Root-Mean-Square Gains of Switched Linear Systems

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Abstract— In this paper we compute the root-mean-square (RMS) gain of a switched linear system when the interval between consecutive switchings is large. The algorithm proposed is based on the fact that a given constant γ provides an upper bound on the RMS gain whenever there is a separation between the stabilizing and the antistabilizing solutions to a set of γ-dependent algebraic Riccati equations. The motivation for this problem is the application of robust stability tools to the analysis of hybrid systems.

I. INTRODUCTION

A switched linear system is defined by a linear differential equation for which the matrices that appear in the right-hand-side are piecewise constant. The times at which these matrices are discontinuous are called "switching times." Formally, to define a switched linear system one considers a family of n-dimensional, m-input, k-output minimal realizations

$$\{(A_p, B_p, C_p, D_p) : p \in \mathcal{P}\}, \tag{1}$$

parameterized by an index set $\mathcal{P}$, together with a family of piecewise constant signals $\mathcal{S} := \{\sigma : [0, \infty) \to \mathcal{P}\}$. For a given signal $\sigma \in \mathcal{S}$, one can then define the following linear time-varying system

$$\dot{x} = A_{\sigma(t)}x + B_{\sigma(t)}u, \qquad y = C_{\sigma(t)}x + D_{\sigma(t)}u. \tag{2}$$

The signal σ is called a *switching signal* and the times at which it is discontinuous are called *switching times*. Between switching times the dynamics of (2) are time-invariant. Typically, switching signals are generated by a supervisory logic and different values of σ correspond to distinct modes of operation [1]. We consider here switched systems for which all the matrices A_p, $p \in \mathcal{P}$ are Hurwitz. For simplicity of notation, in the sequel we drop the time-dependence in $\sigma(t)$ when the switching signal is used as a subscript to the matrices in (1).

For different switching signals σ, the system (2) has different properties. However, it is useful to study properties of this system that hold for every σ in a particular set of switching signals $\mathcal{S}$. Stability is a property that has been extensively investigated for several sets of switching signals: We say that (2) is *uniformly exponentially stable over* $\mathcal{S}$ when the state transition matrix $\Phi_\sigma(t, \tau)$ of the switched system $\dot{z} = A_\sigma z$ can be bounded by[1]

$$\|\Phi_\sigma(t, \tau)\| \le ce^{-\lambda(t-\tau)}, \qquad \forall t, \tau \ge 0, \quad \forall \sigma \in \mathcal{S},$$

where the constants $c, \lambda > 0$ are chosen independently of σ in $\mathcal{S}$. It is well known that (2) is uniformly exponentially stable as long as all signals in $\mathcal{S}$ have consecutive switching times separated by a sufficiently long interval. In particular, defining $\mathcal{S}[\tau_D]$, $\tau_D > 0$ to be the set of switching signals with interval between consecutive discontinuities no smaller than τ_D, there exists a $\tau_D^* > 0$ so that (2) is uniformly exponentially stable over any $\mathcal{S}[\tau_D]$, $\tau_D \ge \tau_D^*$. The minimum interval τ_D between consecutive switching times is called the *dwell-time*. In [2] it is shown that uniform exponential stability is preserved when the set of switching signals is enlarged to contain signals that occasionally have dwell times smaller than τ_D, provided

[1]Given a vector a and a matrix A we denote by $\|a\|$ and $\|A\|$ the Euclidean norm of a and the largest singular value of A, respectively. Given a vector or matrix-valued signal z defined on $[0, \infty)$, we denote by $\|z\|_2$ its $\mathcal{L}_2$-norm $\left(\int_0^\infty \|z(t)\|^2 dt\right)^{\frac{1}{2}}$. The set of signals with finite $\mathcal{L}_2$-norm is denoted by $\mathcal{L}_2$.

that this does not happen on-the-average (*slow-switching on-the-average*). The reader is referred to [1], [3] for a discussion of these and related results.

Surprisingly, there has been relatively little work on the study of input-output properties that hold over sets of switching signals. A notable exception is the study of passivity-like conditions for switched systems in [4]. In this paper we study the root-mean-square (RMS) gain of a switched system [2], [5], [6]. RMS gains are typically needed to evaluate the performance of closed-loop systems with hybrid controllers or processes. This analysis also finds application in the use of robust control techniques in the stability analysis of hybrid systems.

Given a set of switching signals $\mathcal{S}$, we define the *RMS gain of* (2) *over* $\mathcal{S}$ to be the smallest scalar $\mathfrak{g}_\mathcal{S} \ge 0$ for which

$$\mathfrak{g}_\mathcal{S} := \inf\{\gamma \ge 0 : \|y\|_2 \le \gamma\|u\|_2, \ \forall u \in \mathcal{L}_2, \ \sigma \in \mathcal{S}\},$$

where y is computed along solutions to (2) with $x(0) = 0$. The RMS gain $\mathfrak{g}_\mathcal{S}$ can be viewed as a "worst-case" energy amplification gain *over all possible input and switching signals*.

