Stabilization of Nonholonomic Integrators via Logic-Based Switching*

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João Hespanha
hespanha@yale.edu

A. S. Morse
morse@sysc.eng.yale.edu

Department of Electrical Engineering
Yale University, New Haven, CT 06520
Phone: (203) 432-2211 Fax: (203) 432-7481

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Abstract

This paper demonstrates how to stabilize a nonholonomic integrator using a hybrid control law employing switching and logic. Results concerning asymptotic stability and exponentially fast convergence to the origin are derived. The methodology used seems to be generalizable to a larger class of control problems related to nonholonomic systems.

1 Introduction

Over the last decade there has been a great deal of research on the problem of stabilizing systems which are locally null controllable but fail to meet Brockett's condition for smooth stabilizability [1]: *Given the system*

$$\dot{x} = f(x,u)\,, \quad x(t_0) = x_0\,, \quad f(0,0) = 0 \tag{1}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *continuously differentiable. If* (1) *is smoothly stabilizable, i.e. there exists a continuously differentiable function* $g(x)$ *such that the origin is an asymptotically stable equilibrium point of* $\dot{x} = f(x, g(x))$*, with stability defined in the Lyapunov sense, then the image of* f *must contain an open neighborhood of the origin.*

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In [2] it is noted that this condition extends to the class of time-invariant feedback laws that are only locally Lipschitz and recently it was shown [3] [4] that Brockett's condition also extends to a even larger class which includes a wide variety of time-invariant discontinuous feedback laws. A prototype example of a system that is not smoothly stabilizable is the so called "nonholonomic integrator" [1]:

$$\dot{x}_1 = u_1 , \quad \dot{x}_2 = u_2 , \quad \dot{x}_3 = x_1 u_2 - x_2 u_1 \tag{2}$$

where $x \triangleq [x_1, x_2, x_3]' \in \mathbb{R}^3$ and $u \triangleq [u_1, u_2]' \in \mathbb{R}^2$. Since the image of the mapping $[x, u]' \mapsto [u_1, u_2, x_1 u_2 - x_2 u_1]'$ does not contain the point $[0, 0, \epsilon]'$ for any $\epsilon \neq 0$, Brockett's condition implies that there is no time-invariant continuously differentiable control law that asymptotically stabilizes the origin.

Possible control laws that avoid the difficulties implied by Brockett's condition are time-varying periodic controllers [5] [6] [7] [8] [9] [10] [11], stochastic control laws [9], and sliding modes control laws [12] [13]. The solution proposed in this paper falls in the class of hybrid control laws, namely those employing both continuous dynamics and discrete logic [14] [15] [16] [17] [18] [19] [20] . Applications of these type of control laws to nonholonomic systems can be found in [22] [23] [24] [25] [26]. In the first two references global convergence to the origin is achieved in finite time; however, these controls may result in chattering in the presence of unmodeled dynamics. The reader is referred to [27] for an extensive survey of recent results in the control of nonholonomic systems. The present paper continues a line of research that was started with [24], where a hybrid control law that achieves algebraically fast convergence to the origin was proposed. The main contribution of the current paper is a new control law that guarantees exponentially fast convergence.

The remaining of the paper is organized as follows. In section 2, motivation and a formal specification for the control law is presented. Section 3 contains the main results of the paper and includes the stability analysis of the hybrid system. Section 4 presents generalizations for higher dimensional systems. Finally, section 5 contains a brief discussion of the results achieved so far and some directions for future research.

2 Switching Controller

Consider again the nonholonomic integrator (2). Whatever control law is chosen, whenever x_1 and x_2 are both zero, $\dot{x}_3$ will also be zero and x_3 will remain constant, until x_1 and/or x_2 became non zero. Furthermore, whenever x_1 and x_2 are "small", only "large" control signals will be able to produce significant changes in x_3. A plausible strategy to make the origin an attractor of the close-loop system is to keep the state away from the axis $x_1 = x_2 = 0$ while x_3 is large and, as x_3 decreases, to let x_1 and x_2 became small. Several control laws can achieve the aforementioned type of behavior. The one presented in this paper has the virtue of being easy to analyze, not only in terms of stability, but also in terms of performance. The control law proposed is constructed as follows:

1. Pick four continuous functions $\pi_k : [0, +\infty) \to [0, +\infty)$, $k \in \mathcal{I} \triangleq \{1, 2, 3, 4\}$ with the following properties:

(i) $\pi_k(0) = 0$, for each $k \in \mathcal{I}$;
(ii) $0 < \pi_1(w) < \pi_2(w) < \pi_3(w) < \pi_4(w)$, for all $w > 0$
(iii) π_1 and π_2 are bounded;
(iv) π_2 and π_3 are monotone nondecreasing;
(v) π_2 is such that

$$\lim_{w \to 0} \frac{w}{\pi_2(w)} = 0$$

Furthermore, if $w \to 0$ exponentially fast then $\frac{w}{\pi_2(w)} \to 0$ exponentially fast.

