$\mathcal{L}_2$-induced Gains of Switched Systems and Classes of Switching Signals

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***Abstract*— This paper addresses the $\mathcal{L}_2$-induced gain analysis for switched linear systems. We exploit non-conservative necessary and sufficient conditions for the induced gain to lie below a prescribed positive constant and discuss on the induced gains of switched systems obtained for different classes of switching signals, which distinct regularity assumptions are placed on. We particularly show that the induced gain that is obtained for the class of every piecewise constant switching signal can also be attained by the more restricted classes of switching signals.**

I. Introduction

Hybrid dynamical systems whose behavior can be described using a mixture of event-based logic and differential or difference equations have been attracting significant interest. This is motivated by the observation that a wide variety of artificial/man-made and physical systems/processes are naturally modeled in a hybrid dynamical framework. Switched systems typically arise in the context of hybrid dynamical systems when it is possible to describe the behavior in each mode through a differential or difference equation and the event-based transitions as discontinuous switchings.

The stability of switched system has been extensively studied and several key results can be found in the survey papers [1], [2], [3] and references therein. An important distinction in the studies of properties of switched systems such as stability, rate of convergence, etc. is that these properties crucially depend on admissible classes of switching signals, which each the switching signal ranges over. Distinct regularity assumptions placed on the family of switching signals introduce specific admissible classes of switching signals [4], [5]. The impact of regularity properties on the switching signal class on the type of stability such as asymptotic stability, uniformity of the rate of convergence was explored [5].

The input-output properties of dynamical systems, especially $\mathcal{L}_2$-induced gains are fundamental tools for robust control theories in linear [6], and nonlinear [7] settings, particularly in $\mathcal{H}^\infty$ control problems. In spite of their important roles, the progress on the study of input-output properties for switched systems has been difficult [8]. Non-conservative necessary and sufficient conditions that can be used to establish the value of the induced gain have been only available for special switched systems. A separation property between all the stabilizing and all the anti-stabilizing solutions to a set of algebraic Riccati equations of the systems being switched provides a complete solution to the induced gain analysis in the case of *slow-switching* signals, where slow-switching refers to the limit as the intervals between consecutive switchings grows to infinity [9], and the technically related results were also obtained in the studies of Hankel operators of switched systems [10] or systems with a failure [11]. The variational approaches in [12] also provide a complete and non-conservative characterization of the induced gain for single-input single-output first-order systems. In the recent papers [13] and [14], a complete characterization of the induced gain for general switched linear systems was obtained, where it was proved that the existence of a *common* storage function that is a solution to a set of Hamilton-Jacobi inequalities, one equation for each system being switched, is *equivalent* to the induced gain of a switched system to lie below a prescribed value.

This paper further pursues non-conservative induced gain estimates for switched systems. We exploit switching signal classes, which distinct regularity assumptions placed on and were used in the stability analysis of switched systems [5], and investigate the induced gains obtained for these switching signal classes. The paper utilizes non-conservative induced gain estimate conditions obtained in [13], [14] and show that the induced gain that is obtained for the class of every piecewise constant switching signal remains the *same* for the more *restricted switching signal classes.*

The remainder of the paper is organized as follows: Section II describes the problem that is considered here, and required preliminary results are summarized. Section III contains main results. Section III-A describes switching signal classes which distinct regularity properties placed on. Section III-B shows that the induced gain that is obtained for the class of every piecewise constant switching signal can also be attained by the more restricted switching signal classes. Section IV contains some concluding remarks.

II. Problem Descriptions

This section formulates the problem that is considered here. Some regularity properties of switching signal classes which are needed in this paper are described in Section II-C. Required preliminary results which were obtained in [13] and [14] are summarized in Section II-D.

A. Switched Systems

The switched systems under consideration are represented by equations of the form

$$\dot{x} = A_s x + B_s u \qquad y = C_s x + D_s u \tag{1a}$$

where $s : [0, \infty) \to \mathcal{P}$ denotes a piecewise constant switching signal that selects appropriate quadruples

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(A_p, B_p, C_p, D_p) from a parametrized family

$$\{(A_p, B_p, C_p, D_p)|\ p \in \mathcal{P}\} \tag{1b}$$

of n-dimensional, m-input, k-output state space realizations, where $\mathcal{P}$ denotes an index set. Throughout the paper, the set of matrices in (1b) is assumed to be compact.