When (2) is uniformly exponentially stable over $\mathcal{S}$, the RMS gain $\mathfrak{g}_\mathcal{S}$ is known to be finite. Since both switching with large dwell-time and slow-switching on-the-average result in exponential stability, either of these forms of slow switching result in a finite RMS gain $\mathfrak{g}_\mathcal{S}$. In [2], [5], [6] one can find upper bounds on RMS gains over appropriately defined sets of switching signals that satisfy the slow-switching properties mentioned above. However, these bounds are typically very conservative and prompted us to compute the precise value of the RMS gain under slow-switching.

The RMS gain over the set of switching signals $\mathcal{S}[\tau_D]$ with dwell-time $\tau_D > 0$ is a monotone decreasing function of τ_D. This is simply because, given two dwell-times $\tau_{D_1} \le \tau_{D_2}$, we have that $\mathcal{S}[\tau_{D_1}] \supset \mathcal{S}[\tau_{D_2}]$. One can then ask: *What is the smallest RMS gain that can be obtained for* (2) *by increasing the dwell-time?* To answer this question we define the *slow-switching RMS gain*, by

$$\mathfrak{g}_{\text{slow}} := \inf_{\tau_D > 0} \mathfrak{g}_{\mathcal{S}[\tau_D]}. \tag{3}$$

Since every $\mathcal{S}[\tau_D]$, $\tau_D > 0$ contains the constant switching signals $\sigma(t) = p$, $t \ge 0$, $p \in \mathcal{P}$, $\mathfrak{g}_{\text{slow}}$ is necessarily larger or equal to the RMS gains of all the "unswitched systems," i.e.,

$$\mathfrak{g}_{\text{slow}} \ge \mathfrak{g}_{\text{static}} := \sup_{p \in \mathcal{P}} \|C_p(sI - A_p)^{-1}B_p + D_p\|_\infty, \tag{4}$$

where $\|T\|_\infty$ denotes the $\mathcal{H}_\infty$-norm of a transfer matrix T, i.e., $\|T\|_\infty := \sup_{\Re[s] \ge 0} \|T(s)\|$. We recall that $\|T\|_\infty$ is numerically equal to the RMS gain of any linear time-invariant system with transfer matrix T. Until recently, it was believed that the inequality in (4) was actually an equality. This would mean that, by switching sufficiently slow, one would recover the RMS gains of the "unswitched systems," which are realization independent. It turns out that this is not true and the above inequality is often strict. The following simple example illustrates this and also provides some intuition on the difficulties introduced by switching: Consider a SISO switched system

$$\dot{x} = A_\sigma x + b_\sigma u, \qquad y = c_\sigma x, \tag{5}$$

for which the switching signal σ takes values in the set $\mathcal{P} := \{1, 2\}$ and both unswitched systems have $\mathcal{H}_\infty$-norm (and therefore RMS gain) equal to one, i.e.,

$$\|c_p(sI - A_p)^{-1}b_p\|_\infty = 1, \qquad \forall p \in \mathcal{P}. \tag{6}$$

Defining

$$\sigma(t) := \begin{cases} 1 & t < 1 \\ 2 & t \geq 1 \end{cases} \qquad u(t) := \begin{cases} 1 & t < 1 \\ 0 & t \geq 1 \end{cases} \qquad (7)$$

the switching signal σ has a single discontinuity and therefore belongs to every family of switching signals $\mathcal{S}[\tau_D]$, $\tau_D > 0$; and the probe input u has $\mathcal{L}_2$-norm equal to one. The output of (5) for u and σ in (7) is given by

$$y(t) = \begin{cases} c_1 \big(\int_0^t e^{A_1(t-\tau)} d\tau \big) b_1 & 0 \leq t < 1 \\ c_2 e^{A_2(t-1)} \big(\int_0^1 e^{A_1(1-\tau)} d\tau \big) b_1 & t \geq 1. \end{cases}$$

It turns out that c_2 can be chosen arbitrarily large without violating (6), provided that b_2 is sufficiently small. Therefore, by appropriate choice of c_2, the $\mathcal{L}_2$-norm of y can actually be very large (in particular larger than 1) for a unit-norm input, *even though we are switching between two systems with RMS gain equal to one.* The intuition behind this example is that for $t < 1$ the input energy is "stored" in the system through one realization (perhaps through a "large" b_1) and then it is "released" to the output for $t \geq 1$ through a different realization (perhaps through a "large" c_2). From this example, we conclude that even for very slow switching (in the extreme a single switch), the RMS gain of a switched system can be arbitrarily larger than the RMS gains of the "unswitched systems." Moreover, the RMS gain of a switched system is realization dependent and cannot be determined just from the transfer functions of the systems being switched. The goal of this paper is then to compute the slow-switching RMS gain $\mathfrak{g}_{\text{slow}}$ defined by (3).