(vi) π_4 is smooth on some non-empty, open-half interval $(0, c]$ and

$$\frac{\mathrm{d}\pi_4}{\mathrm{d}w}(w) < \frac{\pi_4(w)}{w}, \qquad w \in (0, c]$$

2. Partition $\mathbb{R}^3$ into 4 overlapping regions

$$\begin{aligned}
\mathcal{R}_1 &= \left\{x \in \mathbb{R}^3 : 0 \le {x_1}^2 + {x_2}^2 < \pi_2\left({x_3}^2\right)\right\} \\
\mathcal{R}_2 &= \left\{x \in \mathbb{R}^3 : \pi_1\left({x_3}^2\right) < {x_1}^2 + {x_2}^2 < \pi_4\left({x_3}^2\right)\right\} \\
\mathcal{R}_3 &= \left\{x \in \mathbb{R}^3 : \pi_3\left({x_3}^2\right) < {x_1}^2 + {x_2}^2\right\} \\
\mathcal{R}_4 &= \{(0,0,0)\}
\end{aligned}$$

3. Define the control law

$$u = g_\sigma(x) \tag{3}$$

where σ is a piecewise constant switching signal taking values in $\mathcal{I}$ and

$$g_1(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; g_2(x) = \begin{bmatrix} x_1 + \frac{x_2 x_3}{{x_1}^2 + {x_2}^2} \\ x_2 - \frac{x_1 x_3}{{x_1}^2 + {x_2}^2} \end{bmatrix}, \; g_3(x) = \begin{bmatrix} -x_1 + \frac{x_2 x_3}{{x_1}^2 + {x_2}^2} \\ -x_2 - \frac{x_1 x_3}{{x_1}^2 + {x_2}^2} \end{bmatrix}, \; g_4(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4}$$

The signal σ is defined to be continuous from the right at every point and it is determined recursively by

$$\sigma(t) = \phi(x(t), \sigma^-(t)), \qquad \sigma^-(t_0) = \sigma_0 \in \mathcal{I} \tag{5}$$

where $\sigma^-(t)$ denotes the limit of $\sigma(\tau)$ as $\tau \to t$ from below, and $\phi : \mathbb{R}^3 \times \mathcal{I} \to \mathcal{I}$ is the transition function

$$\phi(x, \sigma) = \begin{cases} \sigma & \text{if } x \in \mathcal{R}_\sigma \\ \max\{k \in \mathcal{I} : x \in \mathcal{R}_k\} & \text{otherwise} \end{cases} \tag{6}$$

σ can be thought as being generated by a computer diagram like the one on Figure 1.

Example. A typical choice for the functions π_k $(k \in \mathcal{I})$ is $\pi_1(w) = (1 - e^{-\sqrt{w}})$, $\pi_2 = 2\,\pi_1$, $\pi_3 = 3\,\pi_1$, and $\pi_4 = 4\,\pi_1$. Figure 2 shows the projection of the corresponding region $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ into the $({x_3}^2, {x_1}^2 + {x_2}^2)$-space.

The type of control proposed is similar as that in [16] and can be seen as an extension of the hysteresis switching algorithm considered in [15]. Its appeal comes from the fact that it naturally excludes the possibility of infinitely fast chattering and therefore does not require the concept of generalized solution in Filippov's sense [28] [12].

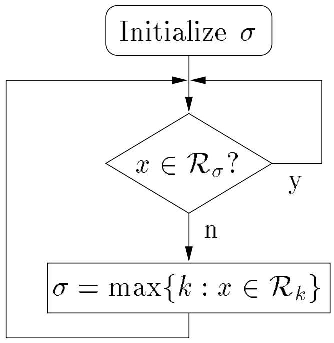

Figure 1: Switching Logic

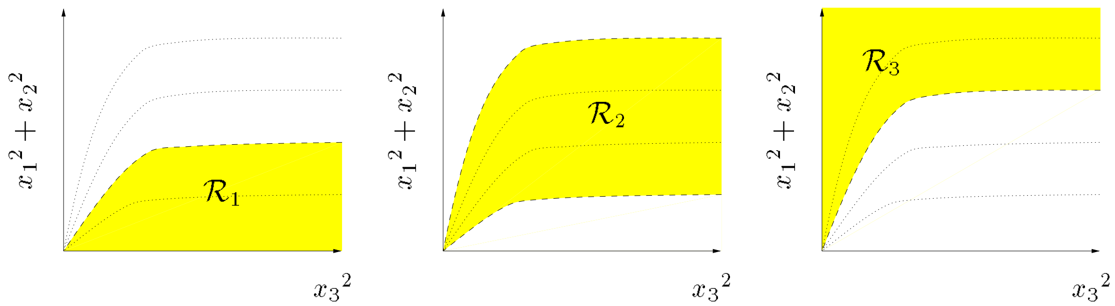

Figure 2: Projection of the regions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ into the $(x_3{}^2, x_1{}^2 + x_2{}^2)$-space.