The set of all piecewise constant switching signals is denoted by

$$\mathcal{S} = \{s:\ [\,0, \infty\,) \to \mathcal{P}\}$$

and, by a piecewise constant signal, we mean a signal that exhibits a finite number of discontinuities on every bounded time interval and that is constant between consecutive discontinuities. The time instances at which s is discontinuous are called switching times. By convention, each piecewise constant signal s is assumed to be continuous from above, i.e., for any $t \geq 0$, the limit from above of $s(\tau)$ as $\tau \downarrow t$ is equal to $s(t)$.

A function $x:\ [0, \infty) \to \mathbb{R}^n$ is said to be a solution to (1) if it is continuous and piecewise continuously differentiable and there exists a switching signal $s \in \mathcal{S}$ and an input signal $u \in \mathcal{L}_2^m$ such that the time-varying differential equation $\dot{x}(t) = A_{s(t)}x(t) + B_{s(t)}u(t)$ holds for almost everywhere on $t \geq 0$. Here, we denote by $\mathcal{L}_2^m$ the set of square integrable functions with values on $\mathbb{R}^m$ defined on $[\,0, \infty\,)$.

B. Stability and Induced Gains

Our main interest is to determine the induced gain of a switched system, but to proceed we need to introduce appropriate stability definitions. We recall that a function $\alpha:\ [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}$, and we write $\alpha \in \mathcal{K}$, if α is continuous, strictly increasing and $\alpha(0) = 0$. A function $\beta:\ [\,0, \infty\,) \times [\,0, \infty\,) \to [\,0, \infty\,)$ is said to be of class $\mathcal{KL}$, and we write $\beta \in \mathcal{KL}$, if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

Definition 1: Let $\mathcal{S}' \subset \mathcal{S}$. The system (1) is said to be *uniformly asymptotically stable over* $\mathcal{S}'$ if there exists a function β of class $\mathcal{KL}$ such that $\|x(t)\| \leq \beta(\|x(0)\|, t)$ for all $t \geq 0$, all $s \in \mathcal{S}'$ and all $x(0) \in \mathbb{R}^n$, where $x(t)$ denotes the solution to (1) obtained for the switching signal s, zero input signal u, and initial condition $x(0)$. □

Definition 2: Let $\mathcal{S}' \subset \mathcal{S}$. The system (1) is said to be *(uniformly) exponentially stable over* $\mathcal{S}'$ if there exist constants $a > 0$ and $\lambda > 0$ such that the function β in Definition 1 can be chosen of the form $\beta(r, t) = ae^{-\lambda t}r$. □

We now state the following definitions of induced gains.

Definition 3: Let $\mathcal{S}' \subset \mathcal{S}$ and $\gamma > 0$. The system (1) is said to have an *$\mathcal{L}_2$-induced gain smaller than or equal to γ uniformly over $\mathcal{S}'$* if

$$\int_0^t \|y(\tau)\|^2 d\tau \leq \gamma^2 \int_0^t \|u(\tau)\|^2 d\tau \tag{2}$$

for all $t \geq 0$, all $s \in \mathcal{S}'$ and all $u \in \mathcal{L}_2^m$, where $y(\tau)$ denotes the output of (1) obtained for the switching signal s, input signal u, and zero initial condition. □

Definition 4: Let $\mathcal{S}' \subset \mathcal{S}$ and $\gamma > 0$. The system (1) is said to have an *$\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}'$* if the system has an $\mathcal{L}_2$-induced gain smaller than or equal to some $\gamma' < \gamma$ uniformly over $\mathcal{S}'$. □

For a given $\mathcal{S}' \subset \mathcal{S}$, we denote by $\gamma_{\mathcal{S}'}$ the *exact* $\mathcal{L}_2$-induced gain of a switched system:

$$\gamma_{\mathcal{S}'} = \inf\{\gamma > 0|\ (2) \text{ holds for all } t \geq 0, \text{ all } s \in \mathcal{S}' \text{ and all } u \in \mathcal{L}_2^m\}$$

C. Regular Switching Signal Class

We next discuss a few important regularity properties of switching signal classes which will be needed to describe induced gains of switched systems. For a given subset $\mathcal{S}'$ of $\mathcal{S}$ and for constants $b > a \geq 0$, possibly with $b = \infty$, we denote by $\mathcal{S}'[\,a, b\,)$ the set of restrictions of all switching signals $s \in \mathcal{S}'$ to the interval $[\,a, b\,)$.

