STABILIZATION OF NONHOLONOMIC INTEGRATORS VIA LOGIC-BASED SWITCHING

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Abstract. This paper demonstrates how to stabilize a nonholonomic integrator using a hybrid control law employing switching and logic. Results concerning stability and performance of the resulting hybrid system are derived. One of the main advantages of this type of controller is that the stability analysis can be made using fairly straightforward arguments. The methodology used is generalizable to a larger class of control problems related to nonholonomic systems.

Keywords. Switching algorithms; Supervisory control; Nonlinear systems; Stability analysis; Performance analysis.

1. INTRODUCTION

Over the last decade there has been a great deal of research on the problem of stabilizing systems which are locally null controllable but fail to meet Brockett's condition for smooth stabilizability (Brockett 1983): *Given the system*

$$\dot{x} = f(x,u)\,, \quad x(t_0) = x_0\,, \quad f(0,0) = 0 \qquad (1)$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *continuously differentiable. If* (1) *is smoothly stabilizable, i.e. there exists a continuously differentiable function* $g(x)$ *such that the origin is an asymptotically stable equilibrium point of* $\dot{x} = f(x,g(x))$*, with stability defined in the Lyapunov sense, then the image of* f *must contain an open neighborhood of the origin.*

In (Sontag 1990) it is noted that this condition extends to the class of time-invariant feedback laws that are only locally Lipschitz and recently it was shown (Ryan 1994) that Brockett's condition also extends to a even larger class which includes a wide variety of time-invariant discontinuous feedback laws. A prototype example of a system that is not smoothly stabilizable is the so called "nonholonomic integrator" (Brockett 1983):

$$\dot{x}_1 \;=\; u_1\,, \quad \dot{x}_2 \;=\; u_2\,, \quad \dot{x}_3 \;=\; x_1\,u_2 - x_2\,u_1 \qquad (2)$$

where $x = [x_1, x_2, x_3]' \in \mathbb{R}^3$ and $u = [u_1, u_2]' \in \mathbb{R}^2$.

Since the image of the mapping $[x,u]' \mapsto [u_1, u_2, x_1\,u_2 - x_2\,u_1]'$ does not contain the point $[0,0,\epsilon]'$ for any $\epsilon \neq 0$, Brockett's condition implies that there is no time-invariant continuously differentiable control law that asymptotically stabilizes the origin.

Possible control laws that avoid the difficulties implied by Brockett's condition are time-varying periodic controllers (Pomet 1992) (M'Closkey and Murray 1993) (Brockett 1993*b*) (Kapitanovsky *et al.* 1993), stochastic control laws (Brockett 1993*b*), and sliding modes control laws (Bloch and Drakunov 1994). The solution proposed in this paper falls in the class of hybrid control laws, namely those employing both con-

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tinuous dynamics and discrete logic (Tavernini 1987), (Morse *et al.* 1992), (Back *et al.* 1993), (Brockett 1993*a*), (Branicky *et al.* 1994) . Applications of these type of control laws to nonholonomic systems can be found in (Bloch *et al.* 1992), (Kolmanovsky *et al.* 1994) where global convergence to the origin is achieved in finite time; however, these controls may result in chattering in the presence of unmodeled dynamics.

This paper is organized as follows. In section 2, motivation and a formal specification for the control law is presented. Section 3 contains the main results of the paper and includes an outline of the stability analysis of the hybrid system. In section 4, closed-loop performance is analyzed in terms of convergence rates and in section 5 generalizations for higher dimensional systems are discussed. The reader is referred to (Hespanha 1995) for detailed proofs of some technical results.

