LIMITS OF PERFORMANCE IN REFERENCE-TRACKING AND PATH-FOLLOWING FOR NONLINEAR SYSTEMS

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Abstract: We investigate limits of performance in reference-tracking and path-following and highlight an essential difference between them. For a class of nonlinear systems, we show that in reference-tracking, the smallest achievable $\mathcal{L}_2$ norm of the tracking error is equal to the least amount of control energy needed to stabilize the zero-dynamics of the error system. We then show that this fundamental performance limitation does not exist when the control objective is to force the output to follow a geometric *path* without a timing law assigned to it. This is true even when an additional desired speed assignment is required to be satisfied asymptotically or in finite time.

Keywords: Limits of performance, non-minimum phase nonlinear systems, path-following, reference-tracking, cheap-control.

1. INTRODUCTION

Fundamental performance limitations in *reference-tracking* for linear systems using feedback control have been quantified with classical Bode integrals and as the limits of cheap optimal control performance [Kwakernaak and Sivan, 1972; Middleton, 1991; Qiu and Davison, 1993; Seron *et al.*, 1999; Chen *et al.*, 2000]. In the absence of unstable zero dynamics (*non-minimum phase zeros*) and if the system is right invertible, perfect tracking of any reference signal is possible, that is, the $\mathcal{L}_2$ norm of the tracking error can be made arbitrarily small. However, this is not the case in the presence of unstable zero dynamics. A formula derived by Qiu and Davison [1993], see also [Su *et al.*, 2003], shows that the tracking error increases as the signal frequencies approach those of the unstable zeros.

For step reference signals, Seron *et al.* [1999] reinterpreted the Qiu-Davison formula and generalized it to a class of nonlinear systems under the assumption that the relative degree is one and the zero dynamics are anti-stable. They showed that the best attainable value of the $\mathcal{L}_2$ norm of the tracking error $e_T(t)$, denoted by J_T, is equal to the lowest control effort, measured by the $\mathcal{L}_2$ norm, needed to stabilize the zero dynamics driven by $e_T(t)$. It is its role as a stabilizing control input that prevents the output $y(t)$ from perfect tracking. Extensions to non-right-invertible systems are given in [Woodyatt *et al.*, 2002; Braslavsky *et al.*, 2002].

In this paper we show that these results hold for the problem of tracking any reference signal generated by a known exosystem. The zero dynamics to be stabilized are those of the error system driven by $e_T(t)$. As before, the best attainable value of J_T is the lowest energy needed to stabilize the zero-dynamics.

Path-following problems are concerned with the design of control laws that drive an object (robot arm, mobile robot, ship, aircraft, etc.) to reach and follow a geometric *path*. A secondary goal is to force the object moving along the path to satisfy some additional dynamic specification such

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as follow the path with some desired velocity. A common approach to the path-following problem is to parameterize the geometric path y_d by a *path variable* θ and then select a *timing law* for θ, [Hauser and Hindman, 1995; Al-Hiddabi and McClamroch, 2002; Skjetne *et al.*, 2004; Aguiar *et al.*, 2004; Aguiar *et al.*, 2004*a*; Aguiar and Hespanha, 2004*b*]. A framework for path-following as a method to avoid some limitations in reference-tracking was described in [Aguiar *et al.*, 2004*a*]. The key idea is to use θ as an additional control input to stabilize the unstable zero-dynamics while the original control variables keep the system on the path.
We show that for a class of nonlinear systems the fundamental performance limitations imposed on reference-tracking by unstable zero dynamics *do not apply* to the path-following problem. Furthermore, the same is true for the speed-assigned path-following problem in which a speed assignment is required to be satisfied asymptotically or in finite time.

In Section 2 we formulate the reference-tracking and path-following problems. In Section 3 we briefly review our recent results for non-minimum phase linear systems, [Aguiar *et al.*, 2004], showing that the path-following problems can be solved with arbitrarily small $\mathcal{L}_2$ norm of the path-following error. Section 4 presents the main results of the paper. Concluding remarks are given in Section 5.