Let $\mathcal{S}' \subset \mathcal{S}$ and $t > 0$. Using the set of restricted switching signals $\mathcal{S}'[t, \infty)$, we can define a new set of switching signals on the whole time interval by

$$\Sigma_t\mathcal{S}'[\,t, \infty\,) = \{\sigma|\ \sigma(\rho) = \mu(\rho + t) \quad \mu \in \mathcal{S}'[\,t, \infty\,), \rho \geq 0\}$$

The switching signal class $\mathcal{S}'$ is said to be *shift-invariant* if $\mathcal{S}' = \Sigma_t\mathcal{S}'[\,t, \infty\,)$ for all $t > 0$.

Let $\mathcal{S}' \subset \mathcal{S}$ and $t > 0$. Suppose that $\sigma \in \mathcal{S}'[\,0, t\,)$ and $\mu \in \mathcal{S}'[\,t, \infty\,)$. We define a switching signal $\sigma \oplus \mu$ by:

$$(\sigma \oplus \mu)(\tau) = \begin{cases} \sigma(\tau) & \tau \in [\,0, t\,) \\ \mu(\tau) & \tau \in [\,t, \infty\,) \end{cases}$$

and also define:

$$\mathcal{S}'[0, t) \bigoplus \mathcal{S}'[t, \infty) = \{\sigma \oplus \mu|\ \sigma \in \mathcal{S}'[0, t),\ \mu \in \mathcal{S}'[t, \infty)\}$$

The switching signal class $\mathcal{S}'$ is said to be *closed under concatenation* if $\mathcal{S}' = \mathcal{S}'[\,0, t\,) \bigoplus \mathcal{S}'[\,t, \infty\,)$ for all $t > 0$.

We now state the following definition.

Definition 5: Let $\mathcal{S}' \subset \mathcal{S}$. The switching signal class $\mathcal{S}'$ said to be *regular* if

1) For every $p \in \mathcal{P}$, the constant switching signal $s^p(t) = p(= \text{constant})$ belongs to $\mathcal{S}'$.
2) The switching signal class $\mathcal{S}'$ is shift-invariant.
3) The switching signal class $\mathcal{S}'$ is closed under concatenation. □

Remark 1: For regular classes of switching signals, there is a close connection between the uniform stability of the switched system over the given class of switching signals and the (usual) uniform stability of the time-varying system that is obtained when we pick a switching signal and use it in (1). To understand this, let $\mathcal{S}' \subset \mathcal{S}$ be regular. Suppose that a switched system (1) is uniformly asymptotically stable over $\mathcal{S}'$. Let us pick a switching signal s in $\mathcal{S}'$. The time-varying system $\dot{x}(\tau) = A_{s(\tau)}x(\tau)$, $\tau \geq 0$ is asymptotically stable, and we have $\|x(\tau)\| \leq \beta(\|x(0)\|, \tau)$ for all $\tau \geq 0$ and all $x(0)$. Let $t \geq 0$ be any constant and consider the switching signal $s(\tau + t)$, $\tau \geq 0$ which also belongs to $\mathcal{S}'$, since a regular switching signal class is shift-invariant. This concludes the asymptotic stability of the time-varying system $\dot{x}(\tau) = A_{s(\tau+t)}x(\tau)$, $\tau \geq 0$, and we have $\|x(\tau)\| \leq \beta(\|x(0)\|, \tau)$ for all $\tau \geq 0$ and all $x(0)$. Let us set $\ell = \tau + t$

and $y(\ell) = x(\ell - t)$, then we have $\|y(\ell)\| \le \beta(\|y(t)\|, \ell - t)$ for all $\ell \ge t$ and all $y(t) = x(0)$, where $y(\ell)$ represents the solution to the time-varying system $\dot{y}(\ell) = A_{s(\ell)}y(\ell)$, $\ell \ge t$ obtained for initial condition $y(t) = x(0)$. Since $t \ge 0$ is arbitrarily, this demonstrates that the uniform asymptotic stability over a regular switching signal class implies the uniform asymptotic stability of the time-varying system $\dot{y}(\ell) = A_{s(\ell)}y(\ell)$, $\ell \ge t \ge 0$ for every switching signal s in $\mathcal{S}'$. □

Necessary and sufficient conditions for the induced gain of a switched system with a regular switching signal class to lie below a prescribed constant $\gamma > 0$ were obtained in [13] and [14], which can be summarized as in the next section.