2. SWITCHING CONTROLLER

Consider again the nonholonomic integrator (2). Whatever control law is chosen, whenever x_1 and x_2 are both zero, $\dot{x}_3$ will also be zero and x_3 will remain constant, until x_1 and/or x_2 became non zero. Furthermore, whenever x_1 and x_2 are "small", only "large" control signals will be able to produce significant changes in x_3. A plausible strategy to make the origin an attractor of the close-loop system is to keep the state away from the axis $x_1 = x_2 = 0$ while x_3 is large, and, as x_3 decreases, to let x_1 and x_2 became small. In this way one may be able to make the state approach the origin without using very large inputs. This reasoning leads to a natural partition of the state into two components: the one that should go to zero as fast as possible (in this case x_3) and the one that should not decay too fast (in this case $\{x_1, x_2\}$). Grouping the norms of these components into a 2-dimensional vector one defines a map W from $\mathbb{R}^3$ into the closed positive quadrant[2] $\Omega \subset \mathbb{R}^2$ by

$$W(x_1, x_2, x_3) = \big((x_3)^2, (x_1)^2 + (x_2)^2\big) \qquad (3)$$

Denoting by $w(t) = (w_1(t), w_2(t))$ the image under W of a trajectory $x(t)$, the basic idea is to pick a control law so as to cause $w_2(t)$ to remain nonzero whenever $w_1(t) \neq 0$, while at the same time insure that both signals tend to zero. A "phase portrait" of such a $w(t)$ is shown in Fig. 1.

This kind of decomposition of the state is very natural for systems that violates Brockett's condition. For example, for a nonholonomic system in the chained power form (Teel *et al.* 1992)

[2] By the closed positive quadrant of $\mathbb{R}^2$ it is meant the set $\Omega = \{(w_1, w_2) \in \mathbb{R}^2 : w_1 \geq 0, w_2 \geq 0\}$.

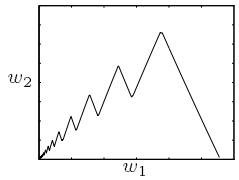

Fig. 1. Image by $W : \mathbb{R}^3 \to \Omega$ of a typical trajectory.

$$\begin{aligned} \dot{x}_1 &= u_1 & \dot{x}_3 &= x_1 u_2 \\ \dot{x}_2 &= u_2 & \dot{x}_4 &= \frac{1}{2}(x_1)^2 u_2 \\ & & &\vdots \\ & & \dot{x}_n &= \frac{1}{(n-2)!}(x_1)^{n-2} u_2 \end{aligned}$$

one would seek to take $[x_2, x_3, \ldots, x_n]'$ to zero as fast as possible, without letting x_1 decrease too fast. For systems not in a special form like the preceding, a more complex decomposition may be required, but it is also clear that there exists a "singular set" embedded in the state space that one wants to avoid. For a general n-dimensional system $\dot{x} = f(x, u)$, this set seems to be the set of points in $\mathbb{R}^n$ that do not belong to the image of $f(x, u)$. For the systems just described, these sets are linear subspaces of the state space: the line $\{[0, 0, x_3]' \in \mathbb{R}^3\}$ for the nonholonomic integrator, and the hyperplane $\{[0, x_2, \ldots, x_n]' \in \mathbb{R}^n\}$ for the chained system.

Several control laws can achieve the aforementioned type of behavior. The one presented in this paper has the virtue of being easy to analyze, not only in terms of stability, but also in terms of performance. The control law proposed is constructed as follows:

(1) Pick four continuous functions $\pi_k : \mathbb{R} \to \mathbb{R}$, $k \in \mathcal{I} \triangleq \{1, 2, 3, 4\}$ with the following properties:
   (a) $\pi_k(0) = 0$ for each $k \in \{1, 2, 3, 4\}$;
   (b) for each $k \in \{1, 2, 3, 4\}$ and for all $w > 0$
   $$0 < \pi_1(w) < \pi_2(w) < \pi_3(w) < \pi_4(w)$$
   (c) π_1 and π_2 are bounded;
   (d) π_2 and π_3 are monotone nondecreasing;
   (e) π_4 is differentiable on some non-empty, open-half interval $(0, c]$ and
   $$\frac{\mathrm{d}\pi_4}{\mathrm{d}w}(w) < \frac{1}{w}, \qquad w \in (0, c]$$

(2) Partition the closed positive quadrant $\Omega \subset \mathbb{R}^2$ into four overlapping regions
   $$\mathcal{R}_1 = \{(w_1, w_2) \in \Omega : 0 \leq w_2 < \pi_2(w_1)\}$$
   $$\mathcal{R}_2 = \{(w_1, w_2) \in \Omega : \pi_1(w_1) < w_2 < \pi_4(w_1)\}$$
   $$\mathcal{R}_3 = \{(w_1, w_2) \in \Omega : w_2 > \pi_3(w_1)\}\}$$
   $$\mathcal{R}_4 = \{(0, 0)\}$$

