Supervision of Families of Nonlinear Controllers*

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Abstract

A high-level controller called a "supervisor," is defined for the purpose of orchestrating the switching of a sequence of candidate controllers into feedback with an imprecisely modeled process so as to stabilize it. The controller selection is made by (i) continuously comparing in real time suitably defined "normed" output estimation errors or "performance signals" and (ii) placing in the feedback-loop, from time to time, that candidate controller whose corresponding performance signal is the smallest. The paradigm is a manifestation of the idea of certainty equivalence from parameter adaptive control.

1 Problem Formulation

The process Σ_P is modeled by a n-dimensional nonlinear dynamical system

$$\dot{x} = A_\omega(x,u), \qquad\qquad y = C_\omega(x) \tag{1}$$

where ω is an uncertain constant parameter taking values in a finite set $\mathcal{P} = \{p_1, \dots, p_m\}$, $u \in \mathbb{R}^k$ is the control input, and $y \in \mathbb{R}^n$ the measurable output. The family of locally Lipschitz functions $\{A_p : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \mid p \in \mathcal{P}\}$ and $\{C_p : \mathbb{R}^n \to \mathbb{R}^n \mid p \in \mathcal{P}\}$ are given and it is assumed that for all $p \in \mathcal{P}$, $C_p(x)$ is invertible[1], i.e.

Assumption 1. There exists a family of functions $\{\chi_p : \mathbb{R}^n \to \mathbb{R}^n \mid p \in \mathcal{P}\}$ such that for every $p \in \mathcal{P}$

$$x = \chi_p\big(C_p(x)\big), \qquad y = C_p\big(\chi_p(y)\big), \qquad \forall x, y \in \mathbb{R}^n$$

We will try, to the extent that is possible, to employ "off-the-shelf" controllers that would stabilize the system (1) if ω was known. Accordingly, we will take as given a family of static state feedback control laws $\{F_p : \mathbb{R}^n \to \mathbb{R}^k \mid p \in \mathcal{P}\}$ which at least stabilize the differential equation $\dot{z} = A_p\big(z, F_p(z)\big)$. In fact we demand more:

Assumption 2. There exist constants $\lambda, \mu > 0$, and two families of nonnegative functions $\{V_p : \mathbb{R}^n \to [0, +\infty) \mid p \in \mathcal{P}\}$, $\{W_p : \mathbb{R}^n \to [0, +\infty) \mid p \in \mathcal{P}\}$ such that for all $p \in \mathcal{P}$, $V_p(\cdot)$ is continuously differentiable, positive definite, and radially unbounded, $W_p(\cdot)$ is Lipschitz continuous, $W_p(0) = 0$, and

$$\frac{\partial V_p}{\partial x}(x)\Big(A_p\big(x-d, F_p(x-d)\big) - \lambda\, d\Big) \le -2\mu V_p(x) + W_p(d)\,, \quad d, x \in \mathbb{R}^n \tag{2}$$

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[1] This requirement does not tantamount to assuming that x can be measured since ω is not presumed to be known.

This requirement not only stipulates that $F_p(\cdot)$ exponentially[2] stabilizes the unperturbed system $\dot{z} = A_p\big(z, F_p(z)\big)$, but also input-to-state stabilizes [1] the perturbed system

$$\dot{z} = A_p\big(z - d, F_p(z - d)\big) - \lambda\, d, \qquad\qquad p \in \mathcal{P}, \tag{3}$$

Note that the control $u \triangleq F_\omega\big(\chi_\omega(y)\big)$ asymptotically stabilizes (1). However, implementation of $u \triangleq F_\omega\big(\chi_\omega(y)\big)$ is not possible, because the value of ω is not presumed to be know. What's to be used instead is the "multi-controller" $\Sigma_C(\sigma)$ defined by the feedback law

$$u \triangleq F_\sigma\big(\chi_\sigma(y)\big) \tag{4}$$

where σ is a suitably defined piecewise-constant *switching signal* taking values in $\mathcal{P}$. The system to be supervised thus consists of the feedback interconnection of $\Sigma_C(\sigma)$ and the process Σ_P modeled by (1). The problem of interest is to develop a provably correct, "high-level" controller called a *supervisor* which, without a-priori knowledge of ω, is capable of choosing σ so as to cause x to tend to zero.

Remark. It will be seen below that the signal d in equation (3) corresponds to a state estimation error that can be measured. This measurement can be used by the control law to ensure that Assumption 2 holds. In this case the family of static feedback laws is of the form $\{F_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k \mid p \in \mathcal{P}\}$ and equation (2) is replaced by

$$\frac{\partial V_p}{\partial x}(x)\,\Big(A_p\big(x - d, F_p(x - d, d)\big) - \lambda\, d\Big) \le -2\mu V_p(x) + W_p(d) \tag{5}$$

The corresponding multi-controller $\Sigma_C(\sigma)$ is defined by the feedback law

$$u \triangleq F_\sigma\big(\chi_\sigma(y), e_\sigma\big) \tag{6}$$

where the family of signals $\{e_p : p \in \mathcal{P}\}$ are state estimation errors to be defined later. For sake of generality, in the sequel, this more general form for the control law will be used.

