Stability of switched systems: a Lie-algebraic condition*

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Abstract

We present a sufficient condition for asymptotic stability of a switched linear system in terms of the Lie algebra generated by the individual matrices. Namely, if this Lie algebra is solvable, then the switched system is exponentially stable for arbitrary switching. In fact, we show that any family of linear systems satisfying this condition possesses a quadratic common Lyapunov function. We also discuss the implications of this result for switched nonlinear systems.

Keywords: Switched system; uniform exponential stability; quadratic common Lyapunov function.

1 Introduction

Suppose that we are given a compact (with respect to the usual topology in $\mathbb{R}^{n\times n}$) set of strictly stable real $n \times n$ matrices $\{A_p : p \in \mathcal{P}\}$, where the index set $\mathcal{P}$ is a subset of a finite-dimensional normed linear vector space, e.g., $\mathbb{R}^m$. Consider the *switched linear system*

$$\dot{x} = A_\sigma x \tag{1}$$

where $x \in \mathbb{R}^n$ and $\sigma : [0, \infty) \to \mathcal{P}$ is a piecewise constant *switching signal*. The problem of finding conditions that guarantee asymptotic stability of (1) for an arbitrary switching signal σ has recently attracted a considerable amount of attention—see the work reported in [2, 3, 8, 9, 10, 11, 14, 15] and the references therein.

Some of the aforementioned results suggest that certain properties of the Lie algebra $\{A_p : p \in \mathcal{P}\}_{LA}$ generated by the matrices A_p may be of relevance to the question of stability

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of (1). In particular, it is well known and easy to show that if these matrices commute pairwise, i.e., the *Lie bracket* $[A_p, A_q] := A_pA_q - A_qA_p$ equals zero for all $p, q \in \mathcal{P}$, and if $\mathcal{P}$ is a finite set, then the system (1) is asymptotically stable for any switching signal σ. An explicit construction of a quadratic common Lyapunov function for the family of linear systems

$$\dot{x} = A_p x, \quad p \in \mathcal{P} \tag{2}$$

in this case is given in [10].

A connection between asymptotic stability of a switched linear system and the properties of the corresponding Lie algebra was first explicitly discussed by Gurvits in [3]. That paper is concerned with the discrete-time counterpart of (1) which takes the form

$$x(k+1) = A_{\sigma(k)}x(k). \tag{3}$$

where σ is a function from nonnegative integers to a finite index set $\mathcal{P}$. Gurvits conjectured that if the Lie algebra $\{A_p : p \in \mathcal{P}\}_{LA}$ is nilpotent (see Section 2 for definitions) then (3) is asymptotically stable for any switching signal σ. He used the Baker-Campbell-Hausdorff formula to prove this conjecture for the particular case when $\mathcal{P} = \{1, 2\}$, the matrices A_1 and A_2 are nonsingular, and their third-order Lie brackets vanish: $[A_1, [A_1, A_2]] = [A_2, [A_1, A_2]] = 0$.

By establishing the existence of a quadratic common Lyapunov function for the family of linear systems (2), we will show that the system (1) is uniformly exponentially stable for an arbitrary switching signal σ if the Lie algebra $\{A_p : p \in \mathcal{P}\}_{LA}$ is solvable. The corresponding statement for the discrete-time case can be derived in a similar fashion. Since every nilpotent Lie algebra is solvable, we obtain a more general result than the one conjectured by Gurvits.

The outline of the paper is as follows. In Section 2 we state some basic facts about Lie algebras that we need in the sequel. Our main result for the linear case is proved in Section 3. In Section 4 we briefly discuss the implications of this result in the nonlinear case. We make concluding remarks and sketch some directions for future research in Section 5.

Throughout the paper we use the following notation. Given a complex number a, we denote by $\overline{a}$ its conjugate. We use the symbols v^* and A^* to denote the conjugate transpose of a vector v and a matrix A, respectively. Given a matrix A, we denote by $(A)_{ij}$ its ij-th element. If A is complex, $\Re A$ is the matrix defined by $(\Re A)_{ij} = \Re[(A)_{ij}]$, where $\Re$ stands for the real part. The matrix $\Im A$, where $\Im$ stands for the imaginary part, is defined similarly.