(3) Define the control law
   $$u(t) = g_{\sigma(t)}\big(x(t)\big) \qquad (4)$$

where $\sigma(t)$ is a piecewise constant switching signal taking values in $\mathcal{I}$ and

$$g_1(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; g_2(x) = \begin{bmatrix} x_1 + x_2 x_3 \\ x_2 - x_1 x_3 \end{bmatrix}, \\ g_3(x) = \begin{bmatrix} -x_1 + x_2 x_3 \\ -x_2 - x_1 x_3 \end{bmatrix}, \; g_4(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

The signal $\sigma(t)$ is defined to be continuous from the right at every point and it is determined recursively by

$$\sigma(t) = \phi(x(t), \sigma^-(t)), \qquad \sigma^-(t_0) = \sigma_0 \in \mathcal{I} \tag{6}$$

where $\sigma^-(t)$ denotes the limit of $\sigma(\tau)$ from below as $\tau \to t$ and $\phi : \mathbb{R}^3 \times \mathcal{I} \to \mathcal{I}$ is the transition function

$$\phi(x, \sigma) = \begin{cases} \sigma & \text{if } W(x) \in \mathcal{R}_\sigma \\ \max\limits_{\mathcal{I}} \{k : W(x) \in \mathcal{R}_k\} & \text{otherwise} \end{cases} \tag{7}$$

Example 1. A typical example would be $\pi_1(w) = (1 - \mathrm{e}^{-w_1})$, $\pi_2 = 2\pi_1$, $\pi_3 = 3\pi_1$ and $\pi_4 = 4\pi_1$. The corresponding regions $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ are shown on Fig. 2.

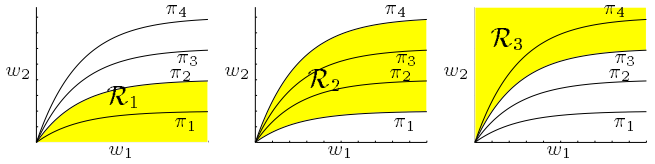

Fig. 2. A typical partition of Ω.

The type of control proposed is similar as that in (Back *et al.* 1993) and can be seen as an extension of the hysteresis switching algorithm considered in (Morse *et al.* 1992). Its appeal comes from the fact that it naturally excludes the possibility of infinitely fast chattering and therefore does not require the concept of generalized solution in Filippov's sense (Filippov 1964) (Bloch and Drakunov 1994).

3. MAIN RESULTS

The aim of this section is to study the closed loop hybrid dynamical system described in the previous section. The relevant equations are

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned} \qquad \begin{aligned} u &= g_\sigma(x) \\ \sigma &= \phi(x, \sigma^-) \end{aligned}$$

where $x = [x_1, x_2, x_3]' \in \mathbb{R}^3$, $u = [u_1, u_2]' \in \mathbb{R}^2$ and $g_\sigma(x)$ and $\phi(x, \sigma)$ are defined in equations (5) and (7) respectively. Although the closed loop system is not globally Lipschitz, global existence of solutions can be easily justified. Indeed, let $[x_1(t), x_2(t), x_3(t)]'$ be the continuous part of the state at time t and define $w_1(t) = x_3(t)^2$ and $w_2(t) = x_1(t)^2 + x_2(t)^2$. Simple algebra shows that

$$\begin{aligned} \dot{w_1}(t) &\le 2(\max\{w_2, 1\} + 1) \max\{w_1, 1\} \\ \dot{w_2}(t) &\le 2 \max\{w_2, 1\} + 2 \end{aligned}$$

Since the bound for $\dot{w_2}(t)$ is globally Lipschitz, $w_2(t)$ must be bounded on any finite time interval. This in turn makes the bound for $\dot{w_1}(t)$ globally Lipschitz which guarantees that $w_1(t)$ will also be bounded on the same interval. Since the distance between two points in Ω where consecutive switchings can occur is always nonzero and all signals are bounded, infinitely fast chattering cannot occur in finite time so global existence of solution is assured. Note however that the interval between discontinuities of $\sigma(t)$ in the whole interval $[t_0, +\infty)$ may not be bounded below by any positive constant.