Remark. The fact the for all $p \in \mathcal{P}$, $V_p(x)$ is continuous, positive definite and radially unbounded with respect to x guarantees {c.f. Lemma 3 in the Appendix} that there exists a family of functions $\{\Gamma_p^V : [0, +\infty) \to [0, +\infty) \mid p \in \mathcal{P}\}$ such that $\Gamma_p^V(\xi) > 0$ iff $\xi > 0$ and

$$\Gamma_p^V\big(V_p(x)\big) \ge \|x\|^2, \qquad \forall x \in \mathbb{R}^n \tag{7}$$

Before pursuing we need the following growth rate assumption

Assumption 3. The functions $A_p(\cdot)$, $C_p(\cdot)$, $\chi_p(\cdot)$, $F_p(\cdot)$, $\Gamma_p^V(\cdot)$ $(p \in \mathcal{P})$ have polynomial growth rate[3]

Remark. Assumption 3 could be made less stringent by requiring $\psi_{p\,q}(x) \triangleq \chi_p\big(C_q(x)\big)$ $(p, q \in \mathcal{P},\ x \in \mathbb{R}^n)$ to have a polynomial growth rate, instead of both $C_p(\cdot)$ and $\chi_p(\cdot)$. This allows, e.g. $C_p(x) \triangleq px^3$.

Feedback Linearizable Systems

Suppose that the family of functions $\{A_p : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \mid p \in \mathcal{P}\}$ is such that for every $p \in \mathcal{P}$ the system

$$\dot{x} = A_p(x, u) \tag{8}$$
$$y = C_p(x) \tag{9}$$

is globally linearizable by static state feedback, and that the linearized system is stabilizable. In particular, assume that there exists a family of static state feedback control law $\{F_p^l : \mathbb{R}^n \to \mathbb{R}^k \mid p \in \mathcal{P}\}$ and a family of smooth change of coordinates $\{\phi_p : \mathbb{R}^n \to \mathbb{R}^n \mid p \in \mathcal{P}\}$ such that:

[2] In order to conclude that the convergence is exponential one also needs the growth rate condition for $\Gamma_p^V(\cdot)$ in Assumption 3 that will be introduced shortly.

[3] A function $f : \mathbb{R}^n \to \mathbb{R}^n$ has *polynomial growth rate* if there exist constants $a_1, a_2 > 0, k > 1$ such that for all $x \in \mathbb{R}^n$, $\|f(x)\| \le a_1\|x\| + a_2\|x\|^k$.

Assumption 4. For all $p \in \mathcal{P}$ and $x \in \mathbb{R}^n$

$$\frac{\partial \phi_p}{\partial x}(x)\, A_p\big(x, F_p^l(x)\big) = A_p\, \phi_p(x) \tag{10}$$

where $\{A_p : p \in \mathcal{P}\}$ is a family of matrices such that $A_p + \mu I$ is a stability matrix for all $p \in \mathcal{P}$.

In the sequel it is assumed that the change of coordinates $\phi(\cdot)$ is the identity and therefore equation (10) simplifies to

$$A_p\big(x, F_p^l(x)\big) = A_p\, x \tag{11}$$

This introduces no loss of generality since one can always apply a parameter dependent change of coordinates to the model given by equations (8)-(9) such that $\phi_p(x) = x$ for all $p \in \mathcal{P}$. Under this assumption the following lemma, whose proof can be found in the Appendix, guarantees input-to-state stabilizability of (8)-(9).

Lemma 1. *If Assumption 4 holds and $\phi_p(x) = x$ ($p \in \mathcal{P}$, $x \in \mathbb{R}^n$) then Assumption 2 also holds for*

$$F_p(x, d) = F_p^l(x), \qquad x, d \in \mathbb{R}^n, p \in \mathcal{P} \tag{12}$$

$$V_p(x) = x'\, Q_p\, x, \tag{13}$$

$$W_p(d) = \|d\|^2, \tag{14}$$

where $\{Q_p : p \in \mathcal{P}\}$ is a family of symmetric positive definite matrices such that

$$(A_p + \mu\, I)' Q_p + Q_p (A_p + \mu\, I) + Q_p (A_p + \lambda I)(A_p + \lambda I)' Q_p \leq 0 \tag{15}$$

2 Estimator-based Supervisor

The supervisor proposed consists of three subsystems:

multi-estimator Σ_E– a dynamical system whose inputs are u and y and whose outputs are *estimation errors* of the form $e_p = x_p - \chi_p(y)$, $p \in \mathcal{P}$; here each x_p is a suitably defined estimate of x which would be asymptotically correct if p were equal to ω.

performance signal generator Σ_Π– a dynamical system whose inputs are the estimation errors and whose outputs $\{\pi_p : p \in \mathcal{P}\}$ are suitably "normed" values of the estimation errors called *performance signals*.

switching logic Σ_S– a hybrid dynamical system whose inputs are the performance signals and whose output is σ.

The underlying decision making strategy used by this supervisor is basically the following: from time to time, select for σ that candidate control index q whose corresponding performance signal π_q is the smallest among π_p, $p \in \mathcal{P}$. Motivation for this idea is as follows: the nominal process models whose associated performance signals are small, "best" approximate what the process is and thus the candidate controller designed on the basis of those models ought to be able to do a good job of controlling the process. The origin of this idea is the concept of certainty equivalence from parameter adaptive control.

