\| + \gamma_{\mathcal{S}}\|u\|)^2 - \gamma^2\|u\|^2 \end{aligned}$$

where we set $\eta = \frac{ca}{\sqrt{2\lambda}}$. From (27), we have

$$v(x,s,u) \leq w(\|x\|,\|u\|) \quad \text{and} \quad v(x,s) = \sup_{u\in\mathcal{L}_2^m} v(x,s,u) \leq \sup_{\|u\|\in\mathcal{R}} w(\|x\|,\|u\|)$$

Since $\gamma > \gamma_{\mathcal{S}}$, applying a square completion, we have

$$\begin{aligned} w(\|x\|,\|u\|) &= (\gamma_{\mathcal{S}}^2 - \gamma^2)\|u\|^2 + 2\eta\gamma_{\mathcal{S}}\|x\|\|u\| + \eta^2\|x\|^2 \\ &= (\gamma_{\mathcal{S}}^2 - \gamma^2)(\|u\| + \frac{\eta\gamma_{\mathcal{S}}}{\gamma_{\mathcal{S}}^2 - \gamma^2}\|x\|)^2 - \frac{\eta^2\gamma_{\mathcal{S}}^2}{\gamma_{\mathcal{S}}^2 - \gamma^2}\|x\|^2 + \eta^2\|x\|^2 \\ &\leq -\frac{\eta^2\gamma_{\mathcal{S}}^2}{\gamma_{\mathcal{S}}^2 - \gamma^2}\|x\|^2 + \eta^2\|x\|^2 = \frac{c^2a^2}{2\lambda}\frac{-\gamma^2}{\gamma_{\mathcal{S}}^2 - \gamma^2}\|x\|^2 = \alpha_2\|x\|^2 \end{aligned}$$

where we set $\alpha_2 = \frac{c^2a^2}{2\lambda}\frac{-\gamma^2}{\gamma_{\mathcal{S}}^2-\gamma^2} > 0$. This concludes $v(x) = \sup_{s\in\mathcal{S}} v(x,s) \leq \sup_{\|u\|\in\mathcal{R}} w(\|x\|,\|u\|) \leq \alpha_2\|x\|^2$ for all $x \in \mathbb{R}^n$. ■

Proof: [Lemma 3] Let $s \in \mathcal{S}$ be fixed. Adopting the temporary notations in the proof of Lemma 2, we have

$$\begin{aligned} v(x,s) &= \sup_{u\in\mathcal{L}_2^m} v(x,s,u) = \sup_{u\in\mathcal{L}_2^m} \lim_{t\to\infty}\int_0^t \|y(\tau)\|^2 - \gamma^2\|u(\tau)\|^2 d\tau \\ &\geq v(x,s,0) = \lim_{t\to\infty}\int_0^t \|y(\tau)\|^2 d\tau \geq 0 \end{aligned}$$

This concludes, for each $s \in \mathcal{S}$, $v(x,s)$ is positive semi-definite. ■

Proof: [Lemma 4] The following lemma will be useful [26, Problem 2.3-1].

Lemma 13: Let $w:\ \mathbb{R}^n \to \mathbb{R}$. A function $w(x)$ to be a quadratic form in x, if and only if,

1) w is continuous in x.

2) $w(kx) = k^2 w(x) \qquad$ for all $k \in \mathbb{R}$

3) $w(x_1) + w(x_2) = \frac{1}{2}[w(x_1 + x_2) + w(x_1 - x_2)]$

Let $s \in \mathcal{S}$ be fixed. We adopt the temporary notations in the proof of Lemma 2, and let $u(t; x)$ denote the *optimal* input for a given initial condition $x(0) = x$, i.e.,

$$v(x, s) = \sup_{u \in \mathcal{L}_2^m} v(x, s, u) = v(x, s, u(\cdot; x))$$

We will see that, for each fixed s, the function $v(x, s)$ satisfies all three conditions in Lemma 13. The equality in item 2) of Lemma 13 can be verified by inequalities