The usual definition of Lyapunov stability extends in a natural way to hybrid systems: the origin is a Lyapunov stable equilibrium point of the system $\dot{x} = f_\sigma(x)$, $\sigma = \phi(x, \sigma^-)$, $t \ge t_0$, $\sigma \in \mathcal{I}$, and $x \in \mathbb{R}^n$ if

(1) $f_\sigma(0) = 0$ for all $\sigma \in \mathcal{I}$ such that $\phi(0, \sigma) = \sigma$, and
(2) for all $\epsilon > 0$ there exists a $\delta > 0$ such that for any set of initial conditions $x(t_0) = x_0 \in \mathbb{R}^n$ and $\sigma^-(t_0) = \sigma_0 \in \mathcal{I}$, if $\|x_0\| < \delta$ then $\|x(t)\| < \epsilon$, $\forall t \ge t_0$

Moreover, if for any initial conditions, the continuous part of the state converges to the origin, then the origin is said to be globally asymptotically stable. The main result of this paper is the following theorem which guarantees global asymptotic stability for overall system.

Theorem 2. Let Σ denote the hybrid system defined by (2), (4) and (7).

(1) For any set of initial conditions $x(t_0) = x_0 \in \mathbb{R}^3$ and $\sigma^-(t_0) = \sigma_0 \in \mathcal{I}$ the solution $x(t)$ of Σ converges to zero as $t \to +\infty$.
(2) The origin is a Lyapunov stable equilibrium point of Σ.

The proof of this theorem requires the notion of "positive invariant" set. A set $\mathcal{S} \subset \mathbb{R}^3$ is *positive invariant* if for any $x(t_0) \in \mathcal{S}$ and any $\sigma^-(t_0) \in \mathcal{I}$, $x(t) \in \mathcal{S}$ for all $t \ge t_0$. Often, it is also convenient to talk of invariance of subsets of Ω with respect to $w(t) \stackrel{\Delta}{=} W(x(t))$. In this case it is said that $\mathcal{R} \subset \Omega$ is positive invariant with respect to $w(t)$ if for any $x(t_0) \in \mathbb{R}^3$ such that $w(t_0) \in \mathcal{R}$ and any $\sigma^-(t_0) \in \mathcal{I}$, $w(t) \in \mathcal{R}$ for all $t \ge t_0$. The relation between the two notions of invariance is easy to establish. Given $\mathcal{R} \subset \Omega$ and its inverse image $\mathcal{S}$ by W, i.e.

$$\mathcal{S} = W^{-1}(\mathcal{R}) \triangleq \{x \in \mathbb{R}^3 : W(x) \in \mathcal{R}\}$$

$\mathcal{R}$ is positive invariant with respect to $w(t)$ if and only if $\mathcal{S}$ is positive invariant. Associating to each region $\mathcal{R}_k \subset \Omega$ ($k \in \{1,2,3,4\}$) its inverse image $\mathcal{S}_k \subset \mathbb{R}^3$ by W, one can speak of invariance of the sets $\mathcal{R}_k$, $\mathcal{S}_k$, its closures, complements, finite unions and/or intersections. In the sequel, the interior, closure, and boundary of a set $\mathcal{R} \subset \mathbb{R}^n$ are denoted by $\text{Int}(\mathcal{R})$, $\overline{\mathcal{R}}$, and $\partial\mathcal{R}$ respectively. For $x \in \mathbb{R}^n$, $B_\delta(x)$ denotes an open ball of radius $\delta > 0$ centered at x.

The following lemma gives a procedure to check invariance of a subset of the state space, just by observing the vector field in a neighborhood of the subset's boundary. The formulation used does not require the vector field to point strictly to the inside of the region.