2.1 Multi-Estimator

The multi-estimator Σ_E consists of a family of n-dimensional estimators of the form

$$\dot{x}_p = -\lambda\big(x_p - \chi_p(y)\big) + A_p\big(\chi_p(y), u\big), \qquad p \in \mathcal{P} \tag{16}$$

$$e_p = x_p - \chi_p(y) \tag{17}$$

Thus Σ_E is a dynamical system with state $x_E = \begin{bmatrix} x'_{p_1} & \cdots & x'_{p_m} \end{bmatrix}'$, and inputs u and y. Note that $\chi_\omega(y) = x$ and therefore, because of (1) and the defining equations (16)-(17),

$$\dot{e}_\omega = -\lambda\, e_\omega \tag{18}$$

Hence, for any input u, the estimation error associated with ω converges to zero exponentially fast.

Suppose q is the value of σ on some interval $[t_1, t_2)$. During this period, the closed-loop differential equations that result from application of the certainty equivalence control law (6) to (1) and (16) yields the closed-loop differential equations

$$\dot{x} = A_\omega\Big(x, F_q\big(\chi_q(y), x_q - \chi_q(y)\big)\Big) \tag{19}$$

$$\dot{x}_p = -\lambda\, x_p + \lambda\, \chi_p(y) + A_p\Big(\chi_p(y), F_q\big(\chi_q(y), x_q - \chi_q(y)\big)\Big), \quad p \in \mathcal{P} \tag{20}$$

$$y = C_\omega(x) \tag{21}$$

Meanwhile from (17) one can get the equation for the relevant output estimation error, namely

$$e_q = x_q - \chi_q(y) \tag{22}$$

Defining $\mathbf{x} \triangleq \begin{bmatrix} x' & x'_{p_1} & \cdots & x'_{p_m} \end{bmatrix}' \in \mathbb{R}^{(m+1)n}$, equations (19)-(22) can be written in the form

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}), \qquad\qquad e_q = \mathbf{C}(\mathbf{x}) \tag{23}$$

With detectability defined in the sense of [2] {see also [3]}, the following lemma states that the system defined by equations (19)-(21) is detectable through the estimation error e_q defined by (22). The proof of this lemma can be found on the Appendix.

Lemma 2. *Under assumptions 1, 2, and 3, there exists a positive constant γ and two nonnegative functions $\Gamma : [0, +\infty) \to [0, +\infty)$ and $\mathbf{W} : \mathbb{R}^{(m+1)n} \to [0, +\infty)$ such that $\mathbf{W}(\cdot)$ is Lipschitz continuous, $\mathbf{W}(0) = 0$, and defining $\mathbf{V}(\mathbf{x}) \triangleq \|\mathbf{x}\|^2 + \Gamma\big(V_q(x_q)\big)$,*

$$\frac{\partial \mathbf{V}}{\partial \mathbf{x}} \mathbf{A}(\mathbf{x}) \leq -\gamma \mathbf{V}(\mathbf{x}) + \mathbf{W}\big(\mathbf{C}(\mathbf{x})\big) \tag{24}$$

This consequence of the fact that (3) is input-to-state stable, is a manifestation of the principle that certainty equivalence implies detectability. These ideas can be traced back to [4]. It is worth noting that detectability of (19)-(22) does *not* depend on ω being an element of $\mathcal{P}$ or, for that matter, on the relation between u and y being given by the equations (1). In other words detectability has nothing to do with "exact matching."

Remark. Defining $\mathbf{x} \triangleq \begin{bmatrix} \chi_q(y) & x'_{p_1} & \cdots & x'_{p_m} \end{bmatrix}' \in \mathbb{R}^{(m+1)n}$, and using the control law (4), the first rows of $\mathbf{A}(\mathbf{x})$ could be written as

$$A_q\Big(\chi_q(y), F_q\big(\chi_q(y)\big)\Big) + \bar{d}$$

where $\bar{d} \triangleq -(\dot{e}_q + \lambda e)$. In this case one could replace inequality (2) by the weaker condition

$$\frac{\partial V_p}{\partial x}(x)\,\Big(A_p\big(x, F_p(x)\big) + \bar{d}\Big) \leq -2\mu V_p(x) + W_p(\bar{d})$$

and get detectability through $\bar{d} \triangleq -(\dot{e}_q + \lambda e)$. However, it is not clear[4] how the switching logic could guarantee that $\dot{e}_q$ is small in a integral sense.

[4] Check !!!

2.2 Performance Signals Generator

The performance signal generator Σ_Π to be considered consists of the following family of one-dimensional differential equations

$$\dot{\pi}_p = \mathbf{W}(e_p), \qquad \pi_p(0) = 0, \qquad p \in \mathcal{P} \tag{25}$$

The definition implies that for each $p \in \mathcal{P}$

$$\pi_p(t) = \int_0^t \mathbf{W}\left(e_p(\tau)\right) d\tau, \qquad t \geq 0 \tag{26}$$

which shows that the performance signal π_p is just an integral norm of the estimation error e_p. Under appropriate assumptions, state-sharing [2, 5] can be used to generate the performance signals more efficiently and extend this formulation to situations in which $\mathcal{P}$ has infinitely many elements.