$$\begin{aligned} v(kx, s) &= v(kx, s, u(\cdot; kx)) \geq v(kx, s, ku(\cdot; x)) \\ &= \lim_{t\to\infty} \int_0^t \|(T_s^0 kx)(\tau) + (T_s^1 ku(\cdot; x))(\tau)\|^2 - \gamma^2 \|ku(\tau; x)\|^2 d\tau \\ &= k^2 v(x, s, u(\cdot; x)) = k^2 v(x, s) \end{aligned}$$

and

$$\begin{aligned} k^2 v(x, s) &= k^2 v(x, s, u(\cdot; x)) \geq k^2 v(x, s, \frac{1}{k} u(\cdot; kx)) \\ &= k^2 \lim_{t\to\infty} \int_0^t \|(T_s^0 x)(\tau) + (T_s^1 \frac{1}{k} u(\cdot; kx))(\tau)\|^2 - \gamma^2 \|\frac{1}{k} u(\tau; kx)\|^2 d\tau \\ &= k^2 \lim_{t\to\infty} \int_0^t \|\frac{1}{k}(T_s^0 kx)(\tau) + (T_s^1 \frac{1}{k} u(\cdot; kx))(\tau)\|^2 - \gamma^2 \|\frac{1}{k} u(\tau; kx)\|^2 d\tau \\ &= k^2 v(\frac{1}{k} kx, s, \frac{1}{k} u(\cdot; kx)) = v(kx, s, u(\cdot; kx)) = v(kx, s) \end{aligned}$$

where the inequalities follow directly from optimality of $v(x, s)$. Hence, we conclude that $v(kx) = k^2 v(x)$ for all $k \in \mathbb{R}$.

Similarly, we have that

$$\begin{aligned} v(x_1, s) + v(x_2, s) &= \frac{1}{4}[v(2x_1, s) + v(2x_2, s)] = \frac{1}{4}[v(2x_1, s, u(\cdot; 2x_1)) + v(2x_2, s, u(\cdot; 2x_2)] \\ &\geq \frac{1}{4}[v(2x_1, s, u(\cdot; x_1 + x_2) + u(\cdot; x_1 - x_2)) + v(2x_2, s, u(\cdot; x_1 + x_2) - u(\cdot; x_1 - x_2))] \\ &= \frac{1}{2}[v(x_1 + x_2, s) + v(x_1 - x_2, s)] \end{aligned}$$

and

$$\begin{aligned} v(x_1 + x_2, s) + v(x_1 - x_2, s) &= v(x_1 + x_2, s, u(\cdot; x_1 + x_2)) + v(x_1 - x_2, s, u(\cdot; x_1 - x_2)) \\ &\geq v(x_1 + x_2, s, u(\cdot; x_1) + u(\cdot; x_2)) + v(x_1 - x_2, s, u(\cdot; x_1) - u(\cdot; x_2)) \\ &= 2(v(x_1, s) + v(x_2, s)) \end{aligned}$$

Hence, we conclude that $v(x_1, s) + v(x_2, s) = \frac{1}{2} v(x_1 + x_2, s) + v(x_1 - x_2, s)$.

To prove continuity of $v(x,s)$, we first point-out that there exits a bound for the optimal input $u(\cdot;x)$. From the definition of $w(\|x\|,\|u\|)$ [see the proof of Lemma 2], we have

$$v(x,s,u) \leq w(\|x\|,\|u\|) < 0$$
$$\text{if} \quad \|u\| > \frac{-2\eta\gamma_{\mathcal{S}}\|x\| - \sqrt{(2\eta\gamma_{\mathcal{S}}\|x\|)^2 - 4(\gamma_{\mathcal{S}}^2-\gamma^2)\eta^2\|x\|^2}}{2(\gamma_{\mathcal{S}}^2-\gamma^2)} = \frac{ca}{\sqrt{2\lambda}}\frac{-1}{\gamma_{\mathcal{S}}-\gamma}\|x\| = \kappa\|x\|$$