For linear time-invariant systems, the RMS gain can be determined by solving algebraic Riccati equations. In particular, it is well known (*Bounded-real Lemma*) that this gain is smaller than a given constant $\gamma > 0$ if and only if an algebraic Riccati equation of the form

$$S(P; \gamma) := A_\gamma' P + P A_\gamma + P R_\gamma P + Q_\gamma = 0, \qquad (8)$$

has a positive definite solution P, where A_γ, Q_γ, and R_γ are appropriately defined γ-dependent $n \times n$ real matrices with Q_γ and R_γ symmetric and positive semidefinite (cf. Theorem 1 in Section III for details). This allows one to determine the precise value of the RMS gain of a system by using a bisection algorithm to find the smallest value of γ for which a positive definite solution to (8) exists [7]. Although in general algebraic Riccati equations have more than one solution, in order to determine the RMS gain of a time-invariant system it is sufficient to find one of these solutions. However, to determine the RMS gain of a switched system, we have to pay special attention to the multiple solutions to the algebraic Riccati equations involved. We shall see below that the condition for the slow-switching RMS gain of a switched system to be smaller than γ requires a separation between *all* the stabilizing and *all* the antistabilizing solutions to the algebraic Riccati equations of the systems being switched.

The remaining of this paper is organized as follows: In Section II we derive a general formula for the RMS gain of a switched system in terms of the solution to a differential Riccati equation. This prompts us to investigate the solutions to differential Riccati equations, which is done in Section III. In Section IV we determine upper and lower bounds on the RMS gain of a switched system over the set $\mathcal{S}_1$ of switching signals with no more than one discontinuity. We then show in Section V that these same bounds are also valid for the slow-switching RMS gain. Section VI illustrates the results presented in the context of an example. Section VII contains final conclusions and directions for future research. A partial version of the results in this paper was presented at the Fifth Hybrid Systems: Computation and Control Workshop, Stanford, March 2002.

II. COMPUTATION OF RMS GAINS

To determine the RMS gain of (2) over $\mathcal{S}$ we consider the following optimization problem:

$$J(\gamma, \sigma) := \lim_{T \to \infty} \sup_{u \in \mathcal{L}_2} \int_0^T \|y(\tau)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau,$$

$\forall \gamma \geq 0$, $\sigma \in \mathcal{S}$, where y is computed along solutions to (2) with $x(0) = 0$. The RMS gain of (2) over $\mathcal{S}$ is then given by

$$\mathfrak{g}_{\mathcal{S}} := \inf\{\gamma \geq 0 : J(\gamma, \sigma) \leq 0, \ \forall \sigma \in \mathcal{S}\}. \qquad (9)$$

Note that because the system is linear, when $J(\gamma, \sigma) > 0$ we must actually have $J(\gamma, \sigma) = +\infty$, which means that $J(\gamma, \sigma) \leq 0$ in (9) could be replaced by $J(\gamma, \sigma) < \infty$. It is also sufficient to consider the case $\gamma > \|D_p\|$, $p \in \mathcal{P}$ as the RMS gain is always larger or equal to any of the high frequency gains $\|D_p\|$ because of (4).

We shall use dynamic programming to compute $J(\gamma, \sigma)$ for a given $\sigma \in \mathcal{S}$ and $\gamma > \|D_p\|$, $p \in \mathcal{P}$. To this effect let us define the following finite-horizon "cost-to-go:"

$$V(x_0, t; T, \gamma, \sigma) := \sup_{u \in \mathcal{L}_2, x(t) = x_0} \int_t^T \|y(\tau)\|^2 - \gamma^2 \|u(\tau)\|^2 d\tau.$$

The corresponding dynamic-programming equation is given by[2]

$$\begin{aligned} -V_t &= \sup_{u \in \mathbb{R}} \|C_\sigma x_0 + D_\sigma u\|^2 - \gamma^2 \|u\|^2 + V_x \big(A_\sigma x_0 + B_\sigma u\big) \\ &= V_x A_\sigma x_0 + x_0' C_\sigma' C_\sigma x_0 \\ &\quad + \Big(\frac{1}{2} V_x B_\sigma + x_0' C_\sigma' D_\sigma\Big) X^{-1} \Big(\frac{1}{2} B_\sigma' V_x' + D_\sigma' C_\sigma x_0\Big), \end{aligned} \qquad (10)$$

where $X_p := \gamma^2 I - D_p' D_p > 0$. Since for a fixed σ we are dealing with a linear time-varying system, we can restrict our attention to costs-to-go of the form $V(x_0, t; T, \gamma, \sigma) = x_0' P(t) x_0$, $t \geq 0$ with $P(t)$ real symmetric. Replacing this in (10) we obtain

$$\begin{aligned} 0 = x_0' \big(\dot{P} + P A_\sigma + A_\sigma' P + C_\sigma' C_\sigma \\ + (P B_\sigma + C_\sigma' D_\sigma) X_\sigma^{-1} (B_\sigma' P + D_\sigma' C_\sigma)\big) x_0, \quad \forall x_0 \in \mathbb{R}^n, \end{aligned}$$

which (because of symmetry) actually implies that

$$\begin{aligned} -\dot{P} &= P A_\sigma + A_\sigma' P + C_\sigma' C_\sigma \\ &\quad + (P B_\sigma + C_\sigma' D_\sigma) X_\sigma^{-1} (B_\sigma' P + D_\sigma' C_\sigma) = S_\sigma(P; \gamma), \quad (11) \end{aligned}$$

where

$$S_p(P; \gamma) := P \bar{A}_p + \bar{A}_p P + Q_p + P R_p P, \qquad p \in \mathcal{P}, \qquad (12)$$