2.3 Switching Logic

The switching logic Σ_S we shall consider was originally devised for switching between the members of a finite family of parameter adaptive controllers [6, 7]. To specify Σ_S it is necessary to first pick a positive number h called the *hysteresis constant*. The switching signal σ is defined recursively by the equation

$$\sigma = \begin{cases} \sigma^- & \text{if } \pi_{\sigma^-} < \pi_p + h, \ p \in \mathcal{P} \\ \rho & \text{otherwise} \end{cases}$$

where $\sigma^-(t)$ is the limit of $\sigma(\tau)$ from below as $\tau \to t$, and

$$\rho(t) = \min_{i \in \{1,\ldots,m\}} \{i : \pi_{p_i}(t) \leq \pi_p(t), \ p \in \mathcal{P}\}$$

The functioning of Σ_S is roughly as follows. Suppose that at some time t_i, Σ_S has just changed the value of σ to $q \in \mathcal{P}$. The switching signal σ is then held fixed at this value unless, and until, there is a time $t_{i+1} > t_i$ at which $\pi_p + h \leq \pi_q$ for some $p \in \mathcal{P}$. If this occurs, σ is set equal to p and so on.

3 Main Result

The following theorem, whose proof can be found on the Appendix, is a straightforward consequence of the Hysteresis Switching Lemma [7], and Lemma 2.

Theorem 1. *Under assumptions 1, 2, and 3, there exists a time T^* beyond which the value of σ becomes constant and $\mathbf{x} \to 0$ as $t \to +\infty$.*

4 Concluding Remarks

The use of supervisory control to feedback stabilize an imprecisely modeled process seems to have important advantages over conventional adaptive techniques: Unlike conventional adaptive control, supervisory control can deal with nonlinearly parameterized models. Supervisory control makes use of off-the-shelf {non-adaptive} controllers rather than highly specialized controllers tailored to the specific needs of continuously-tuned conventional adaptive algorithms.

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A Appendix

Lemma 3. *Given a continuous, positive definite, and radially unbounded function $V : \mathbb{R}^n \to [0, +\infty)$, there exists a function $\Gamma^V : [0, +\infty) \to [0, +\infty)$ such that $\Gamma^V(\xi) > 0$ iff $\xi > 0$ and*

$$\Gamma^V\big(V(x)\big) \geq \|x\|^2, \qquad \forall x \in \mathbb{R}^n$$

Proof of Lemma 3. Defining

$$\Gamma_p^V(\xi) \triangleq \sup_{y \in \mathbb{R}^n} \left\{ \|y\|^2 : V_p(y) \leq \xi \right\}, \quad \xi \in [0, +\infty)$$

for any given $x \in \mathbb{R}^n$,

$$\Gamma_p^V\big(V_p(x)\big) = \sup_{y \in \mathbb{R}^n} \left\{ \|y\|^2 : V_p(y) \leq V_p(x) \right\}$$

but x belongs to the set over which the supremum is being taken, therefore

$$\Gamma_p^V\big(V_p(x)\big) = \sup_{y \in \mathbb{R}^n} \left\{ \|y\|^2 : V_p(y) \leq V_p(x) \right\} \geq \|x\|^2$$

On the other hand, the continuity, positive definiteness, and radial unboundedness of $V(\cdot)$ guarantees that $\Gamma^V(\xi) \geq 0$ for all $\xi \in [0, +\infty)$, with equality just for $\xi = 0$. ∎

Before proceeding to the proof of Lemma 2, let us note that the vector-valued functions $\mathbf{A}(\cdot)$ and $\mathbf{C}(\cdot)$ in equation (23) are given by

$$\mathbf{C}(\mathbf{x}) \triangleq x_q - \psi_{q\,\omega}(x), \qquad \mathbf{A}(\mathbf{x}) \triangleq -\lambda\,\mathbf{x} + \Phi\big(\chi_q(y), \mathbf{C}(\mathbf{x})\big) \tag{27}$$

where $\psi_{p\,q}(x) \triangleq \chi_p\big(C_q(x)\big)$ $(p, q \in \mathcal{P},\ x \in \mathbb{R}^n)$, and

$$\Phi(x, d) \triangleq \begin{bmatrix} \lambda\,\psi_{\omega\,q}(x) + A_\omega\big(\psi_{\omega\,q}(x), F_q(x, d)\big) \\ \lambda\,\psi_{p_1\,q}(x) + A_{p_1}\big(\psi_{p_1\,q}(x), F_q(x, d)\big) \\ \vdots \\ \lambda\,\psi_{p_m\,q}(x) + A_{p_m}\big(\psi_{p_m\,q}(x), F_q(x, d)\big) \end{bmatrix} \qquad x, d \in \mathbb{R}^n$$

The following lemma is a straightforward (but tedious) consequence of Assumption 3.

Lemma 4. *There exists a function $\Gamma^\Phi : [0,+\infty) \to [0,+\infty)$ such that*

(i) $\Gamma^\Phi(0) = 0$,

(ii) $\Gamma^\Phi(.)$ *is monotonous nondecreasing,*

(iii) for all $x, d \in \mathbb{R}^{(m+1)n}$,

$$\|\Phi(x-d,d)\|^2 \leq \Gamma^\Phi\big(\|x\|^2\big) + \Gamma^\Phi\big(\|d\|^2\big) \tag{28}$$

(iv) for every $p \in \mathcal{P}$, $\Gamma^\Phi\big(\Gamma_p^V(\cdot)\big)$ *has polynomial growth rate.*