where we set $\kappa = \frac{ca}{\sqrt{2\lambda}}\frac{-1}{\gamma_{\mathcal{S}}-\gamma}$[8]. From Lemma 3, we know that $v(x,s) \geq 0$, hence an optimal input that attains $v(x,s) = \sup_{u\in\mathcal{L}_2^m} v(x,s,u) = v(x,s,u(\cdot;x))$ necessarily satisfies

$$\|u(\cdot;x)\| \leq \kappa\|x\| \tag{28}$$

Let us assume that for any $\epsilon > 0$, there exist $\|\delta x\| \leq \epsilon$ and $\delta > 0$ such that

$$|v(x+\delta x,s) - v(x,s)| > \delta$$

Hence, $v(x,s)$ is assumed to be discontinuous at the point x.

We first consider the case $v(x+\delta x,s) - v(x,s) > 0$, i.e.,

$$v(x+\delta x,s) > \delta + v(x,s) > v(x,s) \tag{29}$$

In the following straightforward calculations, $(\cdot,\cdot)$ denotes inner product on $\mathcal{L}_2^m \times \mathcal{L}_2^m$, i.e.,

$$(y_1,y_2) = \lim_{t\to\infty}\int_0^t y_1^{\mathrm{T}}(\tau)y_2(\tau)d\tau \qquad y_1,y_2 \in \mathcal{L}_2^m$$

then we have

$$\begin{aligned}
v(x+\delta x,s) &= v(x+\delta x,s,u(\cdot;x+\delta x)) = \|T_s^0(x+\delta x) + T_s^1u(\cdot;x+\delta x)\|^2 - \gamma^2\|u(\cdot;x+\delta x)\|^2 \\
&= \|T_s^0x\|^2 + (T_s^0x, T_s^1u(\cdot;x+\delta x)) + (T_s^1u(\cdot;x+\delta x), T_s^0x) + \|T_s^1u(\cdot;x+\delta x)\|^2 - \gamma^2\|u(\cdot;x+\delta x)\|^2 \\
&\quad + (T_s^0x, T_s^0\delta x) + (T_s^0\delta x, T_s^0x) + \|T_s^0\delta x\|^2 + (T_s^0\delta x, T_s^1u(\cdot;x+\delta x)) + (T_s^1u(\cdot;x+\delta x), T_s^0\delta x) \\
&= v(x,s,u(\cdot;x+\delta x)) \\
&\quad + (T_s^0x, T_s^0\delta x) + (T_s^0\delta x, T_s^0x) + \|T_s^0\delta x\|^2 + (T_s^0\delta x, T_s^1u(\cdot;x+\delta x)) + (T_s^1u(\cdot;x+\delta x), T_s^0\delta x) \\
&\leq v(x,s,u(\cdot;x+\delta x)) \\
&\quad + (T_s^0x, T_s^0\delta x) + (T_s^0\delta x, T_s^0x) + \|T_s^0\delta x\|^2 + 2\|T_s^0\delta x\|\cdot\|T_s^1\|\cdot\kappa\|x+\delta x\|
\end{aligned}$$

[8]Since $w(\|x\|,\|u\|)$ is a quadratic polynomial function of $\|u\|$, solutions to $w(\|x\|,\|u\|) = 0$ have analytical expressions.

where the last inequality comes from Cauchy-Schwartz inequality and (28). Since δx, and hence $\epsilon > 0$, can be made arbitrarily small, then we conclude that

$$v(x, s, u(\cdot; x + \delta x)) \geq v(x + \delta x, s)$$

From (29), we further have

$$v(x, s, u(\cdot; x + \delta x)) \geq v(x + \delta x, s) > v(x, s) = v(x, s, u(\cdot; x))$$

This contradicts optimality of $u(\cdot; x)$.