Lemma 3. Given the system

$$\dot{x}(t) = f(x(t),t),\ x(t_0) = x_0,\ f(0,t) = 0\ t \geq t_0 \qquad (8)$$

with $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ locally Lipschitz with respect to x and piecewise continuous with respect to t. Let $x : [t_0, T) \to \mathbb{R}^n$ be a solution of (8) and $\mathcal{S}$ a closed subset of $\mathbb{R}^n$ such that

(1) $x(t_0)$ belong to $\mathcal{S}$ and
(2) for every $\bar{t}$ such that $x(\bar{t}) \in \partial\mathcal{S}$, there exist 3 positive constants δ_1, δ_2, and δ_3 such that for all $\epsilon \in [0, \delta_1]$ and all $\tilde{x} \in B_{\delta_2}(x(\bar{t})) \setminus \mathcal{S}$

$$x(\bar{t}) + \epsilon f(\tilde{x}, t) \in \mathcal{S}\,, \quad t \in [\bar{t}, \bar{t} + \delta_3]$$

Then $x(t) \in \mathcal{S}$ for all $t \in [t_0, T)$.

Condition 2. can be interpreted as requiring the velocity vector to point to the inside of $\mathcal{S}$ in a neighborhood[3] of $x(\bar{t})$ during some interval of positive length starting at $\bar{t}$. This lemma is a straight forward consequence of the mean value theorem (Hespanha 1995).

Outline of Theorem 2's Proof: A detailed proof of Theorem 2 can be found in (Hespanha 1995).

1. Convergence: Denoting by $w(t)$ the image by $W : \mathbb{R}^3 \to \Omega$ of an arbitrary trajectory $x(t)$, while the state is in the region $\mathcal{S}_1$ {and therefore $\sigma(t) = 1$ or 2}, $w_2(t)$ grows without bound and therefore at some finite time t_1 this region will be abandoned. Furthermore, observing the direction of the velocity vector over its boundary, one concludes by Lemma 3 that the state will never get into $\mathcal{S}_1$ again. Pursuing the same reasoning, one concludes that after some finite time t_2 the state will be trapped into $\overline{\mathcal{S}}_3$. At this point the switching signal $\sigma(t)$ will always be equal to 2, 3 or 4 and therefore

$$\dot{w_1} = -2w_2w_1\,, \quad \forall t \geq t_1$$

but since the state is in $\overline{\mathcal{S}}_3$, $w_2 \geq \pi_2(w_1)$ and therefore

$$\dot{w_1} \leq -2\pi_2(w_1)w_1\,, \quad \forall t \geq t_1$$

hence $w_1(t) \to 0$ as $t \to +\infty$. In order to show that $w_2(t)$ also converges to zero, another invariant region will be used. Analyzing the direction of the velocity vectors, it is possible to conclude that once the state gets into $\overline{\mathcal{R}_2 \cap \mathcal{R}_3}$ it will never leave this region, therefore two possibilities exist. Either the state will get into $\overline{\mathcal{R}_2 \cap \mathcal{R}_3}$ at some finite time t_3 and will never leave this region again {which means that $w_2(t) \leq \pi_4(w_1(t))$ for all $t \geq t_3$}; or it will remain in $\overline{\mathcal{S}}_3 \setminus \overline{\mathcal{S}}_2$ which means that $\sigma(t)$ will always be equal to 3 and therefore

$$\dot{w_2} = -2w_2\,, \quad \forall t \geq t_0 \qquad (9)$$

In either case convergence to zero it achieved. It is actually possible to show that the latter possibility can never occur and that after some finite time t_3 the state will never leave $\overline{\mathcal{R}_2 \cap \mathcal{R}_3}$. The reason for this is that equation (9) would lead to a decay of w_2 too fast and the state would eventually exit the region $\overline{\mathcal{S}}_3$ which can not happen. □

2. Stability: Let us start be assuming that $x(t)$ starts outside $\mathcal{S}_1$. It was argued before that in this case the state will never get into this region again, therefore $\sigma(t)$ will always be equal to 2, 3 or 4 and

$$\dot{w_1} = -2w_2w_1 \quad \text{and} \quad \dot{w_2} = \pm 2w_2$$

This shows that $w_1(t)$ is ever decreasing and $w_2(t)$ will be decreasing outside $\overline{\mathcal{R}_2 \cap \mathcal{R}_3}$ and smaller than $\pi_4(w_1(t))$ once it gets inside $\overline{\mathcal{R}_2 \cap \mathcal{R}_3}$, therefore

$$w_1(t) \leq w_1(t_0) \ \text{ and } \ w_2(t) \leq \max\{w_2(t_0), \pi_4(w_1(t_0))\}$$