with $\bar{A}_p := A_p + B_p X_p^{-1} D_p' C_p$, $R_p := B_p X_p^{-1} B_p'$, $Q_p := C_p'(I + D_p X_p^{-1} D_p') C_p$. Moreover,

$$P(T) = 0, \qquad (13)$$

since $V(x_0, T; T, \gamma, \sigma) = 0$ for every x_0. Since (11) is locally Lipschitz with respect to P, we conclude that $J(\gamma, \sigma) < \infty$ if and only if the unique solution to the final value problem (11)–(13) exists on every interval $[0, T)$, $T > 0$ (cf. [8] for technical details). Therefore

$$\mathfrak{g}_{\mathcal{S}} = \inf \Big\{\gamma \geq 0 : \text{solution to (11)–(13)}$$
$$\text{exists on } [0, T), \ \forall T > 0, \ \sigma \in \mathcal{S}\Big\}. \quad (14)$$

To compute the RMS gain of a switched system we must then study the existence of solution to a differential Riccati equation. We do this in the following section.

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[2]For simplicity we dropped the dependence of $(x_0, t; T, \gamma, \sigma)$ on V. We denote the partial derivatives of V with respect to t and x by V_t and V_x, respectively.

III. DIFFERENTIAL RICCATI EQUATIONS

Consider the following *algebraic Riccati equation* (ARE)

$$S(P) := A'P + PA + PRP + Q = 0, \qquad (15)$$

where $A, Q, R \in \mathbb{R}^{n \times n}$ with Q and R symmetric. Associated with this equation one defines a $2n \times 2n$ *Hamiltonian matrix* by

$$H := \begin{bmatrix} A & R \\ -Q & -A' \end{bmatrix}.$$

In the sequel, we say that a matrix P^- is a *stabilizing* solution to the ARE (15) if P^- is real, symmetric, positive definite, and $A + RP^-$ is Hurwitz. We say that P^+ is an *antistabilizing* solution to (15) if P_+ is real, symmetric, positive definite, and $-(A + RP^+)$ is Hurwitz. The following theorem is a consequence of several well known results on AREs [8].

Theorem 1: Consider a minimal realization $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ with $\bar{A}$ Hurwitz and $\|\bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}\|_\infty < \gamma$ for some $\gamma > 0$. Then $\|\bar{D}\| < \gamma$ and there exist stabilizing and antistabilizing solutions P^- and P^+, respectively, to the ARE (15) with

$$A := \bar{A} + \bar{B}\bar{X}^{-1}\bar{D}'\bar{C}, \qquad R := \bar{B}\bar{X}^{-1}\bar{B}',$$
$$Q := \bar{C}'(I + \bar{D}\bar{X}^{-1}\bar{D}')\bar{C}, \qquad \bar{X} := \gamma^2 I - \bar{D}'\bar{D}.$$

Moreover, $P_+ - P_- > 0$ [3] and these solutions can be computed by

$$P^- = P_2^-(P_1^-)^{-1}, \qquad P^+ = P_2^+(P_1^+)^{-1}, \qquad (16)$$

where the columns of $\begin{bmatrix} P_1^{-\prime} & P_2^{-\prime} \end{bmatrix}'$ and $\begin{bmatrix} P_1^{+\prime} & P_2^{+\prime} \end{bmatrix}'$ are chosen to form bases for the stable and antistable invariant[4] subspaces of H, respectively. The square matrices P_1^- and P_1^+ above are guaranteed to be nonsingular. □

Consider now the final-value solution to the following differential Riccati equation

$$-\dot{P} = S(P) = A'P + PA + PRP + Q, \quad P(t_0) = P_0, \qquad (17)$$

for $t \le t_0$. The following theorem is inspired by results from [9] and [10, Chapter 3] and its proof can be found in [8].

Theorem 2: Suppose that there exist stabilizing and antistabilizing solutions to (15) with $P^+ - P^- > 0$. The following statements hold:

a) When $P_0 - P^+$ is nonsingular, the solution $\Pi(t; P_0)$ to (17) is given by

$$\Pi(t; P_0) = P^+ + \Lambda(t)^{-1}, \qquad \forall t \le t_0,$$

with

$$\Lambda(t) := e^{(A+RP^+)(t-t_0)}\Big((P_0 - P^+)^{-1} +$$
$$(P^+ - P^-)^{-1}\Big)e^{(A+RP^+)'(t-t_0)} - (P^+ - P^-)^{-1}$$

on the interval $\mathcal{I}$ on which Λ is nonsingular.

b) When $R \ge 0$ and $P_0 - P^+ < 0$ the solution $\Pi(t; P_0)$ to (17) exists globally for $t \le t_0$ and $\Pi(t; P_0) \to P^-$ as $t \to -\infty$. The same result holds when $R \le 0$ and $P_0 - P^+ > 0$.

c) When $R \ge 0$ and $P_0 - P^+ \not\le 0$, the solution $\Pi(t; P_0)$ to (17) has finite escape time. The same result holds when $R \le 0$ and $P_0 - P^+ \not\ge 0$. □

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[3] Given a symmetric matrix Q we write $Q > 0$, $Q \ge 0$, $Q < 0$, and $Q \le 0$ when Q is positive definite, positive semidefinite, negative definite, and negative semidefinite, respectively. We write $Q \not\ge 0$ when Q is nonsingular and not positive semidefinite. Similarly, $Q \not\le 0$ means that Q is nonsingular and not negative semidefinite.