2 Solvable Lie algebras

Given a Lie algebra $\mathfrak{g}$, the descending sequence of ideals $\mathfrak{g}^{(k)}$ is defined inductively as follows: $\mathfrak{g}^{(1)} := \mathfrak{g}$, $\mathfrak{g}^{(k+1)} := [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] \subset \mathfrak{g}^{(k)}$. If $\mathfrak{g}^{(k)} = 0$ for k sufficiently large, then $\mathfrak{g}$ is called *solvable*. Similarly, one defines the descending sequence of ideals $\mathfrak{g}^k$ by $\mathfrak{g}^1 := \mathfrak{g}$, $\mathfrak{g}^{k+1} := [\mathfrak{g}, \mathfrak{g}^k] \subset \mathfrak{g}^k$, and calls $\mathfrak{g}$ *nilpotent* if $\mathfrak{g}^k = 0$ for k sufficiently large. For example, if $\mathfrak{g}$ is a Lie algebra generated by two matrices A and B, i.e., $\mathfrak{g} = \{A, B\}_{LA}$, then we have: $\mathfrak{g}^{(1)} = \mathfrak{g}^1 = \mathfrak{g} = \text{span}\{A, B, [A, B], [A, [A, B]], \dots\}$, $\mathfrak{g}^{(2)} = \mathfrak{g}^2 = \text{span}\{[A, B], [A, [A, B]], \dots\}$, $\mathfrak{g}^{(3)} = \text{span}\{[[A, B], [A, [A, B]]], \dots\} \subset \mathfrak{g}^3 = \text{span}\{[A, [A, B]], [B, [A, B]], \dots\}$, and so on. Every nilpotent Lie algebra is solvable, but the converse is not true.

The following result plays a key role in our subsequent developments. It is known as Lie's Theorem and can be found in most textbooks on the theory of Lie algebras (see, e.g., [12]).

Proposition 1 *Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field, and let ρ be a representation of $\mathfrak{g}$ on a vector space V of finite dimension n. Then there exists a basis $\{v_1, \ldots, v_n\}$ of V such that for each $X \in \mathfrak{g}$ the matrix of $\rho(X)$ in that basis takes the upper-triangular form*

$$\begin{pmatrix} \lambda_1(X) & \ldots & * \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_n(X) \end{pmatrix}$$

($\lambda_1(X), \ldots, \lambda_n(X)$ being its eigenvalues).

In our context, this means that if $\{A_p : p \in \mathcal{P}\}_{LA}$ is solvable, then there exists a nonsingular complex matrix T such that

$$A_p = T^{-1}\widetilde{A}_p T, \quad p \in \mathcal{P} \tag{4}$$

where the complex matrices $\widetilde{A}_p$ are upper-triangular.

3 Uniform exponential stability

We will say that the system (1) is *globally uniformly exponentially stable* if there exist positive constants c and μ such that the solution of (1) for any initial state $x(0)$ and any switching signal σ satisfies

$$\|x(t)\| \leq ce^{-\mu t}\|x(0)\| \quad \forall t \geq 0 \tag{5}$$

(here $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^n$). Following [6], we use the word "uniform" in this definition to describe uniformity with respect to switching signals. Our main result is as follows.

Theorem 2 *If the Lie algebra $\{A_p : p \in \mathcal{P}\}_{LA}$ is solvable, then the system (1) is globally uniformly exponentially stable.*

In view of (4), we would like to show that the system

$$\dot{x} = \widetilde{A}_\sigma x, \quad x \in \mathbb{C}^n$$

is globally uniformly exponentially stable (with respect to the standard norm in $\mathbb{C}^n$). We could do this directly by starting with the bottom component of the vector x and working our way up, aided by the upper-triangular structure of the matrices $\widetilde{A}_p$ and the fact that the state of an exponentially stable linear system with an exponentially decaying input decays exponentially. To see how it works, suppose that $\mathcal{P} = \{1, 2\}$ and $x \in \mathbb{R}^2$. For $k = 1, 2$ let $\lambda_k := \min\{-(\Re\widetilde{A}_1)_{kk}, -(\Re\widetilde{A}_2)_{kk}\} > 0$. We have