Proof of Lemma 4. By virtue of Assumption 3, there exists constants $a_1, a_2 > 0$, $k > 1$ such that

$$\begin{aligned}
\|A_p(x_1,x_2)\| &\leq a_1\big(\|x_1\| + \|x_2\|\big) + a_2\big(\|x_1\|^k + \|x_2\|^k\big)\\
\|C_p(x_1)\| &\leq a_1\|x_1\| + a_2\|x_1\|^k\\
\|\chi_p(x_1)\| &\leq a_1\|x_1\| + a_2\|x_1\|^k\\
\|F_p(x_1,x_2)\| &\leq a_1\big(\|x_1\| + \|x_2\|\big) + a_2\big(\|x_1\|^k + \|x_2\|^k\big)\\
\Gamma_p^V(\xi) &\leq a_1\,\xi + a_2\,\xi^k
\end{aligned}$$

Therefore, for all $p, q \in \mathcal{P}$, $x, d \in \mathbb{R}^n$, and $\xi \in [0,+\infty)$, [5]

$$\begin{aligned}
\|\psi_{p\,q}(x)\| &\leq a_1\|C_q(x)\| + a_2\|C_q(x)\|^k\\
&\leq a_1^2\|x\| + \big(a_1a_2 + 2^k a_1^k a_2\big)\,\|x\|^k + 2^k a_2^{k+1}\|x\|^{k^2}\\
&\leq a_1^2\|x\| + \big(a_1a_2 + 2^k a_1^k a_2\big)\,\|x\|^k + 2^k a_2^{k+1}\|x\|^{k^2}\\
&\leq (a_1^2 + a_1a_2 + 2^k a_1^k a_2)\|x\| + (2^k a_2^{k+1} + a_1a_2 + 2^k a_1^k a_2)\|x\|^{k^2}\\
\|F_q(x,d)\|^k &\leq 2^k a_1^k(\|x\| + \|d\|)^k + 2^k a_2^k(\|x\|^k + \|d\|^k)^k\\
&\leq 2^{2k} a_1^k(\|x\|^k + \|d\|^k) + 2^k a_2^{2k}(\|x\|^{k^2} + \|d\|^{k^2})\\
\|A_p\big(\psi_{p\,q}(x), F_q(x,d)\big)\| &\leq (a_1^3 + a_1^2a_2 + 2^k a_1^{k+1}a_2)\|x\|\\
&\quad + (2^k a_1 a_2^{k+1} + a_1^2 a_2 + 2^k a_1^{k+1} a_2)\|x\|^{k^2}\\
&\quad + a_1^2\|x\| + a_1^2\|d\| + a_1a_2\|x\|^k + a_1a_2\|d\|^k\\
&\quad + 2^k a_2(a_1^2 + a_1a_2 + 2^k a_1^k a_2)^k\|x\|^k\\
&\quad + 2^k a_2(2^k a_2^{k+1} + a_1a_2 + 2^k a_1^k a_2)^k\|x\|^{k^3}\\
&\quad + 2^{2k} a_1^k a_2(\|x\|^k + \|d\|^k) + 2^k a_2^{2k+1}(\|x\|^{k^2} + \|d\|^{k^2})\\
&\leq \Big(a_1^3 + a_1^2a_2 + 2^k a_1^{k+1}a_2 + 2^k a_1 a_2^{k+1} + a_1^2a_2 + 2^k a_1^{k+1}a_2 + a_1^2\\
&\quad + a_1a_2 + 2^k a_2(a_1^2 + a_1a_2 + 2^k a_1^k a_2)^k + 2^{2k} a_1^k a_2 + 2^k a_2^{2k+1}\Big)\|x\|\\
&\quad + \Big(2^k a_1 a_2^{k+1} + a_1^2 a_2 + 2^k a_1^{k+1} a_2 + a_1a_2\\
&\quad + 2^k a_2(2^k a_2^{k+1} + a_1a_2 + 2^k a_1^k a_2)^k + 2^k a_2(a_1^2 + a_1a_2 + 2^k a_1^k a_2)^k\\
&\quad + 2^{2k} a_1^k a_2 + 2^k a_2^{2k+1}\Big)\|x\|^{k^3}\\
&\quad + (a_1^2 + a_1a_2 + 2^{2k} a_1^k a_2)\|d\| + (a_1a_2 + 2^{2k} a_1^k a_2 + 2^k a_2^{2k+1})\|d\|^{k^2}\\
\|\lambda\psi_{p\,q}(x) + A_p\big(\psi_{p\,q}(x), F_q(x,d)\big)\| &\leq
\end{aligned}$$

[5] For every $a, b, k \geq 0$, $(a+b)^k \leq \big(2\max\{a,b\}\big)^k \leq 2^k\max\{a^k, b^k\} \leq 2^k(a^k + b^k)$.
For $a, b, c, d > 0$, $\alpha \geq \beta \geq \gamma \geq 0$, $ba^\alpha + ca^\beta + da^\gamma \leq ba^\alpha + c\max\{a^\alpha, a^\gamma\} + da^\gamma \leq (b+c)a^\alpha + (d+c)a^\gamma$.