[4] Let H be a matrix with characteristic polynomial $\alpha(s) = \alpha^-(s)\alpha^0(s)\alpha^+(s)$, where all roots of the polynomials $\alpha^-(s)$, $\alpha^+(s)$, and $\alpha^0(s)$ have negative, positive, and zero real parts, respectively. The *stable* and *antistable* invariant subspaces of H are defined to be $\mathrm{Ker}\,\alpha^-(H)$ and $\mathrm{Ker}\,\alpha^+(H)$, respectively.

IV. RMS GAIN OF SINGLE-SWITCH SYSTEMS

In this section we consider the set $\mathcal{S}_1$ of switching signals with at most one discontinuity. Using the results from the previous section we compute bounds on the RMS gain $\mathfrak{g}_{\mathcal{S}_1}$ of (2) over $\mathcal{S}_1$:

Theorem 3: Assume that the realizations in (1) are minimal. Given any $\gamma > \mathfrak{g}_{\text{static}}$,

$$\exists p, q \in \mathcal{P} \quad P_q^+ - P_p^- \not\ge 0 \quad \Rightarrow \quad \mathfrak{g}_{\mathcal{S}_1} \ge \gamma \qquad (18)$$
$$\forall p, q \in \mathcal{P} \quad P_q^+ - P_p^- > 0 \quad \Rightarrow \quad \mathfrak{g}_{\mathcal{S}_1} \le \gamma, \qquad (19)$$

where P_p^+ and P_p^-, $p \in \mathcal{P}$ respectively denote the stabilizing and antistabilizing solutions to $S_p(P; \gamma) = 0$, with S_p defined by (12). □

Before proving Theorem 3, it is interesting to note that the condition in (19) that gives the upper bound on the RMS gain requires a separation between *all* the stabilizing and *all* the antistabilizing solutions to the AREs of the systems being switched. Note that for $p = q$, the fact that the qth system has RMS gain smaller than γ already guarantees that $P_q^+ - P_q^- > 0$ (cf. Theorem 1). However, this is not enough for $\mathfrak{g}_{\mathcal{S}_1}$ to be smaller or equal to γ. Indeed, we need all the stabilizing solutions P_p^- to be "smaller" than all the antistabilizing solutions P_q^+.

Proof: [Theorem 3] To prove (18) we show that when $P_q^+ - P_p^- \not\ge 0$ for some $p, q \in \mathcal{P}$, the RMS gain $\mathfrak{g}_{\mathcal{S}_1}$ given by (14) must be larger or equal to γ because the solution to (11)–(13) does not exist for some $T > 0, \sigma \in \mathcal{S}$. To this effect suppose that there are $p, q \in \mathcal{P}$ for which $P_q^+ - P_p^- \not\ge 0$ and take

$$\sigma(t) := \begin{cases} q & t < T_{\text{switch}} \\ p & t \ge T_{\text{switch}} \end{cases} \qquad \forall t \ge 0, \qquad (20)$$

for some $T_{\text{switch}} > 0$. Because $R_p \ge 0$ and $-P_p^+ < 0$ (cf. Theorem 1), we conclude from Theorem 2b) that the solution to (11)–(13) exists for every $t \in [T_{\text{switch}}, T)$. Moreover, if σ where equal to p for all times, we would actually have $P(t) \to P_p^-$ as $t \to -\infty$. This means that by choosing T sufficiently large, it is possible to have $P(T_{\text{switch}})$ sufficiently close to P_p^- so that $P_q^+ - P(T_{\text{switch}}) \not\ge 0$. Here, we are using the fact that the set $\{P : P_q^+ - P \not\ge 0\}$ is open and therefore if $P(T_{\text{switch}})$ is sufficiently close to the element P_p^- of this set, it must be inside the set. Once T is be chosen such that $P_q^+ - P(T_{\text{switch}}) \not\ge 0$, by applying Theorem 2 to solve (11)–(13) for $t \le T_{\text{switch}}$, we conclude from c) that the solution is actually only defined in an interval $(T_{\text{escape}}, T_{\text{switch}}]$ of finite length. Because of time invariance, we can choose T sufficiently large so that $T_{\text{escape}} > 0$. So the solution to (11)–(13) does not exist globally. This finishes the proof of (18).

To prove (19) we show directly that when $P_q^+ - P_p^- > 0$ for every $p, q \in \mathcal{P}$, we have $\|y\|_2 \le \gamma\|u\|_2$ for every input and switching signals. To this effect pick arbitrary $u \in \mathcal{L}_2$, $\sigma \in \mathcal{S}_1$ and define

$$v(t) := V(x(t), t) + \int_0^t \|y(\tau)\|^2 - \gamma^2\|u(\tau)\|^2 d\tau, \quad \forall t \ge 0,$$

where x and y are evaluated along a solution to (2) and $V(x, t) := x'P(t)x$, $t \ge 0$. Here, $P(t)$, $t \in [0, \infty)$ denotes a time-varying continuous real symmetric matrix with the property that on any interval on which σ is constant we have