D. Preliminary Results

The following result was obtained in [13] and provides necessary and sufficient conditions for a switched system to admit an induced gain *smaller than or equal to* a given constant $\gamma > 0$ uniformly over a regular switching signal class $\mathcal{S}' \subset \mathcal{S}$.

Proposition 1: Let $\mathcal{S}' \subset \mathcal{S}$ be regular and $\gamma > 0$. Suppose that the system (1) is uniformly asymptotically stable over $\mathcal{S}'$. The following statements are equivalent:

1) The system (1) has an $\mathcal{L}_2$-induced gain smaller than or equal to γ uniformly over $\mathcal{S}'$.
2) The positive constant γ satisfies $\gamma^2 I - D_p^{\mathrm{T}} D_p > 0$ for all $p \in \mathcal{P}$, and there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is locally Lipschitz, and zero at zero, which satisfies

$$\begin{aligned}&\frac{\partial V}{\partial x}(x) A_p x + x^{\mathrm{T}} C_p^{\mathrm{T}} C_p x \\ &+ \left(\frac{1}{2} B_p^{\mathrm{T}} \frac{\partial V^{\mathrm{T}}}{\partial x}(x) + D_p^{\mathrm{T}} C_p x\right)^{\mathrm{T}} \left(\gamma^2 I - D_p^{\mathrm{T}} D_p\right)^{-1} \\ &\quad \times \left(\frac{1}{2} B_p^{\mathrm{T}} \frac{\partial V^{\mathrm{T}}}{\partial x}(x) + D_p^{\mathrm{T}} C_p x\right) \le 0 \qquad (3)\end{aligned}$$

for all $p \in \mathcal{P}$ and almost everywhere in x.
3) The positive constant γ satisfies $\gamma^2 I - D_p^{\mathrm{T}} D_p > 0$ for all $p \in \mathcal{P}$, and there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is convex, zero at zero, and homogeneous of degree two, which satisfies (3) for all $p \in \mathcal{P}$ and almost everywhere in x.
4) The positive constant γ satisfies $\gamma^2 I - D_p^{\mathrm{T}} D_p > 0$ for all $p \in \mathcal{P}$, and there exists a compact set $K \subset \mathbb{R}^{n\times n}$ of positive semi-definite matrices such that the piecewise quadratic function $V(x) = \max_{Q\in K} x^{\mathrm{T}} Q x$ satisfies (3) for all $p \in \mathcal{P}$ and almost everywhere in x. □

We recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous of degree q if $f(kx) = k^q f(x)$ for all $x \in \mathbb{R}^n$ and all $k \in \mathbb{R}$. Each function V in the statements 2), 3) and 4) is differentiable almost everywhere in x, but not necessarily everywhere. Therefore, the quantification over x in (3) should be interpreted as for every $x \in \mathbb{R}^n$ except for the zero-measure set of points at which $x \mapsto V(x)$ is not differentiable.

Each statement 2), 3) and 4) refers to the existence of a *common* solution V that satisfies the inequalities in (3), one for every p in the index set $\mathcal{P}$. The function V can be regarded as a *common storage function* for all the systems being switched. It is probably not surprising that the existence of a common storage function suffices to guarantee that the induced gain is smaller than or equals to γ. It is perhaps more unexpected that this is actually a necessary condition.

Proposition 1 assumes that a switched system is uniformly asymptotically stable over a regular switching signal class $\mathcal{S}' \subset \mathcal{S}$. This is needed because the inequalities in (3) do not necessarily imply the stability of a switched system. The following result was obtained in [14] and provides necessary and sufficient conditions for a switched system to be (uniformly) *exponentially stable* and admit an induced gain *strictly smaller* than a given constant $\gamma > 0$ uniformly over a regular switching signal class $\mathcal{S}' \subset \mathcal{S}$. Another important feature appears in the following statement 4), which shows that the storage function admits a *finite parametrization*.