$$
\begin{aligned}
&\le \Big(\lambda(a_1^2 + a_1a_2 + 2^k a_1^k a_2) + \lambda(2^k a_2^{k+1} + a_1a_2 + 2^k a_1^k a_2) \\
&\quad + a_1^3 + a_1^2a_2 + 2^k a_1^{k+1} a_2 + 2^k a_1 a_2^{k+1} + a_1^2 a_2 + 2^k a_1^{k+1} a_2 + a_1^2 \\
&\quad + a_1a_2 + 2^k a_2 (a_1^2 + a_1a_2 + 2^k a_1^k a_2)^k + 2^{2k} a_1^k a_2 + 2^k a_2^{2k+1}\Big)||x|| \\
&\quad + \Big(\lambda(2^k a_2^{k+1} + a_1a_2 + 2^k a_1^k a_2) + 2^k a_1 a_2^{k+1} + a_1^2 a_2 + 2^k a_1^{k+1} a_2 \\
&\quad + a_1a_2 + 2^k a_2(2^k a_2^{k+1} + a_1a_2 + 2^k a_1^k a_2)^k \\
&\quad + 2^k a_2 (a_1^2 + a_1a_2 + 2^k a_1^k a_2)^k + 2^{2k} a_1^k a_2 + 2^k a_2^{2k+1}\Big)||x||^{k^3} \\
&\quad + (a_1^2 + a_1a_2 + 2^{2k} a_1^k a_2)||d|| + (a_1a_2 + 2^{2k} a_1^k a_2 + 2^k a_2^{2k+1})||d||^{k^2} \\
&= b_1||x|| + b_2||x||^{k^3} + b_3||d|| + b_4||d||^{k^2}
\end{aligned}
$$

where b_1, b_2, b_3, and b_4 are appropriately defined constants. Defining

$$\Gamma^\Phi(\xi) \triangleq 4^k(m+1)\max\{b_1^2, b_3^2\}\xi + 4^k(m+1)\max\{b_2^2, b_4^2\}\xi^{k^3}, \quad \xi \in [0, +\infty)$$

one concludes that[6]

$$
\begin{aligned}
||\Phi(x-d, d)||^2 &\le 4^k(m+1)\big(b_1^2||x-e||^2 + b_2^2||x-e||^{2k^3} + b_3^2||e||^2 + b_4^2||e||^{2k^2}\big) \\
&\le 4^k(m+1)\big(2b_1^2||x||^2 + 2^{2k^3} b_2^2 ||x||^{2k^3} \\
&\quad + (2b_1^2 + b_3^2 + b_4^2)||e||^2 + (2^{2k^3} b_2^2 + b_4^2)||e||^{2k^3}\big) \\
&\le \Gamma^\Phi(||x||^2) + \Gamma^\Phi(||d||^2)
\end{aligned}
$$

where $\Gamma^\Phi(\xi) \triangleq c_1\xi + c_2\xi^{k^3}$ ($\xi \in [0, +\infty)$), with $c_1 \triangleq 4^k(m+1)(2b_1^2 + b_3^2 + b_4^2)$, and $c_2 \triangleq 4^k(m+1)(2^{2k^3} b_2^2 + b_4^2)$. Finally

$$
\begin{aligned}
\Gamma^\Phi\big(\Gamma_p^V(\xi)\big) &\le c_1\Gamma_p^V(\xi) + c_2\Gamma_p^V(\xi)^{k^3} \\
&\le c_1a_1\xi + c_1a_2\xi^k + 2^{k^3}c_2(a_1^{k^3}\xi^{k^3} + a_2^{k^3}\xi^{k^4}) \\
&\le (c_1a_1 + c_1a_2 + 2^{k^3}c_2a_1^{k^3})\xi + (c_1a_2 + 2^{k^3}a_1^{k^3}c_2 + 2^{k^3}a_2^{k^3}c_2)\xi^{k^4}
\end{aligned}
$$

which concludes the proof. ∎

Remark. It is possible to get a function $\Gamma^\Phi(\cdot)$ that correspond to bounds less conservative than the ones given in the proof of Lemma 4. In general to get these less conservative bounds one can exploit the specific structure of the functions $A_p(\cdot,\cdot)$, $C_p(\cdot)$, $\chi_p(\cdot)$, $F_p(\cdot,\cdot)$, and $\Gamma_p^V(\cdot)$ ($p \in \mathcal{P}$).

Proof of Lemma 2. By virtue of Lemma 4, there exist constants $a_1, a_2 > 0, k > 1$ such that

$$\Gamma^\Phi\big(\Gamma_p^V(\xi)\big) \le \bar{a}_1\,\xi + \bar{a}_2\,\xi^{\bar{k}}, \quad \xi \in [0, +\infty)$$

Let ϵ_2 to be any positive constant such that

$$\epsilon_2 \ge \max\left\{2\lambda - \frac{2k\mu}{3}, 2\lambda - \mu\right\} \quad \text{and} \quad \epsilon_2 < 2\lambda \tag{29}$$

and let a_3, a_4, and ϵ_1 be positive constants such that

$$a_3 \ge \frac{a_1}{\mu\epsilon_2}, \qquad a_4 \ge \frac{3a_2}{2k\mu\epsilon_2}, \qquad \epsilon_1 \le \frac{2k\mu}{3(k-1)}(ka_4)^{-\frac{1}{k-1}}. \tag{30}$$

[6] For all $a, b, c, d, k \ge 0$, $(a+b+c+d)^k \le 4^k(a^k + b^k + c^k + d^k)$.