$$|x_2(t)| \leq e^{-\lambda_2 t}|x_2(0)|$$

where $|\cdot|$ stands for complex magnitude. Moreover,

$$
\begin{aligned}
|x_1(t)| &\le e^{-\lambda_1 t}|x_1(0)| + \int_0^t e^{-\lambda_1(t-\tau)}e^{-\lambda_2\tau}|x_2(0)|d\tau \\
&= \begin{cases} e^{-\lambda_1 t}|x_1(0)| + \frac{1}{\lambda_1-\lambda_2}(e^{-\lambda_2 t} - e^{-\lambda_1 t})|x_2(0)| & \text{if } \lambda_1 \ne \lambda_2 \\ e^{-\lambda_1 t}|x_1(0)| + te^{-\lambda_1 t}|x_2(0)| & \text{if } \lambda_1 = \lambda_2 \end{cases}
\end{aligned}
$$

The required estimate (5) can be easily deduced from the above inequalities.

However, to prove Theorem 2 in the general case we find it more convenient to proceed by showing that the family of linear systems (2) has a quadratic common Lyapunov function, a fact that is of interest in its own right. To do this, we make use of the two results stated below. The first one is a special case of [5, Corollary 4.2], while the second one is basic and can be found, e.g., in [4].

Lemma 3 *If there exist real symmetric positive definite matrices Q and R such that*

$$-QA_p - A_p^T Q \ge R, \quad p \in \mathcal{P} \tag{6}$$

then the system (1) is globally uniformly exponentially stable.

In other words, the existence of a quadratic common Lyapunov function for the family of linear systems (2) guarantees global uniform exponential stability of (1).

Lemma 4 *All leading principal minors of a Hermitian matrix are real. A Hermitian matrix H is positive definite (i.e., $x^*Hx > 0\ \forall x \in \mathbb{C}^n$) if and only if all its leading principal minors are positive.*

Proof of Theorem 2. According to Lemma 3, it suffices to find real symmetric positive definite matrices Q and R that satisfy the inequalities (6). We start by showing that there exists a positive definite matrix $\widetilde{Q}$ such that

$$-\widetilde{Q}\widetilde{A}_p - \widetilde{A}_p^*\widetilde{Q} > 0, \quad p \in \mathcal{P}. \tag{7}$$

Let us look for $\widetilde{Q}$ that takes the form of a real diagonal matrix $\widetilde{Q} = \operatorname{diag}(q_1, \dots, q_n)$. We have

$$-\widetilde{Q}\widetilde{A}_p - \widetilde{A}_p^*\widetilde{Q} = \begin{pmatrix} -2q_1(\Re\widetilde{A}_p)_{11} & -q_1(\widetilde{A}_p)_{12} & \dots & -q_1(\widetilde{A}_p)_{1n} \\ -q_1\overline{(\widetilde{A}_p)_{12}} & -2q_2(\Re\widetilde{A}_p)_{22} & \dots & -q_2(\widetilde{A}_p)_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -q_1\overline{(\widetilde{A}_p)_{1n}} & -q_2\overline{(\widetilde{A}_p)_{2n}} & \dots & -2q_n(\Re\widetilde{A}_p)_{nn} \end{pmatrix} \tag{8}$$

Choose an arbitrary $q_1 > 0$. Since the eigenvalues of the matrices $\widetilde{A}_p$ have negative real parts, we have $-2q_1(\Re\widetilde{A}_p)_{11} > 0\ \forall p \in \mathcal{P}$. Now, suppose that $q_1, \dots, q_k > 0$ have been chosen so that the leading principal minors of the matrix on the right-hand side of (8) up to order k are larger than some positive number ϵ for all $p \in \mathcal{P}$. It is not hard to see that by choosing q_{k+1} sufficiently large we can make the $(k+1)\times(k+1)$ leading principal minors positive for all $p \in \mathcal{P}$. Indeed, since an $m \times m$ determinant is given by a sum of $m!$ terms, any q_{k+1} such that

$$q_{k+1} > \frac{k!k2^{k-1} \max_{1\le i\le k}\{q_i^{k+1}\} \max_{1\le i,j\le k+1,\ p\in\mathcal{P}}\{|(\widetilde{A}_p)_{ij}|^{k+1}\}}{2\epsilon \min_{p\in\mathcal{P}}\{-(\Re\widetilde{A}_p)_{k+1,k+1}\}}$$

will serve the purpose. The above expression is well defined and finite by virtue of compactness of the set $\{A_p : p \in \mathcal{P}\}$.