On the other hand, we consider the case $v(x + \delta x, s) - v(x, s) < 0$, i.e.,

$$v(x, s) > \delta + v(x + \delta x, s) > v(x + \delta x, s) \tag{30}$$

For each fixed $u \in \mathcal{L}_2^m$, $v(x, s, u)$ is clearly continuous in x. We can make δx, and hence $\epsilon > 0$, arbitrarily small, then this concludes that

$$u(x + \delta x, s, u(\cdot; x)) = u(x, s, u(\cdot; x))$$

We further have

$$v(x + \delta x, s) = \sup_{u \in \mathcal{L}_2^m} v(x + \delta x, s, u) \geq v(x + \delta x, s, u(\cdot; x)) = v(x, s, u(\cdot; x)) = v(x, s)$$

This contradicts (30). Both of the above arguments prove continuity of $v(x, s)$ for each $s \in \mathcal{S}$.

For each fixed $s \in \mathcal{S}$, all three conditions in Lemma 13 are confirmed to $v(x, s)$, which proves the lemma. ■

Proof: [Lemma 5] 1): It is clear from Lemmas 2, 3 and the definition of $v(x)$.

2): For any $k \in \mathcal{R}$, $v(kx) = \max_{Q \in K} (kx)^{\mathrm{T}} Qkx = k^2 \max_{Q \in K} x^{\mathrm{T}} Qx = k^2 v(x)$.

3): Let x_1, $x_2 \in \mathcal{R}^n$ and $x = \theta x_1 + (1 - \theta) x_2$ for some $\theta \in [\,0, 1\,]$. Since a quadratic form $x^{\mathrm{T}} P x$ with a positive semi-definite matrix P is convex, we have $x^{\mathrm{T}} Q x \leq \theta x_1^{\mathrm{T}} Q x_1 + (1 - \theta) x_2^{\mathrm{T}} Q x_2$ for each $Q \in K$. Taking maximum of both sides, we obtain $v(x) = \max_{Q \in K} x^{\mathrm{T}} Q x \leq \max_{Q \in K} [\theta x_1^{\mathrm{T}} Q x_1 + (1 - \theta) x_2^{\mathrm{T}} Q x_2] \leq \max_{Q_1 \in K} \theta x_1 Q_1 x_1 + \max_{Q_2 \in K} (1 - \theta) x_2^{\mathrm{T}} Q_2 x_2 = \theta v(x_1) + (1 - \theta) v(x_2)$.

4): Convex functions on $\mathcal{R}^n$ are Lipschitz, continuous, and differentiable almost everywhere; see, e.g., [20], [21]. ■

Proof: [Lemma 8] Let $q \in \mathcal{P}$ and (C_q, A_q) be an observable pair. Suppose that $\xi(h)$, $h \geq \tau \geq 0$ denotes a solution to the time-invariant system (18) obtained for the constant switching signal $s^q(h) =$

$q(=\text{constant})$, $h \geq \tau \geq 0$, and initial condition $\xi(\tau) = x$. Let $H > 0$ be any fixed constant, we have

$$w(x) = \sup_{s \in \mathcal{S}[\,\tau,\infty\,)} \lim_{t\to\infty} \int_{\tau}^{t} \|y(h)\|^2 dh \geq \sup_{s \in \mathcal{S}[\,\tau,\tau+H\,)} \int_{\tau}^{\tau+H} \|y(h)\|^2 dh$$
$$\geq \int_{\tau}^{\tau+H} \|C_q \xi(h)\|^2 dh = x^{\mathrm{T}} \left(\int_{\tau}^{\tau+H} e^{A_q^{\mathrm{T}} h} C_q^{\mathrm{T}} C_q e^{A_q h} dh \right) x$$