$$-\dot{P} = S_\sigma(P; \gamma). \qquad (21)$$

To construct such $P(t)$ we use the fact that because $\sigma \in \mathcal{S}_1$, the switching signal must be of the form (20) (possibly with $p = q$ if it is actually constant). Suppose then that for $t > T_{\text{switch}}$ we set $P(t)$ constant and equal to P_p^-, and for $t \le T_{\text{switch}}$, $P(t)$ is defined by the final value problem

$$-\dot{P} = S_q(P; \gamma), \qquad P(T_{\text{switch}}) = P_p^-.$$

The function $P(t)$ is well defined on $t \in [0, T_{\text{switch}}]$ because $P_p^- - P_q^+ < 0$ (cf. Theorem 2b) and also on $t \in (T_{\text{switch}}, +\infty)$ because we are simply setting it to be equal to P_p^- on this interval. By construction, this choice of $P(t)$ is continuous and does satisfy (21) on every interval on which σ is constant. Because of (21), on any such interval we have

$$\dot{v} = x'\dot{P}x + V_x(A_\sigma x + B_\sigma u) + \|C_\sigma x + D_\sigma u\|^2 - \gamma^2\|u\|^2$$
$$\leq x'(\dot{P} + S_\sigma(P; \gamma))x = 0.$$

(cf. equations (10)–(11)). We then conclude that v decreases between discontinuities of σ and is continuous at the switching time T_{switch}. Therefore $v(t) \leq v(0) = x(0)'P(0)x(0), \forall t \geq 0$. For a zero initialization of (2) we then conclude that $v(t) \leq 0, \forall t \geq 0$, or equivalently that

$$\int_0^t \|y(\tau)\|^2 - \gamma^2\|u(\tau)\|^2 d\tau \leq -V(x(t), t), \qquad \forall t \geq 0.$$

Since $u \in \mathcal{L}_2$, $\lim_{t\to\infty} x(t) = 0$ and therefore $\int_0^\infty \|y(\tau)\|^2 - \gamma^2\|u(\tau)\|^2 d\tau \leq 0$. This concludes the proof since it shows that indeed $\|y\|_2 \leq \gamma\|u\|_2$. ∎

V. SLOW-SWITCHING RMS GAIN

We show next that the bounds provided by Theorem 3 on the RMS gain $\mathfrak{g}_{\mathcal{S}_1}$ of (2) over $\mathcal{S}_1$ also hold for the slow-switching RMS gain $\mathfrak{g}_{\text{slow}}$ of (2). For simplicity, we assume that the set $\mathcal{P}$ is finite, i.e., that we are switching among a finite number of systems. However, it would be straightforward to replace this assumption by uniformity conditions on the systems being switched, e.g., compactness of the set of realizations $\{A_p, B_p, C_p, D_p : p \in \mathcal{P}\}$.

Theorem 4: Assume that the realizations in (1) are minimal and $\mathcal{P}$ is finite. Given any $\gamma > \mathfrak{g}_{\text{static}}$

$$\exists p, q \in \mathcal{P} \quad P_q^+ - P_p^- \not\geq 0 \qquad \Rightarrow \qquad \mathfrak{g}_{\text{slow}} \geq \gamma \qquad (22)$$
$$\forall p, q \in \mathcal{P} \quad P_q^+ - P_p^- > 0 \qquad \Rightarrow \qquad \mathfrak{g}_{\text{slow}} \leq \gamma, \qquad (23)$$

where P_p^+ and P_p^-, $p \in \mathcal{P}$ respectively denote the stabilizing and antistabilizing solutions to $S_p(P; \gamma) = 0$, with S_p defined by (12). ☐

Proof: [Theorem 4] To prove (22) note that for any $\tau_D > 0$, $\mathcal{S}_1 \subset \mathcal{S}[\tau_D]$ and therefore $\mathfrak{g}_{\mathcal{S}_1} \leq \mathfrak{g}_{\text{slow}}$. From Theorem 3 we then conclude that

$$\exists p, q \in \mathcal{P} \; P_q^+ - P_p^- \not\geq 0 \;\Rightarrow\; \mathfrak{g}_{\mathcal{S}_1} \geq \gamma \;\Rightarrow\; \mathfrak{g}_{\text{slow}} \geq \mathfrak{g}_{\mathcal{S}_1} \geq \gamma.$$

To prove (23) assume that $P_q^+ - P_p^- > 0$ for every $p, q \in \mathcal{P}$ and therefore that we can select a positive constant ϵ and matrices $P_{(p,q)}$, $p, q \in \mathcal{P}$ such that

$$\|P - P_p^-\| \leq \epsilon \quad \Rightarrow \quad P_q^+ > P_{(p,q)} \geq P, \quad \forall p, q \in \mathcal{P}. \quad (24)$$

In practice, $P_{(p,q)}$ can be any matrix sufficiently close to P_p^- so that $P_q^+ > P_{(p,q)} > P_p^-$ and ϵ a sufficiently small positive constant so that for any P such that $\|P - P_p^-\| \leq \epsilon$ we still have $P_{(p,q)} \geq P$.