Proceeding in this fashion, we can construct a positive definite diagonal matrix $\widetilde{Q}$ satisfying (7). Combined with (4) this implies that

$$-T^*\widetilde{Q}TA_p - A_p^T T^*\widetilde{Q}T > 0, \quad p \in \mathcal{P}.$$

Let us denote $T^*\widetilde{Q}T$ by $\widehat{Q}$ and $-T^*\widetilde{Q}TA_p - A_p^T T^*\widetilde{Q}T$ by R_p to obtain

$$-\widehat{Q}A_p - A_p^T\widehat{Q} = R_p$$

or, more explicitly (recall that the matrices A_p are real)

$$-(\Re\widehat{Q} + \sqrt{-1}\,\Im\widehat{Q})A_p - A_p^T(\Re\widehat{Q} + \sqrt{-1}\,\Im\widehat{Q}) = \Re R_p + \sqrt{-1}\,\Im R_p.$$

It follows that

$$-\Re\widehat{Q}A_p - A_p^T\Re\widehat{Q} = \Re R_p.$$

But for any $x \in \mathbb{R}^n$ we have $0 < x^T\widehat{Q}x = x^T\Re\widehat{Q}x$ because $\widehat{Q}$ is Hermitian hence $\Im\widehat{Q}$ is skew-symmetric. Similarly, $0 < x^T R_p x = x^T \Re R_p x$ for any $x \in \mathbb{R}^n$ and any $p \in \mathcal{P}$. Moreover, since the set $\{A_p : p \in \mathcal{P}\}$ is assumed to be compact, the set of positive definite matrices $\{R_p : p \in \mathcal{P}\}$ is also compact, and therefore there exists a real symmetric positive definite matrix R such that $\Re R_p \ge R$ for all $p \in \mathcal{P}$. Thus we see that the matrices $Q := \Re\widehat{Q}$ and R satisfy the assumptions of Lemma 3, which completes the proof of the theorem. □

Remark. The existence of a quadratic common Lyapunov function for a family of linear systems whose matrices can be simultaneously put into the upper-triangular form has been pointed out before (see, e.g., [8, 15] and related earlier work in [1]). It is important to recognize, however, that while it is a nontrivial matter to find a basis in which all matrices take the triangular form or even decide whether such a basis exists, the Lie-algebraic condition given by Theorem 2 is formulated *in terms of the original data* and can always be checked in a finite number of steps if $\mathcal{P}$ is a finite set.

Note that although we showed that a quadratic common Lyapunov function exists, its actual construction depends on the knowledge of the matrix T. Standard numerical methods can be employed to compute this matrix, but it might in fact be more efficient to solve the linear matrix inequalities (6) directly [2].

4 Switched nonlinear systems

Consider the family of nonlinear systems

$$\dot{x} = f_p(x), \quad p \in \mathcal{P} \tag{9}$$

where $f_p : D \to \mathbb{R}^n$ is continuously differentiable with $f_p(0) = 0$ for each $p \in \mathcal{P}$ and D is a neighborhood of the origin in $\mathbb{R}^n$. Consider also the corresponding family of linearized systems

$$\dot{x} = F_p x, \quad p \in \mathcal{P} \tag{10}$$

where $F_p = \dfrac{\partial f_p}{\partial x}(0)$. Assume that the matrices F_p are strictly stable, that $\mathcal{P}$ is a compact set, and that $\dfrac{\partial f_p}{\partial x}(x)$ depends continuously on p for each $x \in D$. We will say that a smooth function $V : D \to \mathbb{R}^n$ is a *common local Lyapunov function* for the family (9) if $V(0) = 0$, $V(x) > 0 \; \forall x \in D \setminus \{0\}$, and there exists an open set $\bar{D} \subset D$ containing the origin such that the derivative of V along solutions of each system in (9) is negative for all $x \in \bar{D} \setminus \{0\}$.