2. REFERENCE-TRACKING AND PATH-FOLLOWING PROBLEMS

2.1 Reference-tracking

The classical reference-tracking problem is to design a feedback controller such that the closed-loop system is asymptotically stable and, for any reference signal $r(t)$ in a prescribed family, the plant output converges to $r(t)$.
For linear systems

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{1}$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^q$, and reference signals $r(t) \in \mathbb{R}^q$ generated by a known *exosystem*

$$\dot{w} = Sw, \quad r = Qw, \tag{2}$$

Davison [1976] and Francis [1977] show that the so-called *servomechanism or regulator problem* is solvable if and only if (A, B) is stabilizable, (C, A) is detectable, the number of inputs is at least as large as the number of outputs ($m \geq q$), and the zeros of (A, B, C, D) do not coincide with the eigenvalues of S. The *internal model approach*, [Francis and Wonham, 1976; Francis, 1977], designs the reference-tracking controller

$$u(t) = Kx(t) + (\Gamma - K\Pi)w(t),$$

where K is such that $(A + BK)$ is Hurwitz, and Π and Γ satisfy

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma, \\ 0 &= C\Pi + D\Gamma - Q. \end{aligned}$$

Then, the *tracking error* defined as

$$e_T(t) := y(t) - r(t)$$

converges to zero, and the transients

$$\tilde{x} := x - \Pi w, \qquad \tilde{u} := u - \Gamma w \tag{3}$$

are governed by $\dot{\tilde{x}} = (A + BK)\tilde{x}$, $\tilde{u} = K\tilde{x}$.
For the nonlinear regulator problem

$$\dot{x} = f(x, u), \qquad y = h(x, u), \tag{4}$$
$$\dot{w} = s(w), \qquad r = q(w), \tag{5}$$

where $f(0, 0) = 0$, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control, $y \in \mathbb{R}^q$ the output, $w \in \mathbb{R}^p$ the state of the exosystem, and $r \in \mathbb{R}^q$ the reference signal, Isidori and Byrnes [1990] show that it is solvable if and only if there exist smooth maps $\Pi(w)$, and $c(w)$, satisfying

$$\begin{aligned} &\frac{\partial \Pi}{\partial w} s(w) = f(\Pi(w), c(w)), \quad \Pi(0) = 0, \\ &h(\Pi(w), c(w)) - q(w) = 0, \quad c(0) = 0. \end{aligned} \tag{6}$$

Krener [1992] presents necessary and sufficient conditions for local solvability of (6) when either the exosystem has a semisimple pole structure or the plant has a semisimple zero structure.

2.2 Path-following

In path-following, the output $y(t)$ is required to reach and follow a geometric path $y_d(\theta)$, where $\theta \in \mathbb{R}$ is the path variable. In this paper, the path $y_d(\theta)$ is assumed to be generated by the exosystem

$$\begin{aligned} &\frac{d}{d\theta} w(\theta) = s(w(\theta)), \quad w(\theta_0) = w_0 \\ &y_d(\theta) = q(w(\theta)), \end{aligned} \tag{7}$$

where $w \in \mathbb{R}^p$, $y_d \in \mathbb{R}^q$, and $\theta_0 := \theta(0)$. For a given *timing law* $\theta(t)$, the *path-following error* is defined as

$$e_P(t) := y(t) - y_d(\theta(t)). \tag{8}$$

We consider the following two path-following problems:
Geometric path-following: For a desired path $y_d(\theta)$, design a controller that achieves:

i) *boundedness:* the state $x(t)$ is uniformly bounded for all $t \geq 0$ and for every $(x(0), w(\theta(0))) = (x_0, w_0)$, in some neighborhood of $(0, 0)$,

ii) *error convergence:* the path-following error $e_P(t)$ converges to zero as $t \to \infty$, and

iii) *forward motion:* $\dot{\theta}(t) > c$ for all $t \geq 0$, where c is a positive constant.

Speed-assigned path-following: In addition to the geometric path-following task, a constant speed $v_d > 0$ is assigned and it is required that either $\dot{\theta}(t) \to v_d$ as $t \to \infty$, or $\dot{\theta}(t) = v_d$ for all $t \geq T$ and some $T \geq 0$.

As illustrated by Skjetne *et al.* [2004], these path-following problems provide natural settings for many engineering applications. From a theoretical standpoint our main interest is to determine whether the freedom to select a timing law $\theta(t)$ can be used to achieve an arbitrarily small $\mathcal{L}_2$ norm of the path-following error, that is, whether $\delta^\star > 0$ in

$$\int_0^\infty \|e_P(t)\|^2 \, dt \leq \delta^\star \tag{9}$$

can be made arbitrarily small.