Proposition 2: Let $\mathcal{S}' \subset \mathcal{S}$ be regular and $\gamma > 0$. Suppose that there exists at least one system index $q \in \mathcal{P}$ such that (C_q, A_q) is an observable pair. The following statements are equivalent:

1) The system (1) is (uniformly) exponentially stable over $\mathcal{S}'$ and has an $\mathcal{L}_2$-induced gain strictly smaller than γ uniformly over $\mathcal{S}'$.
2) The positive constant γ satisfies $\gamma^2 I - D_p^{\mathrm{T}} D_p > 0$ for all $p \in \mathcal{P}$, and there exists a constant $\epsilon > 0$ and a function $V : \mathbb{R}^n \to \mathbb{R}$ that is strictly convex, zero at zero, and homogeneous of degree two, which satisfies

$$\begin{aligned}&\frac{\partial V}{\partial x}(x) A_p x + x^{\mathrm{T}} C_p^{\mathrm{T}} C_p x \\ &+ \left(\frac{1}{2} B_p^{\mathrm{T}} \frac{\partial V^{\mathrm{T}}}{\partial x}(x) + D_p^{\mathrm{T}} C_p x\right)^{\mathrm{T}} \left(\gamma^2 I - D_p^{\mathrm{T}} D_p\right)^{-1} \\ &\quad \times \left(\frac{1}{2} B_p^{\mathrm{T}} \frac{\partial V^{\mathrm{T}}}{\partial x}(x) + D_p^{\mathrm{T}} C_p x\right) \le -\epsilon \|x\|^2 \qquad (4)\end{aligned}$$

for all $p \in \mathcal{P}$ and almost everywhere in x.
3) The positive constant γ satisfies $\gamma^2 I - D_p^{\mathrm{T}} D_p > 0$ for all $p \in \mathcal{P}$, and there exists a constant $\epsilon > 0$ and a compact set $K \subset \mathbb{R}^{n\times n}$ of positive definite matrices such that the piecewise quadratic function $V(x) = \max_{Q\in K} x^{\mathrm{T}} Q x$ satisfies (4) for all $p \in \mathcal{P}$ and almost everywhere in x.
4) The positive constant γ satisfies $\gamma^2 I - D_p^{\mathrm{T}} D_p > 0$ for all $p \in \mathcal{P}$, and there exists a constant $\epsilon > 0$, a constant integer M satisfying $M \ge n$ and a set of constant vectors $\ell_i \in \mathbb{R}^n$, $i = 1 \ldots, M$ such that the matrix $L \equiv [\ \ell_1 \ \cdots \ \ell_M\ ]^{\mathrm{T}} \in \mathbb{R}^{M\times n}$ has full rank, $\mathrm{rank} L = n$, and the piecewise quadratic function $V(x) = \max_{i=1,\ldots,M} (\ell_i^{\mathrm{T}} x)^2$ satisfies (4) for all $p \in \mathcal{P}$ and almost everywhere in x. □

From (4), we have

$$\frac{\partial V}{\partial x}(x) A_p x \le -\epsilon \|x\|^2 \qquad \text{for all } p \in \mathcal{P}$$

almost everywhere in x, and therefore the function V could be a candidate *common* Lyapunov function, provided that it is positive definite. However, this is not always the case. The conditions in the statement 2), strict convexity, $V(0)=0$ and homogeneity with degree two, imply that the function V is positive definite, $V(x)>0$, $x\neq 0$[1]. On the other hand, a finite gain property in the statement 1) and the existence of at least one observable pair imply that the storage function becomes strictly positive.

III. Induced Gains and Switching Signal Classes

This section reviews a few classes of switching signals that will be used, and explores the induced gains of the switched systems with these switching signal classes.

A. Switching Signal Classes

The switching signal classes which will be exploited in this paper were used in the stability analysis of switched systems [4], [5]. The set of all piecewise constant switching signals is denoted by $\mathcal{S}=\{s:\ [\,0,\infty\,)\to\mathcal{P}\}$, where $\mathcal{P}$ denotes an index set, and, by a piecewise constant signal, we mean a signal that exhibits a finite number of discontinuities on each bounded time interval and that is constant between consecutive discontinuities. By convention, we take piecewise constant switching signals to be continuous from above. All classes of switching signals considered in this paper are subsets of the class $\mathcal{S}$.