This shows that making $w(t_0)$ small enough one can keep the whole trajectory arbitrarily close to the origin. The only case that was not covered by the above argument was when $\sigma(t_0) = 1$, which can only happen when $x(t)$ starts in $\mathcal{S}_1$. But it was also argued before that once $x(t)$ leaves this region it will never enter it again therefore one can only have problems in the interval $[t_0, t_1)$ for which $\sigma(t) = 1$. Since while $\sigma(t)$ is equal to 1 the system is linear, it is not surprising to discover that nothing can go wrong. A careful study of this situation can be found in (Hespanha 1995). Fig. 3 shows the "phase portrait" of $w(t)$ for a typical trajectory. □

[3] According to the lemma, only the portion of the neighborhood outside $\mathcal{S}$ (therefore, excluding the boundary) needs to be considered.

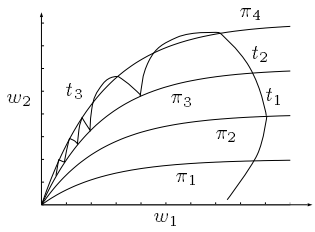

Fig. 3. Image by $W : \mathbb{R}^3 \to \Omega$ of a typical trajectory.

4. PERFORMANCE

One of the interesting features of the proposed control law is that it is simple enough to allow an analysis of the closed loop hybrid system in terms of performance. It is argued in the proof of Theorem 2 that there exists a finite time $t_3 \geq t_0$ such that for all $t \geq t_3$

$$\pi_3\big(w_1(t)\big) \leq w_2(t) \leq \pi_4\big(w_1(t)\big) \qquad (10)$$

Assuming that $\pi_3(.)$ and $\pi_4(.)$ are differentiable at 0 with distinct nonzero derivatives then there exist positive constants δ, k_1 and k_2 $(k_1 < k_2)$ such that

$$\pi_3(w_1) \geq k_1 w_1\,, \quad \pi_4(w_1) \leq k_2 w_1\,, \qquad \forall w_1 \in [0, \delta]$$

Since $w_1(t)$ converges to zero, there exists a finite time after which $w_1(t) \leq \delta$. Without loss of generality assume that this time is t_3. Therefore for all $t \geq t_3$

$$k_1 w_1(t) \leq w_2(t) \leq k_2 w_1(t)$$

But in the region defined by (10), $\dot{w}_1 = -2w_1 w_2$ which means that $-2k_2 w_1^2(t) \leq \dot{w}_1(t) \leq -2k_1 w_1^2(t)$, therefore (Hespanha 1995)

$$\frac{w_1(t_3)}{1 + 2k_2 w_1(t_3)(t - t_3)} \leq w_1(t) \leq \frac{w_1(t_3)}{1 + 2k_1 w_1(t_3)(t - t_3)}$$
$$\frac{k_1 w_1(t_3)}{1 + 2k_2 w_1(t_3)(t - t_3)} \leq w_2(t) \leq \frac{k_2 w_1(t_3)}{1 + 2k_1 w_1(t_3)(t - t_3)}$$

This shows that this system is not exponentially stable, in fact, $\|x(t)\|$ decays with $1/\sqrt{t}$. However, simple changes in the control laws and the map W can provide faster decay rates. Defining the map $W : \mathbb{R}^3 \to \Omega$ by

$$W(x_1, x_2, x_3) = \big(|x_3|^\alpha, (x_1)^2 + (x_2)^2\big)$$

where α is any fixed constant larger than 1, and

$$g_1(x) = \begin{bmatrix}1 & 1\end{bmatrix}', \; g_4(x) = \begin{bmatrix}0 & 0\end{bmatrix}',$$
$$g_{2,3}(x) = \begin{bmatrix}\pm x_1 + x_2 x_3 \\ x_2 - x_1 \operatorname{sgn}(x_3)|x_3|^\beta\end{bmatrix} \qquad (11)$$

where β is any positive constant, it is possible to show that the decay rate of $\|x(t)\|$ would be $t^{-\alpha/2(\alpha+\beta-1)}$ therefore by making $\alpha \to 1$ and $\beta \to 0$ it is possible to get faster (but still algebraic) convergence rates. Figs. 4 and 5 show simulations with the control law defined by equations (4), (7), and (11) for $\alpha = 1.1$ and $\beta = 0.1$. The corresponding decay rate for the norm of the state is 2.75.