3. LIMITS OF PERFORMANCE FOR LINEAR SYSTEMS

3.1 Reference-tracking

In reference-tracking, to satisfy the requirement that

$$\int_0^\infty \|e_T(t)\|^2 \, dt \leq \delta^\star, \tag{10}$$

for an arbitrary $\delta^\star > 0$, while keeping the closed-loop system stable, the zeros of (1) must lie in the open left half-plane $\mathbb{C}^-$, [Kwakernaak and Sivan, 1972].

The non-minimum phase zeros, that is the zeros in $\mathbb{C}^+$, impose a fundamental limitation on the attainable tracking performance (10). This is revealed by the fact, [Kwakernaak and Sivan, 1972], that the limit J_T, as $\epsilon \to 0$, of the optimal value of the cheap control cost functional

$$J_\epsilon := \min_{\tilde{u}} \int_0^\infty \left[\|e_T(t)\|^2 + \epsilon^2 \|\tilde{u}(t)\|^2 \right] dt \tag{11}$$

with $\tilde{u}$ defined in (3), is strictly positive. Qiu and Davison [1993] showed that the locations of the zeros in $\mathbb{C}^+$ determine the best attainable performance, that is the value of J_T.

Theorem 1. (Qiu and Davison [1993]). Let $r(t) := \eta_1 \sin \omega t + \eta_2 \cos \omega t$, $x(0) = 0$, and assume that (1) is right-invertible, has no zeros at $j\omega$, and its non-minimum phase zeros are $z_1, z_2, \ldots, z_p$. Then the best attainable performance $J_T := \lim_{\epsilon \to 0} J_\epsilon$ is given by

$$J_T = \eta' M \eta, \quad \eta = \mathrm{col}(\eta_1, \eta_2),$$

where M is positive semi-definite and its trace satisfies

$$\operatorname{trace} M = \sum_{i=1}^{p} \left(\frac{1}{z_i - j\omega} + \frac{1}{z_i + j\omega} \right).$$

□

For more general reference signals, [Su *et al.*, 2003] give explicit formulas which show the dependence of J_T on the non-minimum phase zeros and their frequency-dependent directional information.

3.2 Path-following

In contrast to reference-tracking, the attainable performance for path-following is not limited by non-minimum phase zeros, [Aguiar *et al.*, 2004]. Let the desired path

$$y_d(\theta) := \sum_{k=1}^{n_d} \left[a_k e^{j\omega_k \theta} + a_k^\star e^{-j\omega_k \theta} \right], \tag{12}$$

where the $\omega_k > 0$ are real numbers and the a_k are non-zero complex vectors, be generated by exosystem (7). Considering, as in [Qiu and Davison, 1993; Woodyatt *et al.*, 2002; Braslavsky *et al.*, 2002; Su *et al.*, 2003], that $x(0) = 0$, [Aguiar *et al.*, 2004] prove:

Theorem 2. (Aguiar *et al.* [2004]). Consider the geometric path-following problem for (1) where (A, B) is stabilizable. Then for any given positive constant $\delta^\star$ there exist constant matrices K and L, and a timing law $\theta(t)$ such that the feedback law

$$u(t) = Kx(t) + Lw(\theta(t)) \tag{13}$$

achieves (9). □

It is important to stress that the stabilizability of (A, B) is the *only* condition (necessary and sufficient) for the solvability of the geometric path-following problem using (13).
Furthermore, an arbitrarily small $\mathcal{L}_2$ norm of the path-following error is attainable even when the speed assignment v_d is specified beforehand.

Theorem 3. (Aguiar *et al.* [2004]). For the speed-assigned path-following problem, let v_d be specified so that the eigenvalues of $v_d S$ do not coincide with the zeros of (4), and assume that (A, B) is stabilizable. Then, (9) can be satisfied for any $\delta^\star > 0$ with a timing law $\theta(t)$ and a controller of the form (13) but with time-varying piecewise-constant matrices K and L. □

4. LIMITS OF PERFORMANCE FOR NONLINEAR SYSTEMS

In the first part of this section, we present an internal model analog of the results in [Seron *et al.*, 1999; Braslavsky *et al.*, 2002] for the reference-tracking problem. In the second part, we present our main result for the path-following problem. We show that, in contrast to reference-tracking, the path-following problems can be solved with arbitrarily small $\mathcal{L}_2$ norm of the path-following error.