Since $P_{(p,q)} - P_q^+ < 0$, we conclude from Theorem 2b) that the solution to the final value problem

$$-\dot{P} = S_q(P; \gamma), \qquad P(T) = P_{(p,q)}, \qquad (25)$$

exists globally and converges to P_q^- as $t \to -\infty$. Pick now a constant τ_D sufficiently large so that $\|P(t) - P_q^-\| \leq \epsilon, \forall t \leq T - \tau_D$ regardless of the choices for p and q in (25). We show next that the RMS gain over $\mathcal{S}[\tau_D]$ is smaller or equal to γ and therefore so is the slow-switching RMS gain $\mathfrak{g}_{\text{slow}}$.

Pick arbitrary $u \in \mathcal{L}_2$, $\sigma \in \mathcal{S}_1$ and define

$$v(t) := V(x(t), t) + \int_0^t \|y(\tau)\|^2 - \gamma^2\|u(\tau)\|^2 d\tau, \quad \forall t \geq 0,$$

where x and y are evaluated along a solution to (2) and $V(x, t) := x'P(t)x, t \geq 0$. Here, $P(t)$ denotes a time-varying symmetric matrix with the property that on any interval (t_1, t_2) on which σ is constant and equal to q we have

$$-\dot{P} = S_q(P; \gamma), \qquad P(t_2) = P_0, \qquad (26)$$

where P_0 is chosen equal to P_q^- in case $t_2 = \infty$ and equal to $P_{(p,q)}$ when $t_2 < \infty$, with p being the value of σ immediately after t_2. Note that in the former case (which occurs only when t_1 is the last switching time), $P(t)$ is constant for $t > t_1$. In the latter case, the solution to (26) is well defined for every $t \leq t_2$ because $P_{(p,q)} - P_q^+ < 0$ (cf. Theorem 2b). This construction yields a time-varying matrix $P(t)$ that is well defined for all $t \in (0, \infty)$ and satisfies (21) on every interval on which σ is constant. However, now $P(t)$ is not necessarily a continuous function of t.

As in the proof of Theorem 3, we conclude that $\dot{v} \leq 0$ on any interval on which σ is constant but now $v(t)$ is discontinuous at switching times. In particular, if σ switches from q to p at time t_2, we have

$$\lim_{\tau \downarrow t_2} v(\tau) - \lim_{\tau \uparrow t_2} v(\tau) = x(t_2)'(\bar{P} - P_{p,q})x(t_2),$$

where $\bar{P}$ is either equal to P_p^- in case t_2 is the last switching time or the solution at time t_2 of a final value problem of the form

$$-\dot{P} = S_p(P; \gamma), \qquad P(t_3) = P_{(\bar{p}, p)},$$

for some $\bar{p} \in \mathcal{P}$ and with $t_3 \geq t_2 + \tau_D$. However, we selected τ_D sufficiently large so that $\|P(t_2) - P_p^-\| \leq \epsilon$. Therefore we always have $\|\bar{P} - P_p^-\| \leq \epsilon$ whether or not t_2 is the last switching time. From this and (24) we conclude that $\bar{P} - P_{p,q} \leq 0$ and therefore

$$\lim_{\tau \downarrow t_2} v(\tau) - \lim_{\tau \uparrow t_2} v(\tau) = x(t_2)'(\bar{P} - P_{p,q})x(t_2) \leq 0.$$

This means that even at points of discontinuity of v, this signal decreases and therefore also here $v(t) \leq v(0), \forall t \geq 0$. As in the proof of Theorem 3 we can then conclude that $\|y\|_2 \leq \gamma\|u\|_2$ for a zero initialization of (2). This finishes the proof since we showed that the RMS gain over $\mathcal{S}[\tau_D]$ is indeed no larger than γ. ∎

Theorem 4 suggests the bisection algorithm in Table I to compute the slow-switching RMS gain $\mathfrak{g}_{\text{slow}}$ of (2). The verification of the matrix-inequality conditions in step 3 is computationally simple when $\mathcal{P}$ is a finite set. However, if $\mathcal{P}$ is a continuum (i.e., when one can switch among an infinitely large family of systems) the above algorithm may not be practical.

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1) *Select a tolerance ϵ, an upper bound γ_{up} and a lower bound γ_{low} on* $\mathfrak{g}_{\text{slow}}$ *(e.g., $\gamma_{\text{low}} := \mathfrak{g}_{\text{static}}$ and γ_{up} taken from [2])*
2) *If $\gamma_{\text{up}} - \gamma_{\text{low}} \leq 2\epsilon$ then stop (the RMS gain $\mathfrak{g}_{\text{slow}}$ is away from $\frac{\gamma_{\text{up}} + \gamma_{\text{low}}}{2}$ by less than ϵ)*
3) *Otherwise set $\gamma = \frac{\gamma_{\text{up}} + \gamma_{\text{low}}}{2}$ and*

   If $\exists p, q \in \mathcal{P}$ $P_q^+ - P_p^- \not\geq 0$ then set $\gamma_{\text{low}} = \gamma$
   If $\forall p, q \in \mathcal{P}$ $P_q^+ - P_p^- > 0$ then set $\gamma_{\text{up}} = \gamma$

   and go back to step 2

TABLE I
BISECTION ALGORITHM TO COMPUTE THE SLOW-SWITCHING RMS GAIN

VI. EXAMPLE

In this section we illustrate the results presented above in the context of a simple example. We consider here the control of the roll angle of an aircraft and use the following process model taken