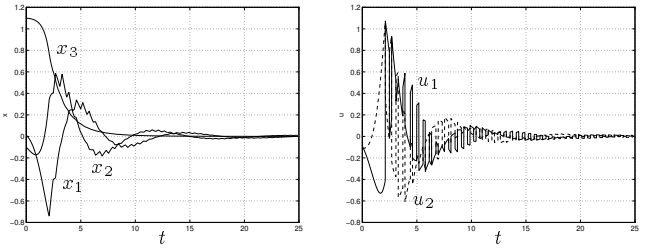

Fig. 4. Simulation of the hybrid system (2), (4), (7), and (11) for $\alpha = 1.1$ and and $\beta = 0.1$.

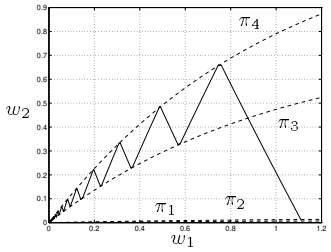

Fig. 5. Image by $W : \mathbb{R}^3 \to \Omega$ of the trajectory shown in Fig.4.

5. GENERALIZATION TO THE VECTOR CASE

The results presented before generalize to the following higher dimensional nonholonomic integrator

$$\dot{x}_1 = u_1\,, \quad \dot{x}_2 = u_2\,, \quad \dot{x}_3 = x_1' u_2 - x_2' u_1$$

where $x_1, x_2, u_1, u_2 \in \mathbb{R}^n$ and $x_3 \in \mathbb{R}$. In this case, the map $W : \mathbb{R}^{2n+1} \to \Omega$ must be

$$W(x) = \big((x_3)^2, x_1'\, x_1 + x_2'\, x_2\big)$$

and the control laws

$$g_1(x) = \begin{bmatrix}e_1 \\ e_1\end{bmatrix}, \; g_{2,3}(x) = \begin{bmatrix}\pm x_1 + x_2 x_3 \\ \pm x_2 - x_1 x_3\end{bmatrix}, \; g_4(x) = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

where $e_1 = \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}'$. Another possible generalization to a higher dimensional nonholonomic integrator is (Bloch and Drakunov 1994)

$$\dot{x}_1 = u_1\,, \quad \dot{x}_2 = u_2\,, \quad \dot{x}_3 = x_1\, u_2 - x_2\, u_1$$

where $x_1, u_1 \in \mathbb{R}$ and $x_2, x_3, u_2 \in \mathbb{R}^n$. In this case the mapping $W : \mathbb{R}^{2n+1} \to \Omega$ must be

$$W(x) = (x_3'\, x_3, (x_1)^2 + x_2'\, x_2)$$

and the control laws

$$g_1(x) = \begin{bmatrix}1 \\ e_1\end{bmatrix}, \; g_{2,3}(x) = \begin{bmatrix}\pm x_1 + x_2'\, x_3 \\ \pm x_2 - x_1\, x_3\end{bmatrix}, \; g_4(x) = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

Note that in either case, while $\sigma = 2$ or 3 one still has $\dot{w}_1 = -2w_1 w_2$ and $\dot{w}_2 = \pm 2w_2$, which is enough for all the proofs to go through.