For a given $s\in\mathcal{S}$, we denote by N_s and t_s^k, $k=0,1,\dots,N_s$ the number of discontinuities and consecutive switching times of s, respectively, where $t_s^0=0$ and possibly $N_s=\infty$. Let $t\geq\tau\geq 0$ be given two constants. We denote by $N_s(t,\tau)$ the number of discontinuities of s in the interval (τ,t). We have $N_s(t,\tau)<\infty$ for all $s\in\mathcal{S}$, and all $t\geq\tau\geq 0$.

We consider particular subsets of the class $\mathcal{S}$ of every piecewise constant switching signal: The set $\mathcal{S}_{\text{dwell}}[\tau_D]$, $\tau_D>0$ consists of switching signals for which any consecutive discontinuities of a switching signal are separated by no less than a *dwell-time* τ_D:

$$\mathcal{S}_{\text{dwell}}[\,\tau_D\,]=\{s\in\mathcal{S}|\ t_s^{k+1}-t_s^k\geq\tau_D \quad \text{for all } k=0,1,\dots,N_s-1\};$$

The set $\mathcal{S}_{\text{average}}[\,\tau_D,N_0\,]$, $\tau_D>0$, $N_0>0$ consists of switching signals for which the number of discontinuities in any open interval is bounded by the length of the interval normalized by an *average dwell-time* τ_D plus a *chatter bound* $N_0>0$:

$$\mathcal{S}_{\text{average}}[\,\tau_D,N_0\,]=\{s\in\mathcal{S}|\ N_s(t,\tau)\leq N_0+\frac{t-\tau}{\tau_D} \quad \text{for all } t\geq\tau\geq 0\};$$

The set $\mathcal{S}_{\text{p-dwell}}[\,\tau_D,T\,]$, $\tau_D>0$, $T\in[\,0,\infty\,]$ consists of switching signals for which there exists a sequence of $0=\tau_0<t_0\leq\tau_1<t_1\leq\tau_2<t_2\leq\cdots$ such that the length of each interval is no smaller than a *persistent dwell-time* τ_D, $t_k-\tau_k\geq\tau_D$, on which a switching signal s takes constant value, $s(t)=p_s^k$ for some $p_s^k\in\mathcal{P}$ and all $t\in[\,\tau_k,t_k\,)$, and consecutive intervals with this property are separated by no more than a *period of persistence* T, $\tau_{k+1}-t_k\leq T$:

$$\begin{aligned}\mathcal{S}_{\text{p-dwell}}[\,\tau_D,T\,]=\{s\in\mathcal{S}|\ &\text{there exist a sequence of}\\ &0=\tau_0<t_0\leq\tau_1<t_1\leq\tau_2<t_2\leq\cdots\\ &\text{such that } t_k-\tau_k\geq\tau_D,\ \tau_{k+1}-t_k\leq T,\\ &\text{and for all } k=0,1,2,\dots \text{ and all } t\in[\,\tau_k,t_k\,)\\ &s(t)=p_s^k \text{ for some } p_s^k\in\mathcal{P}\}\end{aligned}$$

The following classes are limiting cases of the ones described above. Although they lack "uniformly", they still exhibit sufficient regularity for our purpose: The set $\mathcal{S}_{\text{finite}}$, where each switching signal is restricted to have a finite number of discontinuities:

$$\begin{aligned}\mathcal{S}_{\text{finite}}&=\bigcup_{N_0>0}\mathcal{S}_{\text{average}}[\,\infty,N_0\,]\\ &=\{s\in\mathcal{S}|\ s\in\mathcal{S}_{\text{average}}[\,\infty,N_{0,s}\,]\quad\text{for some } N_{0,s}>0\}\\ &=\{s\in\mathcal{S}|\ \text{there exists a } N_{0,s}>0 \text{ such that}\\ &\qquad N_s(t,\tau)\leq N_{0,s} \text{ for all } t\geq\tau\geq 0\};\end{aligned}$$