Defining $\Gamma(\xi) \triangleq a_3\xi + a_4\xi^k$ it is easy to check that for all $\xi \in [0,+\infty)$, $\Gamma'(\xi) \geq \Gamma'(0) \geq 0$, and

$$\alpha\epsilon_1\big(\Gamma'(\xi)-\Gamma'(0)\big)^{\frac{1}{\alpha}} - 2\mu\Gamma'(\xi)\xi + (2\lambda-\epsilon_2)\Gamma(\xi) + \epsilon_2^{-1}\Gamma^{\Phi}\big(\Gamma_q^V(\xi)\big) \leq 0 \tag{31}$$

where $\alpha \triangleq (k-1)/k$. Indeed, using Assumption 3 and inequalities (29)-(30) one concludes that

$$\begin{aligned}
&\alpha\epsilon_1\big(\Gamma'(\xi)-\Gamma'(0)\big)^{\frac{1}{\alpha}} - 2\mu\Gamma'(\xi)\xi + (2\lambda-\epsilon_2)\Gamma(\xi) + \epsilon_2^{-1}\Gamma^{\Phi}\big(\Gamma_q^V(\xi)\big)\\
&\quad\leq \alpha\epsilon_1\big(ka_4\big)^{\frac{k}{k-1}}\xi^k - 2\mu a_3\xi - 2k\mu a_4\xi^k + (2\lambda-\epsilon_2)\big(a_3\xi + a_4\xi^k\big) + \epsilon_2^{-1}\big(a_1\xi + a_2\xi^k\big)\\
&\quad= \Big(\alpha\epsilon_1\big(ka_4\big)^{\frac{k}{k-1}} + (2\lambda-\epsilon_2)a_4 + \epsilon_2^{-1}a_2 - 2k\mu a_4\Big)\xi^k + \Big((2\lambda-\epsilon_2)a_3 + \epsilon_2^{-1}a_1 - 2\mu a_3\Big)\xi\\
&\quad= -\Bigg(\alpha\big(ka_4\big)^{\frac{k}{k-1}}\left(\frac{2k\mu}{3(k-1)}\big(ka_4\big)^{-\frac{1}{k-1}} - \epsilon_1\right) + a_4\left(\epsilon_2 - 2\lambda + \frac{2k\mu}{3}\right)\\
&\qquad\quad + \frac{2k\mu}{3}\left(a_4 - \frac{3a_2}{2k\mu\epsilon_2}\right)\Bigg)\xi^k - \left(a_3\left(\epsilon_2 - 2\lambda + \mu\right) + \mu\left(a_3 - \frac{a_1}{\mu\epsilon_2}\right)\right)\xi \leq 0
\end{aligned}$$

On the other hand, using the fact that $X_q(y) = x_q - \mathbf{C}(\mathbf{x})$, the rows of $\mathbf{A}(\mathbf{x})$ that corresponds to $\dot{x}_q$ can be written as

$$-\lambda\,\mathbf{C}(\mathbf{x}) + A_q\Big(x_q - \mathbf{C}(\mathbf{x}), f_q\big(x_q - \mathbf{C}(\mathbf{x}), \mathbf{C}(\mathbf{x})\big)\Big) \tag{32}$$

Defining $\xi \triangleq V_q(x_q)$, $d \triangleq \mathbf{C}(\mathbf{x})$; using the fact that for any $\epsilon, a, b, > 0$, $c > 1$, $ab \leq \epsilon\, a^c/c + \epsilon^{-1}\, b^d/d$ where $1/c + 1/d = 1$; and using equations (27), (32), (5), (28), (7), (31); one concludes that

$$\begin{aligned}
\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\mathbf{A}(\mathbf{x}) &= -2\lambda\|\mathbf{x}\|^2 + 2\mathbf{x}'\Phi(x_q - d, d) + \Gamma'(\xi)\\
&\qquad \frac{\partial V_q}{\partial x_q}(x_q)\Big(A_q\big(x_q - d, F_q(x_q - d, d)\big) - \lambda\, d\Big)\\
&\leq -(2\lambda-\epsilon_2)\|\mathbf{x}\|^2 + \epsilon_2^{-1}\|\Phi(x_q - d, d)\|^2 + \Gamma'(\xi)\big(W_q(d) - 2\mu\xi\big)\\
&\leq -(2\lambda-\epsilon_2)\|\mathbf{x}\|^2 + \epsilon_2^{-1}\Gamma^{\Phi}\big(\Gamma_q^V(\xi)\big) + \epsilon_2^{-1}\Gamma^{\Phi}(\|d\|^2)\\
&\qquad + \Gamma'(0)W_q(d) + \big(\Gamma'(\xi) - \Gamma'(0)\big)W_q(d) - 2\mu\Gamma'(\xi)\xi\\
&\leq -(2\lambda-\epsilon_2)\|\mathbf{x}\|^2 + \Gamma'(0)W_q(d) + (1-\alpha)\epsilon_1^{-1}W_q(d)^{\frac{1}{1-\alpha}} + \epsilon_2^{-1}\Gamma^{\Phi}(\|d\|^2)\\
&\qquad + \alpha\epsilon_1\big(\Gamma'(\xi) - \Gamma'(0)\big)^{\frac{1}{\alpha}} - 2\mu\Gamma'(\xi)\xi + \epsilon_2^{-1}\Gamma^{\Phi}\big(\Gamma_q^V(\xi)\big)\\
&\leq -(2\lambda-\epsilon_2)\mathbf{V}(\mathbf{x}) + \Gamma'(0)W_q(d) + (1-\alpha)\epsilon_1^{-1}W_q(d)^{\frac{1}{1-\alpha}} + \epsilon_2^{-1}\Gamma^{\Phi}(\|d\|^2)\\
&\qquad + \alpha\epsilon_1\big(\Gamma'(\xi) - \Gamma'(0)\big)^{\frac{1}{\alpha}} - 2\mu\Gamma'(\xi)\xi + (2\lambda-\epsilon_2)\Gamma(\xi) + \epsilon_2^{-1}\Gamma^{\Phi}\big(\Gamma_q^V(\xi)\big)\\
&\leq -(2\lambda-\epsilon_2)\mathbf{V}(\mathbf{x}) + \Gamma'(0)W_q(d) + (1-\alpha)\epsilon_1^{-1}W_q(d)^{\frac{1}{1-\alpha}} + \epsilon_2^{-1}\Gamma^{\Phi}(\|d\|^2)
\end{aligned}$$