6. CONCLUSIONS

The main purpose of this paper is to show that logic-based switching can be used in the control of nonholonomic systems. The application of this type of control to the nonholonomic integrator suggests a general methodology to stabilize other nonholonomic systems. The example presented shows that straight forward arguments can be used to prove stability and even evaluate the performance of the closed loop system. Similar controllers can be implemented to stabilize other types of nonholonomic systems, in particular chained systems in the power form. A stability analysis for these systems can be made along the same lines as the one presented here. It was shown that the state of the closed loop system obtained does not converge exponentially to zero but only at an algebraic rate. This is a characteristic shared by many controllers proposed for this type of systems. The specific controller proposed has the drawback that the resulting control signals are discontinuous, but it should be emphasized that this is not a general characteristic of logic-based switching controllers. Further effort is being developed into the design of alternative control laws without this problem.

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Appendix A. REFERENCES

Back, A., J. Guckenheimer and M. Myers (1993). A dynamical simulation facility for hybrid systems. In: Grossman *et al.* (1993).

Bloch, A. M. and S. Drakunov (1994). Stabilization of a nonholonomic system via sliding modes. In: *Proc. of the 33rd CDC*. Vol. 3. pp. 2961–2963.

Bloch, A. M., M. Reyhanoglu and N. H. McClamroch (1992). Control and stabilization of nonholonomic dynamic systems. *IEEE Trans. Automat. Contr.* **37**(11), 1746–1757.

Branicky, M. S., V. S. Borkar and S. K. Mitter (1994). A unified framework for hybrid control: Background, model and theory. In: *Proc. of the 33rd CDC*. Vol. 4. pp. 4228–4234.

Brockett, R. W. (1983). Asymptotic stability and feedback stabilization. In: *Differential Geometric Control Theory* (R. W. Brockett, R. S. Millman and H. J. Sussmann, Eds.). pp. 181–191. Birkhäuser. Boston.

Brockett, R. W. (1993*a*). Hybrid models for motion control systems. In: *Essays in Control: Perspectives in the Theory and its Applications* (H. L. Trentelman and J. C. Willems, Eds.). pp. 29–53. Birkhäuser. Boston.

Brockett, R. W. (1993*b*). Pattern generation and feedback control of nonholonomic systems. In: *Proc. of the Workshop on Mechanics, Holonomy and Control*.

Filippov, A. F. (1964). Differential equations with discontinuous righthand side. *AMS Translations* **42**(2), 199–231.

Grossman, R. L., Nerode, A., Ravn, A. P. and Rishel, H., Eds.) (1993). *Hybrid Systems*. Vol. 736 of *Lecture Notes in Computer Science*. Springer-Verlag. New York.

Hespanha, J. P. (1995). Stabilization of nonholonomic integrators via logic-based switching. Technical Report 9501. Lab. for Control Science & Eng., Yale University.

Kapitanovsky, A., A. Goldenberg and J. Mills (1993). Design of smooth dynamic feedback for stabilization of nonholonomic systems. In: *Proc. of the 32nd CDC*. Vol. 1. pp. 965–970.

Kolmanovsky, I. V., M. Reyhanoglu and N. H. McClamroch (1994). Discontinuous feedback stabilization of nonholonomic systems in extended power form. In: *Proc. of the 33rd CDC*. Vol. 3. pp. 3469–3474.

M'Closkey, R. T. and R. M. Murray (1993). Nonholonomic systems and exponential convergence: Some analysis tools. In: *Proc. of the 32nd CDC*. Vol. 1. pp. 943–948.

Morse, A. S., D. Q. Mayne and G. C. Goodwin (1992). Applications of hysteresis switching in parameter adaptive control. *IEEE Trans. Automat. Contr.* **37**(9), 1343–1354.

Pomet, J. P. (1992). Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift. *Systems and Control Letters* **18**, 147–158.

Ryan, E. P. (1994). On brockett's condition for smooth stability and its necessity in a context of nonsmooth feedback. *SIAM J. Contr. Optimization* **32**(6), 1597–1604.

Sontag, E. D. (1990). *Mathematical Control Theory*. Vol. 6 of *Texts in Applied Mathematics*. Springer-Verlag. New York.

Tavernini, L. (1987). Differential automata and their discrete simulators. *Nonlinear Analysis Theory, Methods, and Applications* **11**(6), 665–683.

Teel, A., R. Murray and G. Walsh (1992). Nonholonomic control systems: From steering to stabilization with sinusoids. In: *Proc. of the 31st CDC*. Vol. 2. pp. 1603–1609.