The set $\mathcal{S}_{\text{dwell}}$, where each switching signal is restricted to have a positive dwell-time, but there is no common dwell-time for all switching signals:

$$\begin{aligned}\mathcal{S}_{\text{dwell}}&=\bigcup_{\tau_D>0}\mathcal{S}_{\text{dwell}}[\,\tau_D\,]\\ &=\{s\in\mathcal{S}|\ s\in\mathcal{S}[\,\tau_{D,s}\,]\quad\text{for some } \tau_{D,s}>0\};\end{aligned}$$

The set $\mathcal{S}_{\text{average}}$, where each switching signal is restricted to have a positive average dwell-time and a finite chatter bound, but there is no common average dwell-time or chatter bound for all switching signals:

$$\begin{aligned}\mathcal{S}_{\text{average}}&=\bigcup_{\substack{\tau_D>0\\ N_0>0}}\mathcal{S}_{\text{average}}[\,\tau_D,N_0\,]\\ &=\{s\in\mathcal{S}|\ s\in\mathcal{S}_{\text{average}}[\,\tau_{D,s},N_{0,s}\,]\\ &\qquad\text{for some } \tau_{D,s}>0 \text{ and } T_{D,s}>0\};\end{aligned}$$

The set $S_{\text{p-dwell}}$, where each switching signal is restricted to have a positive persistent dwell-time and a finite period of persistence, but these is no common persistent dwell-time or period of persistence for all switching signals:

$$\begin{aligned}S_{\text{p-dwell}}&=\bigcup_{\substack{\tau_D>0\\ T<\infty}}\mathcal{S}_{\text{p-dwell}}[\,\tau_D,T\,]\\ &=\{s\in\mathcal{S}|\ s\in\mathcal{S}_{\text{p-dwell}}[\,\tau_{D,s},T_s\,]\\ &\qquad\text{for some } \tau_{D,s}>0 \text{ and } T_s<\infty\};\end{aligned}$$

The set $\mathcal{S}_{\text{week-dwell}}$, where each switching signal is restricted to have a positive persistent dwell-time, but can have infinite period of persistence:

$$\begin{aligned}\mathcal{S}_{\text{week-dwell}}&=\bigcup_{\tau_D>0}\mathcal{S}_{\text{p-dwell}}[\,\tau_D,+\infty\,]\\ &=\{s\in\mathcal{S}|\ s\in\mathcal{S}_{\text{p-dwell}}[\,\tau_{D,s},+\infty\,]\\ &\qquad\text{for some } \tau_{D,s}>0\}\end{aligned}$$

[1]Let $\theta\in(0,1)$ and $x\neq 0$. We have $\theta^2V(x)=V(\theta x)=V(\theta x+(1-\theta)0)<\theta V(x)+(1-\theta)V(0)=\theta V(x)$, thus $\theta(1-\theta)V(x)>0$. Since $\theta\in(0,1)$, this implies that $V(x)>0$.

The relation

$$\begin{aligned}\mathcal{S}_{\text{dwell}}[\,\tau_D\,] &= \mathcal{S}_{\text{average}}[\,\tau_D, 1\,]\\ &= \mathcal{S}_{\text{p-dwell}}[\,\tau_D, 0\,] \subset \mathcal{S}_{\text{average}}[\,\tau_D, N_0\,]\\ &\subset \mathcal{S}_{\text{p-dwell}}[\,\delta\tau_D, \delta\tau_D(N_0-\delta)/(1-\delta)\,]\end{aligned}$$

can be checked for all $\tau_D > 0$, all $N_0 \geq 1$ and all $\delta \in (\,0, 1\,)$ [5]. In addition, because of the above relation, one can conclude a strict inclusion relation

$$\mathcal{S}_{\text{finite}} \subset \mathcal{S}_{\text{dwell}} \subset \mathcal{S}_{\text{average}} \subset \mathcal{S}_{\text{p-dwell}} \subset \mathcal{S}_{\text{week-dwell}} \subset \mathcal{S} \quad (5)$$

for non-uniform switching signal classes.

B. $\mathcal{L}_2$-induced Gains for the Classes of Switching Signals

This section provides two corollaries which describe the $\mathcal{L}_2$-induced gains of switched systems effected by the switching signal classes in (5).