4.1 Reference-tracking

We consider the class of nonlinear systems which are locally diffeomorphic to systems in strict-feedback form [3]:

$$\dot{z} = f_0(z) + g_0(z)\xi_1, \tag{14a}$$

$$\dot{\xi}_1 = f_1(z, \xi_1) + g_1(z, \xi_1)\xi_2,$$

$$\vdots$$

$$\dot{\xi}_{r_d} = f_{r_d}(z, \xi_1, \ldots, \xi_{r_d}) + g_{r_d}(z, \xi_1, \ldots, \xi_{r_d})u, \tag{14b}$$

$$y = \xi_1, \tag{14c}$$

where $z \in \mathbb{R}^{n_z}$, $\xi := \mathrm{col}(\xi_1, \ldots, \xi_{r_d})$, $\xi_i \in \mathbb{R}^m$, $\forall i \in \{1, \ldots, r_d\}$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^m$. $f_i(\cdot)$

[3] When convenient we use the compact form (4) for (14). In that case, $f(\cdot)$ denotes the vector field described by the right-hand-side of (14a)–(14b), $h(\cdot)$ the output map described by (14c), and $x = \mathrm{col}(z, \xi_1, \ldots, \xi_{r_d})$.

and $g_i(\cdot)$ are $\mathcal{C}^k$ functions of their arguments (for some large k), $f_i(0,\ldots,0)=0$, and the matrices $g_i(\cdot)$, $\forall i \in \{1,\ldots,r_d\}$ are always nonsingular. We assume that initially the system is at rest, $(z,\xi)=(0,0)$.

When the reference-tracking problem is solvable, there exist maps $\Pi=\mathrm{col}(\Pi_0,\ldots,\Pi_{r_d})$, $\Pi_0:\mathbb{R}^p\to\mathbb{R}^{n_z}$, $\Pi_i:\mathbb{R}^p\to\mathbb{R}^m$, $\forall i\in\{1,\ldots,r_d\}$, and $c:\mathbb{R}^p\to\mathbb{R}^m$ that satisfy (6). The following locally diffeomorphic change of coordinates

$$\tilde{z}=z-\Pi_0(w), \tag{15}$$
$$\tilde{\xi}:=\mathrm{col}(\tilde{\xi}_1,\ldots,\tilde{\xi}_{r_d}), \tag{16}$$
$$\tilde{\xi}_i=\xi_i-\Pi_i(w), \quad i=1,\ldots,r_d \tag{17}$$
$$\tilde{u}=u-c(w), \tag{18}$$

transforms the system (14) into the *error system*

$$\dot{\tilde{z}}=\tilde{f}_0(\tilde{z},w)+\tilde{g}_0(\tilde{z},w)e_T, \tag{19a}$$
$$\begin{aligned}\dot{\tilde{\xi}}_1&=\tilde{f}_1(\tilde{z},\tilde{\xi}_1,w)+\tilde{g}_1(\tilde{z},\tilde{\xi}_1,w)\tilde{\xi}_2,\\ &\vdots\\ \dot{\tilde{\xi}}_{r_d}&=\tilde{f}_{r_d}(\tilde{z},\tilde{\xi}_1,\ldots,\tilde{\xi}_{r_d},w)+\tilde{g}_{r_d}(\tilde{z},\tilde{\xi}_1,\ldots,\tilde{\xi}_{r_d},w)\tilde{u},\end{aligned} \tag{19b}$$
$$e_T=\tilde{\xi}_1, \tag{19c}$$

where

$$\begin{aligned}\tilde{f}_0&:=f_0(\tilde{z}+\Pi_0(w))-f_0(\Pi_0(w))\\ &\quad+\Big[g_0(\tilde{z}+\Pi_0(w))-g_0(\Pi_0(w))\Big]q(w),\\ \tilde{g}_0&:=g_0(\tilde{z}+\Pi_0(w)),\end{aligned}$$

$\tilde{f}_0(0,w)=0$, $\tilde{g}_0(\tilde{z},0)=g_0(\tilde{z})$, and $\tilde{f}_i(\cdot)$, $\tilde{g}_i(\cdot)$, $i=1,\ldots,r_d$ are appropriately defined functions that satisfy $\tilde{f}_i(0,\ldots,0,w)=0$ and $\tilde{g}_i(\tilde{z},\ldots,\tilde{\xi}_i,0)=g_i(\tilde{z},\ldots,\tilde{\xi}_i)$.