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from [11, p. 381]: $H(s) = \frac{-1000}{s(s+.875)(s+50)}$. We designed two feedback controllers to be connected to the process as in Figure 1. In this figure u denotes the control input, y the process output, r a reference signal, d an input disturbance, and n measurement noise. The two
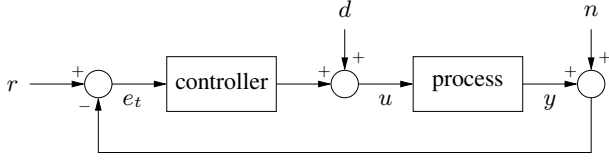

Fig. 1.   Feedback configuration

controllers designed essentially differ by the closed-loop bandwidth that they achieve: Controller K_1 exhibits small closed-loop bandwidth and is therefore not very sensitive to noise but exhibits a slow response, whereas controller K_2 has larger bandwidth and is therefore faster but more sensitive to noise. Both controllers were designed using LQR/LQG and their transfer functions were given by $K_1(s) \approx \frac{-20.73(s+50.03)(s+0.9644)}{(s+49.85)(s^2+17.45s+144.3)}$ and $K_2(s) \approx \frac{-1980(s+51.01)(s+0.9901)}{(s+57.17)(s^2+71.28s+2925)}$ (cf. [8] details). Although both controllers exhibit different bandwidths (cf. Figure 2), their closed-loop RMS gains from the noise n to the output y are relatively similar: 1.008 (.069dB) and 1.01 (.087dB) for controllers K_1 and K_2, respectively.
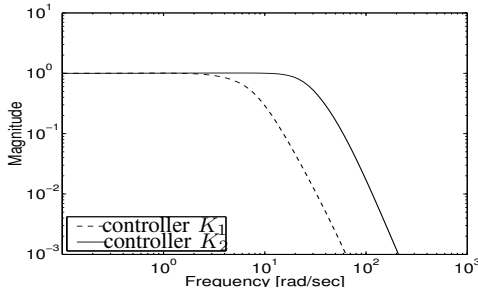

Fig. 2.   Magnitude Bode plots for the closed-loop transfer functions from the noise n to the output y, for the two time-invariant controllers.

Suppose now that we implement the following *multi-controller* [1] to switch between the two controllers :

$$\dot{z} = F_\sigma z + g_\sigma e_t, \qquad u = h_\sigma z,$$

where each $\{F_p, g_p, h_p\}$, $p \in \{1, 2\}$ denotes a balanced realization of the controller transfer function $K_p(s)$, and $\sigma : [0, \infty) \to \{1, 2\}$ a switching signal that effectively determines which controller is connected to the process at each instant of time. The closed-loop system is then a switched system of the form

$$\dot{x} = A_\sigma x + b_\sigma(r - n) + fd, \qquad e_t = c_\sigma x + r - n, \qquad (27)$$

where $A_p := \begin{bmatrix} A & bh_p \\ -g_p c & F_p \end{bmatrix}$, $b_p := \begin{bmatrix} 0 \\ g_p \end{bmatrix}$, $c_p := \begin{bmatrix} -c & 0 \end{bmatrix}$, $f := \begin{bmatrix} b \\ 0 \end{bmatrix}$, $p \in \{1, 2\}$, and $\{A, b, c\}$ denotes a minimal realization of the process transfer matrix $H(s)$. Using the algorithm in Table I, we obtained for (27) a slow-switching RMS gain from the noise n to the output y of 9.04 (19.1 dB). This example shows that noise entering the system can be greatly amplified because of switching. Note that the RMS gain depends on the realizations chosen for the controllers. Here, we chose balanced realizations because they are generally believed to be well conditioned and, in fact, other realizations tried led to larger RMS gains. An interesting question that remains open is the design of controllers and the choice of realizations that minimize the slow-switching RMS gain. The study of controller realizations that are adequate for switching was pursued in [12]. However, this problem was only studied from a stability perspective and the effect of controllers realizations on RMS gains was not addressed.

VII. Conclusions

We presented a method to compute the slow-switching RMS gain of a switched linear system. The algorithm proposed uses the fact that a given constant γ provides an upper bound on the RMS gain whenever there is a separation between *all* the stabilizing and *all* the antistabilizing solutions to the AREs of the systems being switched. We are now in the process of expressing this condition in the form of Linear Matrix Inequalities. The motivation is to construct efficient algorithms that can be used to design switching controllers that minimize the RMS gain of the closed-loop.

One of the implications of this work is the observation that the RMS gain of a switched linear system depends on the realizations of the systems being switched. This is not the case for linear time-invariant systems, for which the RMS gains essentially provide a measure of how much a sinusoidal signal is amplified (ignoring the effects of initial conditions and therefore of realizations). This is because the probe signals that produce maximum energy amplification for linear time-invariant systems can be chosen to be sinusoids. However, for switched systems this is not the case (as demonstrated by the example in the introduction). In fact, for switched systems the probe signals that define the RMS gain may well use transients to maximize energy amplification. Clearly, this must be taken into account when studying questions related to performance and/or robust stability of switched systems.

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