This proofs the lemma with $\gamma \triangleq 2\lambda - \epsilon_2$ and $\mathbf{W}(d) \triangleq \Gamma'(0)\, W_q(d) + \epsilon_1^{-1}(1-\alpha)\, W_q(d)^{\frac{1}{1-\alpha}} + \epsilon_2^{-1}\Gamma^{\Phi}(\|d\|^2)$. ∎

Proof of Lemma 1. The existence of the family of symmetric positive definite matrices $\{Q_p : p \in \mathcal{P}\}$ such that (15) holds is guaranteed by the fact that for every $p \in \mathcal{P}$, $A_p + \mu I$ is a stability matrix. The family of functions $\{\Gamma_p^V : [0,+\infty) \to [0,+\infty) \mid p \in \mathcal{P}\}$ in (7) can be defined by

$$\Gamma_p^V(\xi) = \frac{\xi}{\sigma_{\min}[Q_p]}, \qquad \xi \in [0,+\infty)$$

where $\sigma_{\min}[Q_p]$ denotes the smallest singular value of Q_p $(p \in \mathcal{P})$. Finally, using equations (12)-(14), (11), and (15),

$$\frac{\partial V_p}{\partial x}(x)\left(A_p\big(x - d, F_p(x - d, d)\big) - \lambda\, d\right)$$

$$
\begin{aligned}
&= \frac{\partial V_p}{\partial x}(x)\Big(A_p\big(x-d, F_p^l(x-d)\big) - \lambda\, d\Big)\\
&= \frac{\partial V_p}{\partial x}(x)\Big(A_p x - \big(A_p + \lambda I\big)\, d\Big)\\
&= x'\big(A_p' Q_p + Q_p A_p\big)x + 2x' Q_p \big(A_p + \lambda I\big)\, d\\
&= x'\big(A_p' Q_p + Q_p A_p + Q_p\big(A_p + \lambda I\big)\big(A_p + \lambda I\big)' Q_p\Big)x\\
&\quad + \|d\|^2 - \|d - \big(A_p + \lambda\, I\big)' Q_p x\|^2\\
&\leq -2\mu x' Q_p x + \|d\|^2 \qquad \blacksquare
\end{aligned}
$$

Proof of Theorem 1. Using the performance signals $\{\pi_p : p \in \mathcal{P}\}$ as test functions, Open-Loop Assumption 1 in [7] holds because all of the performance signals are monotone functions of time. Equation (18) guarantees that e_ω goes to zero exponentially fast and, by virtue of the Lipschitz continuity of $\mathbf{W}(\cdot)$, so does $\mathbf{W}(e_\omega)$. Because of (25),

$$
\lim_{t\to\infty} \pi_\omega(t) \triangleq C < \infty
$$

and Open-Loop Assumption 2 in [7] also holds. In view of the Hysteresis Switching Lemma, there must be a time T^* beyond which σ is constant and no more switching occurs. Moreover, denoting by $q \in \mathcal{P}$ the value of σ beyond T^*, π_q must be bounded on the maximal interval of existence $[0, T)$ of solution for the overall systems of equations involved. Let t be an arbitrary time in the interval $[T^*, T)$. Integrating $\mathbf{V}\big(\mathbf{x}(\tau)\big)$ along the solution of (23) on the interval $[T^*, t)$, and using inequality (24), one concludes that

$$
\mathbf{V}\big(\mathbf{x}(t)\big) + \gamma \int_{T^*}^{t} \mathbf{V}\big(\mathbf{x}(\tau)\big)\, d\tau \leq \mathbf{V}\big(\mathbf{x}(T^*)\big) + \int_{T^*}^{t} \mathbf{W}\big(e_q(\tau)\big)\, d\tau
$$

The boundness of π_q guarantees that $\mathbf{V}(\mathbf{x})$ and $\mathbf{x}$ are bounded and therefore that the solution exists globally, i.e. $T = +\infty$. Moreover, since

$$
\mathbf{V}\big(\mathbf{x}(t)\big) \in \mathcal{L}_1 \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}\big(\mathbf{x}(t)\big) \in \mathcal{L}_\infty
$$

one also concludes that $\mathbf{V}(\mathbf{x})$ and $\mathbf{x}$ converge to zero as $t \to +\infty$. $\blacksquare$