We start with recalling Proposition 1. The statements 2), 3) and 4) of this proposition do not depend on the specific class of switching signals $\mathcal{S}'$. We then conclude that every regular switching signal class exhibits the same induced gain. Since the class $\mathcal{S}$ of every piecewise constant switching signal is regular, Proposition 1 essentially shows that for every regular switching signal class $\mathcal{S}' \subset \mathcal{S}$, we have $\gamma_{\mathcal{S}} = \gamma_{\mathcal{S}'}$.

We now consider the switching signal classes in (5). Let us suppose now that a switched system (1) is uniformly asymptotically stable over $\mathcal{S}$. Consequently, it is also uniformly asymptotically stable over every switching signal class in (5). Then, one concludes that

$$\gamma_{\mathcal{S}_{\text{finite}}} \leq \gamma_{\mathcal{S}_{\text{dwell}}} \leq \gamma_{\mathcal{S}_{\text{average}}} \leq \gamma_{\mathcal{S}_{\text{p-dwell}}} \leq \gamma_{\mathcal{S}_{\text{weak-dwell}}} \leq \gamma_{\mathcal{S}} \quad (6)$$

We shall see in the following corollary that (6) essentially holds with equalities.

Corollary 1: Let $\mathcal{S}'$ be any one of the switching signal classes in (5). Suppose that the system (1) is uniformly asymptotically stable over the class $\mathcal{S}$ of every piecewise constant switching signal. Then, we have $\gamma_{\mathcal{S}'} = \gamma_{\mathcal{S}_{\text{finite}}}$. □

Proof: We notice that the class $\mathcal{S}$ of every piecewise constant switching signal and the class $\mathcal{S}_{\text{finite}}$ of all switching signals having a finite number of discontinuities are regular, while the other classes that appear in (5) are not. The switched system is uniformly asymptotically stable over $\mathcal{S}'$, consequently it is also uniformly asymptotically stable over $\mathcal{S}'$ and $\mathcal{S}_{\text{finite}}$. Then, we have $\gamma_{\mathcal{S}_{\text{finite}}} = \gamma_{\mathcal{S}}$. One concludes, from (6), that $\gamma_{\mathcal{S}_{\text{finite}}} = \gamma_{\mathcal{S}'} = \gamma_{\mathcal{S}}$. ■

We now recall Proposition 2. The statements 2), 3) and 4) of the proposition do not depend on the specific class of switching signals $\mathcal{S}'$. We then conclude that if a switched system which having at least one observable pair was (uniformly) exponentially stable over a specific regular switching signal class, then it is also stable over every other regular switching signal class, and every regular switching signal class exhibits the same induced gain $\gamma_{\mathcal{S}}$.

Let us consider the switching signal classes in (5), a result that follows from Proposition 2 and corresponds to a version of Corollary 1 can be summarized as follows:

Corollary 2: Let $\mathcal{S}'$ be any one of the switching signal classes in (5). Suppose that there exists at least one system index $q \in \mathcal{P}$ such that (C_q, A_q) is an observable pair, and the system (1) is (uniformly) exponentially stable over $\mathcal{S}_{\text{finite}}$. Then, the system (1) is (uniformly) exponentially stable over $\mathcal{S}'$ and $\gamma_{\mathcal{S}'} = \gamma_{\mathcal{S}_{\text{finite}}}$. □

Proof: The switched system is (uniformly) exponentially stable over $\mathcal{S}_{\text{finite}}$, and both of the switching signal classes $\mathcal{S}$ and $\mathcal{S}_{\text{finite}}$ are regular, it is consequently stable over the class $\mathcal{S}$. Since $\mathcal{S}' \subset \mathcal{S}$, one concludes that the switched system is also (uniformly) exponentially stable over the class $\mathcal{S}'$. Since every regular switching signal class exhibits the same induced gain and from (6), one concludes that $\gamma_{\text{finite}} = \gamma_{\mathcal{S}'} = \gamma_{\mathcal{S}}$. ■

IV. Conclusions

This paper investigates the induced gains of switched linear systems effected by the switching signal classes which has been used in the stability analysis of the switched systems. We exploit non-conservative necessary and sufficient conditions for the induced gain to lie below a prescribed positive constant and the inclusion relations of the switching signal classes, and show that the induced gain that is obtained for the class $\mathcal{S}$ of all piecewise constant switching signals can also be attained by the more restricted class $\mathcal{S}_{\text{finite}}$ of all the switching signals having a finite number of discontinuities.

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