Observability assumption implies that the matrix $\int_{\tau}^{\tau+H} e^{A_q^{\mathrm{T}} h} C_q^{\mathrm{T}} C_q e^{A_q h} dh$ is positive definite for any $H > 0$ [27]. We conclude that

$$w(x) \geq x^{\mathrm{T}} \left(\int_{\tau}^{\tau+H} e^{A_p^{\mathrm{T}} h} C_p^{\mathrm{T}} C_p e^{A_p h} dh \right) x > 0 \qquad \text{for all } x \neq 0$$

Moreover, we can set $\alpha_1 > 0$ as

$$\int_{\tau}^{\tau+H} e^{A_p^{\mathrm{T}} h} C_p^{\mathrm{T}} C_p e^{A_p h} dh \geq \alpha_1 I$$

then this concludes $w(x) \geq \alpha_1 \|x\|^2$ for all $x \in \mathbb{R}^n$. ■

Proof: [Lemma 9] Let $x \in \mathbb{R}^n$ be a point at which $x \mapsto V(x)$ is differentiable. From (6), we have

$$\frac{\partial V}{\partial x}(x) A_p x \leq -x^{\mathrm{T}} C_p^T C_p x \qquad \text{for all } p \in \mathcal{P} \tag{31}$$

almost everywhere in x.

Let $x(h)$, $h \geq \tau \geq 0$ be a solution to (18) obtained for the switching signal s, and initial condition $x(\tau) = x$. Since the function V is locally Lipschitz, we conclude from (31) and Lemma 1 that, between any consecutive switching times h_{k-1} and h_k of s, the function $h \mapsto V(x(h))$ is absolutely continuous and

$$\frac{dV(x(h))}{dh} \leq -\|C_{s(h)} x(h)\|^2 \tag{32}$$

almost everywhere in $h \in (\,h_{k-1}, h_k\,)$.

Since both of $x \mapsto V(x)$ and $h \mapsto x(h)$ are continuous, even at switching instances $h \mapsto V(x(h))$ is continuous. By integrating both sides of (32), we obtain

$$\lim_{t\to\infty} \int_{\tau}^{t} \|y(h)\|^2 dh \leq V(x)$$

where we use the fact that the function V is positive semi-definite, $V(x) \geq 0$ for all x. Since $s \in \mathcal{S}[\tau,\infty)$ is arbitrary, we conclude that

$$w(x,s) \leq w(x) \leq V(x) \qquad \text{for all } x \in \mathbb{R}^n \text{ and all } s \in \mathcal{S}[\,\tau,\infty\,) \tag{33}$$

Thus $w(x)$ and $w(x,s)$, $s \in \mathcal{S}[\,\tau,\infty\,)$ are bounded by $V(x)$.

Let $s \in \mathcal{S}[\,\tau, \infty\,)$ be fixed. Applying Lemma 13 in the proof of Lemma 4 to $w(x,s)$, we can conclude that $w(x,s)$ is a quadratic form in x. Therefore, for each $s \in \mathcal{S}[\tau,\infty)$, there exists a positive semi-definite[9] matrix $Q(s) \in \mathbb{R}^{n\times n}$ such as $w(x,s) = x^{\mathrm{T}}Q(s)x$.

Let us consider a compact set $\{\xi \in \mathbb{R}^n |\ \|\xi\| = 1\}$. Since V is continuous, there exits an $\alpha_2 \geq 0$ such that

$$\max_{\substack{\xi\in\mathbb{R}^n \\ \|\xi\|=1}} V(\xi) = \alpha_2$$

From (33), we have

$$w(\xi,s) = \xi^{\mathrm{T}}Q(s)\xi \leq V(\xi) \leq \alpha_2 \qquad \text{for all } s \in \mathcal{S}[\,\tau,\infty\,) \text{ and all } \xi \in \mathbb{R}^n \text{ such that } \|\xi\| = 1$$

This leads to

$$w(x,s) = x^{\mathrm{T}}Q(s)x \leq \alpha_2\|x\|^2 \qquad \text{for all } s \in \mathcal{S}[\,\tau,\infty\,) \text{ and all } x \in \mathbb{R}^n$$

Then, we conclude that $w(x) \leq \alpha_2\|x\|^2$ for all $x \in \mathbb{R}^n$. ■

Proof: [Lemma 10] 1): For a give switching signal $s \in \mathcal{S}'$ and for constants $b > a \geq 0$, possibly with $b = \infty$, we denote by $s[\,a,b\,)$ the restriction of s to the interval $[\,a,b\,)$.