Our analysis makes use of the following two optimal control problems.

Cheap control problem: For the system consisting of the error system (19) and the exosystem (5) with initial condition $\big(\tilde{z}(0),\tilde{\xi}(0),w(0)\big)=\big(\tilde{z}_0,\tilde{\xi}_0,w_0\big)$, find the optimal feedback law $\tilde{u}=\alpha^{cc}_{\delta,\epsilon}(\tilde{z},\tilde{\xi},w)$ that minimizes the cost functional

$$\frac{1}{2}\int_0^\infty\left(\|e_T(t)\|^2+\delta\|\tilde{z}(t)\|^2+\epsilon^{2r_d}\|\tilde{u}(t)\|^2\right)dt \tag{20}$$

for $\delta>0$, $\epsilon>0$. We denote by $J^{cc}_{\delta,\epsilon}(\tilde{z}_0,\tilde{\xi}_0,w_0)$ the corresponding optimal value. The best-attainable cheap control performance for reference-tracking is then

$$J_T:=\lim_{(\delta,\epsilon)\to 0}J^{cc}_{\delta,\epsilon}(\tilde{z}_0,\tilde{\xi}_0,w_0). \tag{21}$$

As shown in [Krener, 2001], in some neighborhood of $(0,0,0)$ and for every $\delta>0$, $\epsilon>0$, the value $J^{cc}_{\delta,\epsilon}(\cdot,\cdot,\cdot)$ is $\mathcal{C}^{k-2}$ under the following assumption:

Assumption 1. The linearization around $(z,\xi)=(0,0)$ of system (14) is stabilizable and detectable, and the linearization around $w=0$ of the exosystem (5) is stable.

Minimum-energy problem: For the system

$$\dot{\tilde{z}}=\tilde{f}_0(\tilde{z},w)+\tilde{g}_0(\tilde{z},w)e_T, \quad \tilde{z}(0)=z_0, \tag{22a}$$
$$\dot{w}=s(w), \quad w(0)=w_0, \tag{22b}$$

with e_T viewed as the input, find the optimal feedback law $e_T=\alpha^{me}_\delta(\tilde{z},w)$ that minimizes the cost

$$\frac{1}{2}\int_0^\infty\left(\delta\|\tilde{z}(t)\|^2+\|e_T(t)\|^2\right)dt, \tag{23}$$

for $\delta>0$. We denote by $J^{me}_\delta(\tilde{z}_0,w_0)$ the corresponding optimal value. Under Assumption 1, $J^{me}_\delta(\cdot,\cdot)$ is $\mathcal{C}^{k-2}$ in some neighborhood of $(0,0)$.

Our analysis reveals that J_T is equal to the least control effort needed to stabilize the corresponding zero dynamics system (22) driven by the tracking error e_T.

Theorem 4. Suppose that Assumption 1 holds and that (6) has a solution in some neighborhood of $w=0$. Then, for any $(\tilde{z}(0),\tilde{\xi}(0),w(0))=(\tilde{z}_0,\tilde{\xi}_0,w_0)$ in some neighborhood of $(0,0,0)$ there exists a solution to the cheap control problem and

$$J_T=\lim_{\delta\to 0}J^{me}_\delta \tag{24}$$

Proof. Under Assumption 1 and from the formulations of the cheap control and minimum-energy problems, we conclude that for every $\delta>0$, $\epsilon>0$, and every initial condition $(\tilde{z}_0,\tilde{\xi}_0,w_0)$ in some neighborhood of $(0,0,0)$, the values $J^{me}_\delta(\tilde{z}_0,w_0)$ and $J^{cc}_{\delta,\epsilon}(\tilde{z}_0,\tilde{\xi}_0,w_0)$ exist and satisfy

$$J^{me}_\delta(\tilde{z}_0,w_0)\le J^{cc}_{\delta,\epsilon}(\tilde{z}_0,\tilde{\xi}_0,w_0). \tag{25}$$