If the Lie algebra $\{F_p : p \in \mathcal{P}\}_{LA}$ is solvable, then the family (9) possesses a quadratic common local Lyapunov function. Indeed, the present assumptions guarantee that $\{F_p : p \in \mathcal{P}\}$ is a compact set of strictly stable matrices, hence according to Theorem 2 the linearized family (10) possesses a quadratic common Lyapunov function. One can then apply Lyapunov's first method (see, e.g., [5, Theorem 3.7]) to show that this function is a quadratic common local Lyapunov function for the original family (9).

Now, consider the *switched nonlinear system*

$$\dot{x} = f_\sigma(x) \tag{11}$$

where $\sigma : [0, \infty) \to \mathcal{P}$ is a piecewise constant switching signal. We will say that the system (11) is (locally) *uniformly exponentially stable* if there exist positive constants M, c and μ such that for any switching signal σ the solution of (11) with $\|x(0)\| \leq M$ satisfies

$$\|x(t)\| \leq ce^{-\mu t}\|x(0)\| \quad \forall t \geq 0.$$

The following statement is an immediate consequence of the above.

Corollary 5 *If the Lie algebra $\{F_p : p \in \mathcal{P}\}_{LA}$ is solvable, then the system (11) is uniformly exponentially stable.*

5 Conclusions and future work

We obtained a Lie-algebraic sufficient condition for asymptotic stability of a system that switches between members of a fixed family of asymptotically stable linear systems. We demonstrated, via proving the existence of a quadratic common Lyapunov function for this family, that such a system is globally uniformly exponentially stable under arbitrary switching if the associated matrix Lie algebra is solvable. This relaxes stability conditions found in earlier work; moreover, for our result to hold the given family of systems does not need to be finite. We also presented a sufficient condition for local asymptotic stability of a switched nonlinear system in terms of the Lie algebra generated by the linearization matrices.

In [7] (see also [6]) it is proved that global uniform exponential stability implies the existence of a common Lyapunov function. The paper [6] contains an example which illustrates that even when such a function exists, a *quadratic* one cannot always be found. This clearly shows that the condition presented here is not necessary for uniform exponential stability.

Both exponential stability and existence of a quadratic common Lyapunov function are robust properties in the sense that they are not destroyed by sufficiently small perturbations of the systems' parameters. Regarding perturbations of upper-triangular matrices, one can obtain explicit bounds that have to be satisfied by the elements below the diagonal so that the quadratic common Lyapunov function for the unperturbed systems remains a common Lyapunov function for the perturbed ones [8]. Unfortunately, the condition of Theorem 2 is not robust. In fact, Lie's Theorem suggests that among the families of matrices $\{A_p : p \in \mathcal{P}\}$, those that generate solvable Lie algebras form a nowhere dense set. It is not clear whether it is possible to obtain a more general result than the one given here by characterizing those sets of matrices that give rise to "almost solvable" Lie algebras.

An interesting possible direction for future research is to try to find Lie-algebraic conditions that guarantee uniform exponential stability of (1) when some additional structure is imposed on the matrices A_p. For example, when different state feedback control laws $u = F_p x$, $p \in \mathcal{P}$ are applied to a given linear system $\dot{x} = Ax + Bu$, the matrices of the resulting closed-loop systems take the form $A_p = A + BF_p$.

As for the nonlinear case, we mention here the recent work [13] which directly generalizes the result and the proof technique of [10] to switched nonlinear systems. The main result of [13] states that a finite family of commuting vector fields that give rise to exponentially stable systems has a common Lyapunov function. The commuting condition is formulated in terms of the Lie algebra generated by the original nonlinear vector fields, which opens interesting new possibilities. Without the assumption that all these vector fields satisfy a global Lipschitz condition the asymptotic stability is local, as in our Corollary 5. It remains to be seen whether Lie-algebraic sufficient conditions for *global* asymptotic stability under arbitrary switching can be found in the general nonlinear case.

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