Let $t > 0$. Adopting the above temporally notation, we show that for any $s \in \mathcal{S}'$, one can construct particular switching signals σ and μ belong to $\mathcal{S}'[\,0,t\,)$, which satisfy $s = \sigma[\,0,t\,) \oplus \mu[\,t,\infty\,)$, hence $s \in \mathcal{S}'[\,0,t\,) \bigoplus \mathcal{S}'[\,t,\infty\,)$.

Let us define

$$\sigma(\tau) = \begin{cases} s(\tau) & 0 \leq \tau < t \\ s(t) & t \leq \tau \end{cases} \qquad \mu(\tau) = \begin{cases} s(t) & 0 \leq \tau < t \\ s(\tau) & t \leq \tau \end{cases}$$

Both σ and μ belong to $\mathcal{S}'$, since they have a strictly smaller or equal number of discontinuities than that of s. By the constriction, we have $s = \sigma[\,0,t\,) \oplus \mu[\,t,\infty\,)$, which proves 1) in the lemma.

2): Let $t > 0$ and $s \in \mathcal{S}'$. We show that the switching signal s belongs to ${}^{t}\mathcal{S}'[\,t,\infty\,)$.

Let us define

$$\pi(\tau) = \begin{cases} s(0) & 0 \leq \tau < t \\ s(\tau - t) & t \leq \tau \end{cases}$$

The switching signal π belongs to $\mathcal{S}'$, because s and π has the same number of discontinuities each other, and let t_k^{π}, t_k^{s}, $k = 0,1,\ldots$ be consecutive switching times of s and π, respectively, then we have $t_1^{\pi} - t_0^{\pi} > t_1^{s} - t_0^{s}$ and $t_{k+1}^{\pi} - t_k^{pi} = t_{k+1}^{s} - t_k^{s}$ for $k \geq 1$. This shows that $\pi \in \mathcal{S}'$.

[9] From the definition, $w(x,s) \geq 0$ for all $x \in \mathbb{R}^n$ and all $s \in \mathcal{S}[\,\tau,\infty\,)$.

Let us define the switching signal $\mu \in \mathcal{S}'[\,t,\infty\,)$ as a restriction of π to the interval $[\,t,\infty\,)$, then we have $\mu(\tau) = s(\tau - t)$. This leads to $\mu(\rho + t) = s(\rho)$, $\rho \geq 0$ and concludes that $s \in {}^t\mathcal{S}'[\,t,\infty\,)$, which proves 2) in the lemma. ∎

Proof: [Lemma 11] Suppose that $t > 0$ and $\sigma \in {}^t\mathcal{S}'$. Since the switching signal class $\mathcal{S}'$ is closed under concatenation, we show that there exits a $\tau > 0$ such that $\sigma \in \mathcal{S}'[\,0,\tau\,) \bigoplus \mathcal{S}'[\,\tau,\infty\,) = \mathcal{S}$.

Since $\sigma \in {}^t\mathcal{S}'$, there exists a switching signal $\mu \in \mathcal{S}'$ such that $\sigma(\rho) = \mu(\rho + t)$, $\rho \geq 0$. Let t_1 be the first switching time of σ, and we define

$$\sigma_1(\rho) = \sigma(0) \quad \rho \geq 0 \qquad \sigma_2(\rho) = \begin{cases} \sigma(t_1) & 0 \leq \rho < t_1 \\ \sigma(\rho) & t_1 \leq \rho \end{cases}$$

The constant signal σ_1 belongs to $\mathcal{S}'$, and σ_2 also belongs to $\mathcal{S}'$ since it has a strictly smaller number of discontinuities than that of μ.