On the other hand, from Lemma 7 in Appendix we have

$$J^{cc}_{\delta,\epsilon}(\tilde{z}_0,\tilde{\xi}_0,w_0)\le J^{me}_\delta(\tilde{z}_0,w_0)+O(\epsilon). \tag{26}$$

Therefore, from (25)–(26) we conclude that

$$J^{me}_\delta(\tilde{z}_0,w_0)\le J^{cc}_{\delta,\epsilon}(\tilde{z}_0,\tilde{\xi}_0,w_0)\le J^{me}_\delta(\tilde{z}_0,w_0)+O(\epsilon).$$

The result (24) follows from this and (21) as one makes $(\delta,\,\epsilon)\to 0$. □

4.2 Path-following

For path-following, we define the correspondent cheap control problem by replacing e_T with e_P in (20). We then show that, in contrast to reference-tracking, the path-following problem can be solved with arbitrarily small $\mathcal{L}_2$ norm of e_P.

We let the vector field $s(w)$ and the output map $q(w)$ of the exosystem (7) be linear, $s(w)=Sw$, $q(w)=Qw$, such that all eigenvalues of $S\in\mathbb{R}^{p\times p}$ are non-zero and semisimple.

Theorem 5. Assume that (6) has a solution when $s(w)=v_dSw$, for v_d almost everywhere on $(0,\infty)$. Then, for every $w(\theta(0))=w_0$ in a neighborhood around $w=0$, there exist a timing law for $\theta(t)$ and a feedback law

$$u=c(w)+\alpha_{\delta,\epsilon}(z,\xi,w) \tag{27}$$

which solve the geometric path-following and satisfy (9) for every $\delta^\star>0$.

Proof. With the timing law

$$\dot{\theta}(t) = v_d, \qquad \theta(0) = 0, \tag{28}$$

and $v_d > 0$ a constant to be selected later, the path-following problem becomes the tracking problem of $r(t)$ generated by

$$\dot{w}(t) = v_d S w(t), \quad r(t) = Q w(t), \tag{29}$$

which, upon the substitution in (6), yields

$$\begin{aligned} \frac{\partial \Pi}{\partial w} v_d S w &= f(\Pi(w), c(w)), \\ h(\Pi(w), c(w)) - Qw &= 0. \end{aligned} \tag{30}$$

The function $c(w)$, used in the feedback law (27), solves (30), while $\alpha_{\delta,\epsilon}(z, \xi, w)$ minimizes (20) for the error system (19) together with the exosystem (29) and some small $\delta > 0$, $\epsilon > 0$. With the timing law (28), Theorem 4 allows us to conclude that as $(\epsilon, \delta) \to 0$ we have $J_P = \lim_{\delta \to 0} J_\delta^{me}$.

To prove that J_δ^{me} can be made arbitrarily small by selecting a sufficiently large v_d, we use Lemma 8 in Appendix. It shows that for the minimum-energy problem and every initial condition in some neighborhood of $(\tilde{z}, w) = (0, 0)$, there exist a sufficiently small $\delta > 0$ in (23) and a feedback law $e_T = \hat{\alpha}_\delta^{me}(\tilde{z}, w)$ for which $J_\delta^{me}(\tilde{z}_0, w_0)$ is bounded by

$$J_\delta^{me}(\tilde{z}_0, w_0) \leq \frac{1}{2} \tilde{z}_0' P_0 \tilde{z}_0,$$

where $P_0 > 0$ does not depend on v_d. Observing that $\tilde{z}_0 = \Pi_0(w_0)$, since $z(0) = 0$, the proof is completed using Lemma 9 in Appendix which establishes that $\|\Pi_0(w_0)\|$ can be made arbitrarily small by choosing a sufficiently large v_d. □

Next we show that an arbitrarily small $\mathcal{L}_2$ norm of the path-following error is attainable even when the speed v_d is specified beforehand.

Theorem 6. Consider the speed-assigned path-following problem with v_d specified so that (30) has a solution in some neighborhood of $w = 0$. Then, (9) can be satisfied for any $\delta^\star > 0$ with a suitable timing law $\theta(t)$ and a controller of the form (27) with time-varying piecewise-continuous maps $c(w)$ and $\alpha(z, \xi, w)$.

Proof. To construct a path-following controller that satisfies (9) we start with

$$u = c_\sigma(w) + \alpha_\sigma(z, \xi, w), \tag{31a}$$

$$\dot{\theta} = v_\sigma, \tag{31b}$$

where for each positive constant v_ℓ, $\ell \in \mathcal{I} := \{0, 1, 2, \ldots N\}$, the maps $\Pi_\ell := col(\Pi_{\ell_0}, \Pi_{\ell_\xi})$, $\Pi_{\ell_\xi} := col(\Pi_{\ell_{\xi_1}}, \ldots, \Pi_{\ell_{\xi_{r_d}}})$, $\Pi_{\ell_0} : \mathbb{R}^p \to R^{n_z}$, $\Pi_{\ell_i} : \mathbb{R}^p \to R^m$, $i = 1, \ldots, r_d$, and $c_\ell : \mathbb{R}^p \to \mathbb{R}^m$ satisfy

$$\begin{aligned} \frac{\partial \Pi_\ell}{\partial w} v_\ell S w &= f(\Pi_\ell(w), c_\ell(w)), \\ h(\Pi_\ell(w), c_\ell(w)) - Qw &= 0, \end{aligned} \tag{32}$$

and $\sigma(t) : [t_0 := 0, \infty) \to \mathcal{I}$, is the piecewise constant switching signal

$$\sigma(t) = \begin{cases} i, & t_i \leq t < t_{i+1},\ i = 0, \ldots, N-1 \\ N, & t \geq t_N \end{cases}$$

Each $\alpha_\ell(z, \xi, w)$ is the optimal feedback-law that minimizes

$$\int_0^\infty \left(\|e_P\|^2 + \delta \|z - \Pi_{\ell_0}(w)\|^2 + \epsilon^{2r_d} \|u - c_\ell(w)\|^2 \right) dt,$$

for some small $\delta > 0$, $\epsilon > 0$. Note that (31) is a speed-assignment path-following controller for which $\dot{\theta}(t)$ converges to $v_N = v_d$ in finite time.

We now prove that for any $\delta^\star > 0$, (9) can be satisfied by appropriate selection of a finite sequence $t_0, t_1, \ldots, t_N$ together with $(v_0, \Pi_0, \alpha_0, c_0), (v_1, \Pi_1, \alpha_1, c_1), \ldots, (v_N, \Pi_N, \alpha_N, c_N)$ used in the feedback controller (31). To this end, we show in Lemma 10 in Appendix that J_P is bounded by

$$\begin{aligned} J_P \leq{}& \frac{1}{2} \tilde{z}_0' P_0 \tilde{z}_0 + \gamma \frac{\lambda_{max}(P_0)}{2} \sum_{\ell=1}^{N} (v_{\ell-1} - v_\ell)^2 \\ &+ \lambda_{max}(P_0) \sum_{\ell=1}^{N} \tilde{z}_{\ell-1}(t_\ell)' \left[\bar{z}_{\ell-1}(t_\ell) - \bar{z}_\ell(t_\ell) \right] \\ &+ \frac{\lambda_{max}(P_0)}{2} \sum_{\ell=1}^{N} \|\tilde{z}_{\ell-1}(t_\ell)\|^2, \end{aligned} \tag{33}$$

where $\lambda_{max}(P_0)$ denotes the maximum eigenvalue of $P_0 > 0$, γ is a positive constant, $\tilde{z}_0 := \tilde{z}(0)$, $\bar{z}_\ell := \Pi_{\ell_0}(w)$, and the transient $\tilde{z}_\ell := z - \Pi_{\ell_0}(w)$ converges to zero as $t \to \infty$.

We show that each term of (33) is upper-bounded by $\frac{\delta^\star}{4}$ so that $J_P \leq \delta^\star$. Applying the same arguments as in Theorem 5, the first term in (33) is bounded by $\frac{\delta^\star}{4}$ using a sufficiently large v_0. To prove that the second term in (33) is smaller than $\frac{\delta^\star}{4}$, we select the parameters v_ℓ, $\ell \in \mathcal{I}$ to satisfy

$$v_{\ell-1} - v_\ell = \mu, \quad v_N = v_d, \quad \ell = 1, 2, \ldots, N \tag{34}$$

where $\mu := \frac{2\delta^\star}{\gamma \lambda_{max}(P_0)(v_0 - v_N)}$, and $N := \frac{v_0 - v_N}{\mu}$. Then

$$\begin{aligned} \gamma \frac{\lambda_{max}(P_0)}{2} \sum_{\ell=1}^{N} (v_{\ell-1} - v_\ell)^2 &\leq \gamma \frac{\lambda_{max}(P_0)}{2} N \mu^2 \\ &= \gamma \frac{\lambda_{max}(P_0)}{2} (v_0 - v_N) \mu = \frac{\delta^\star}{4}. \end{aligned}$$