Adopting the temporally notation in the proof of Lemma 10, we have $\sigma = \sigma_1[\,0,t_1\,) \oplus \sigma_2[\,t_1,\infty\,)$, and this conclude that $\sigma \in \mathcal{S}'[\,0,t_1\,) \bigoplus \mathcal{S}'[\,t_1,\infty\,)$. ∎

Proof: [Lemma 12] Because of $\mathcal{S} \subset \mathcal{S}[\,0,t\,) \bigoplus \mathcal{S}[\,t,\infty\,)$ [Lemma 10, 1)], we need to show the inclusion relation in the opposite direction. To this effect, we show that, for any $\pi \in \mathcal{S}[\,0,t\,)$ and $\nu \in \mathcal{S}[\,t,\infty\,)$, the switching signal $\pi \oplus \nu$ belongs to $\mathcal{S}$.

Let $N_\pi(0,t)$ denote the number of discontinuities of the switching signal π in the interval $(\,0,t\,)$. Consecutive switching times of $\pi \oplus \nu$ are give by

$$\begin{aligned} 0 = t_0^{\pi\oplus\nu} = t_0^{\pi} <= t_1^{\pi\oplus\nu} = t_1^{\pi} < \cdots \\ < t_{N_\pi(0,t)}^{\pi\oplus\nu} = t_{N_\pi(0,t)}^{\pi} < t_{N_\pi(0,t)+1}^{\pi\oplus\nu} = t_0^{\nu} = t < t_{N_\pi(0,t)+2}^{\pi\oplus\nu} = t_1^{\nu} < \cdots \end{aligned} \tag{34}$$

Since $\pi, \nu \in \mathcal{S}$, this shows that the number of discontinuities of $\pi \oplus \nu \in \mathcal{S}$ on any bounded interval is finite. We conclude that $\pi \oplus \nu \in \mathcal{S}$, which proves the lemma. ∎

APPENDIX B
EQUIVALENCE OF DEFINITION 3 AND (8)

Assuming that the system has an $\mathcal{L}_2$-induced gain smaller than or equal to $\gamma > 0$ uniformly over $\mathcal{S}'$ and letting $t \to \infty$, we obtain from (2)

$$\lim_{t\to\infty} \int_0^t \|y(\tau)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau \leq 0 \qquad \text{for all } s \in \mathcal{S}' \text{ and all } u \in \mathcal{L}_2^m \tag{35}$$

This implies (8).

On the other hand we assume (8), this implies (35). Let $f:\ [\,0,\infty\,)\rightarrow\mathcal{R}$. For each $T>0$, we define the function f_T:

$$f_T(t)=\begin{cases}f(t) & 0\le t<T\\ 0 & t\ge T\end{cases}$$

We adopt the temporary notation y_u to denote the output of the system (1) obtained for the input u, and zero initial condition. We note that $u\in\mathcal{L}_2^m$ implies $u_T\in\mathcal{L}_2^m$ for all $T>0$, and due to a causality of the system dynamics, we have

$$(y_{u_T})_T=(y_u)_T$$

Let $s\in\mathcal{S}'$ and $u\in\mathcal{L}_2^m$, we obtain from (35) that

$$\begin{aligned}0&\ge\lim_{t\to\infty}\int_0^t\|y_{u_T}(\tau)\|^2-\gamma^2\|u_T(\tau)\|^2d\tau\ge\lim_{t\to\infty}\int_0^t\|(y_{u_T})_T(\tau))\|^2-\gamma^2\|u_T(\tau)\|^2d\tau\\&=\lim_{t\to\infty}\int_0^t\|(y_u)_T(\tau)\|^2-\gamma^2\|u_T(\tau)\|^2d\tau=\int_0^T\|y(\tau)\|^2-\gamma^2\|u(\tau)\|^2d\tau\end{aligned}$$

for all $T>0$. This implies (2).

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