The above selection for the v_ℓ, $\ell \in \mathcal{I}$, it is made under the constraint that the reference-tracking problem for the signal $r(t)$ generated by (29) with v_d replaced by v_ℓ is solvable. This can always be satisfied by appropriately adjusting v_0. Finally, for any given N, each of the last two terms in (33) can be made smaller than $\frac{\delta}{4}$ by choosing t_ℓ, $\ell = 1, 2, \ldots, N$ sufficiently large. □

5. CONCLUSIONS

This paper demonstrates that the task of following a geometric path $y_d(\theta)$ is less restrictive than the task of tracking a reference signal $r(t)$. The reference-tracking problem is subjected to the limitations imposed by the unstable zero-dynamics, a nonlinear analog of the Bode's limitations caused by non-minimum phase zeros. Our analysis revealed that the limitation is due to the need to stabilize the zero-dynamics by the tracking error, which therefore prevents the output $y(t)$ from achieving perfect tracking. In path-following one has available an additional degree of freedom to select a timing law $\theta(t)$ with which a prescribed path $y_d(\theta)$ will be followed. In Theorems 5 and 6, we prove that with an appropriate choice of $\theta(t)$ the $\mathcal{L}_2$ norm of the path-following error can be made arbitrarily small, that is, the path-following problem is not subjected to the limitations of reference-tracking. This conceptual result may be of practical significance, because the path-following formulation is convenient for many applications. Design of path-following controllers for non-minimum phase systems is a topic of current research, [Dačić *et al.*, 2004].

Appendix A

Due to space limitations, the proofs of the following lemmas are omitted. These can be found in [Aguiar *et al.*, 2004*c*].

Lemma 7. Suppose that Assumption 1 holds. For every initial condition $(\tilde{z}_0, \tilde{\xi}_0, w_0)$ for the error system (19) and the exosystem (5) in some neighborhood of $(0,0,0)$, and every $\delta > 0$, there exist a sufficiently small $\epsilon > 0$ and feedback law $\tilde{u} = \hat{\alpha}^{cc}_{\delta,\epsilon}(\tilde{z}, \tilde{\xi}, w)$ for which the value of (20) does not exceed

$$J^{me}_{\delta}(\tilde{z}_0, w_0) + O(\epsilon).$$

Lemma 8. Consider the minimum-energy problem formulated in Section 4. For every initial condition $(\tilde{z}(0), w(0)) = (\tilde{z}_0, w_0)$ for (22) in some neighborhood of $(0,0)$, there exist $\delta > 0$ in (23) and a feedback law $e_T = \hat{\alpha}^{me}_{\delta}(\tilde{z}, w)$ for which (23) does not exceed

$$\frac{1}{2}\tilde{z}_0' P_0 \tilde{z}_0,$$

where $P_0 > 0$ does not depend on v_d.

Lemma 9. In the reference-tracking problem for the nonlinear system (14) let the vector field $s(w)$ and the output map $q(w)$ of the exosystem (5) be $s(w) = v_d S w$, $q(w) = Qw$. Suppose that the eigenvalues of $S \in \mathbb{R}^{p\times p}$ are non-zero and semisimple, and that for some $v_d > 0$, (6) has a solution in some neighborhood of $w = 0$. Then, for any $\rho > 0$, there exists $v_d^\star > 0$ such that the map $\Pi_0 : \mathbb{R}^p \to \mathbb{R}^{n_z}$ satisfying

$$\frac{\partial \Pi_0(w)}{\partial w} S w = \mu\big[f_0(\Pi_0(w)) + g_0(\Pi_0(w))Qw\big],$$

$\mu := \frac{1}{v_d^\star}$, is bounded by

$$\|\Pi_0(w)\| \le \rho.$$

Lemma 10. Under the conditions of Theorem 6, the path-following controller (31) ensures that there exists $\gamma > 0$ such that J_P satisfies (33).

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