3 Main Results

The aim of this section is to study the closed loop hybrid dynamical system described in the previous section. The relevant equations are

$$\begin{aligned} \dot{x}_1 &= u_1\,, & u &= g_\sigma(x)\,, \\ \dot{x}_2 &= u_2\,, & \sigma &= \phi(x, \sigma^-)\,, \\ \dot{x}_3 &= x_1\, u_2 - x_2\, u_1 & & \end{aligned}$$

where $x = [x_1, x_2, x_3]' \in \mathbb{R}^3$, $u = [u_1, u_2]' \in \mathbb{R}^2$, and $g_\sigma(x)$, $\phi(x, \sigma)$ are defined by equations (4) and (6) respectively. Although the closed loop system is not globally Lipschitz, global existence of solutions can be easily justified. Indeed, defining $w_1 \stackrel{\triangle}{=} x_3{}^2$ and $w_2 \stackrel{\triangle}{=} x_1{}^2 + x_2{}^2$, simple algebra shows that

$$\dot{w}_1 \;\leq\; 2\, w_1 + w_2 \quad \text{and} \quad \dot{w}_2 \;\leq\; 2\, w_2 + 2.$$

Since the bounds for $\dot{w}_1$ and $\dot{w}_2$ are globally Lipschitz with respect to w_1 and w_2, these variables must be bounded on any finite interval. Moreover, the distance between two points in the (w_1, w_2)-space where consecutive switchings can occur is always nonzero, therefore the boundness of w_1 and w_2 guarantees that infinitely fast chattering cannot occur and global existence of solution is assured.

The usual definition of Lyapunov stability extends in a natural way to hybrid systems: the origin is a Lyapunov stable equilibrium point of the system $\dot{x} = f_\sigma(x)$, $\sigma = \phi(x, \sigma^-)$, $t \geq t_0$, $\sigma \in \mathcal{I}$, and $x \in \mathbb{R}^n$ if

(i) $f_\sigma(0) = 0$ for all $\sigma \in \mathcal{I}$ such that $\phi(0, \sigma) = \sigma$, and

(ii) for all $\epsilon > 0$ there exists a $\delta > 0$ such that for any set of initial conditions $x(t_0) = x_0 \in \mathbb{R}^n$ and $\sigma^-(t_0) = \sigma_0 \in \mathcal{I}$, if $\|x_0\| < \delta$ then $\|x(t)\| < \epsilon$, $\forall t \geq t_0$

Moreover, if for any initial conditions, the continuous part of the state x converges to the origin, then the origin is said to be globally asymptotically stable. The main result of this paper is the following theorem which guarantees global asymptotic stability for the overall system.

Theorem 1. *Let Σ denote the hybrid system defined by* (2), (3), *and* (5).

1. *For any set of initial conditions $x(t_0) = x_0 \in \mathbb{R}^3$ and $\sigma^-(t_0) = \sigma_0 \in \mathcal{I}$ the continuous solution x of Σ and the control signal u converge to zero exponentially fast.*
2. *The origin is a Lyapunov stable equilibrium point of Σ.*

The proof of this theorem requires the notion of positive invariant set. Given a system $\dot{x} = f(x, t)$, a set $\mathcal{R} \subset \mathbb{R}^3$ is *positive invariant* if for every $x(t_0) \in \mathcal{R}$, $x(t) \in \mathcal{R}$ for all $t \geq t_0$. Lemma 1 [24] gives a procedure to check invariance subsets of the state space, by observing the vector field in a neighborhood of the subset's boundary. The formulation used does not require smoothness of the boundary nor the vector field to point strictly to the inside of the subset.

In the sequel, the closure, and boundary of a set $\mathcal{R} \subset \mathbb{R}^n$ are denoted by $\overline{\mathcal{R}}$ and $\partial\mathcal{R}$ respectively. Given a positive constant δ and $x \in \mathbb{R}^n$, $B_\delta(x)$ denotes an open ball of radius $\delta > 0$ centered at x.

Lemma 1. *Given the system*

$$\dot{x} = f(x, t)\,, \quad x(t_0) = x_0\,, \quad f(0, t) = 0, \quad t \geq t_0 \tag{7}$$

with $f : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ locally Lipschitz with respect to x and piecewise continuous with respect to t. Let $x : [t_0, T) \to \mathbb{R}^n$ be a solution of (7) *and $\mathcal{R}$ a closed subset of $\mathbb{R}^n$ such that*

1. *$x(t_0)$ belong to $\mathcal{R}$*
2. *for every $\bar{t}$ such that $x(\bar{t}) \in \partial\mathcal{R}$, there exist 3 positive constants δ_1, δ_2, δ_3 such that*

$$x(\bar{t}) + \epsilon f(\tilde{x}, t) \in \mathcal{R}\,, \qquad \forall \epsilon \in [0, \delta_1],\ \forall \tilde{x} \in B_{\delta_2}(x(\bar{t})) \setminus \mathcal{R},\ t \in [\bar{t}, \bar{t} + \delta_3]$$

Then $x(t) \in \mathcal{R}$ for all $t \in [t_0, T)$.

Condition 2. can be interpreted as requiring the velocity vector to point to the inside of $\mathcal{R}$ in a neighborhood[1] of $x(\bar{t})$ during some interval of positive length starting at $\bar{t}$. This lemma is a straight forward consequence of the mean value theorem. Just for completeness, its proof is included in the appendix.

Outline of Theorem 1's Proof. A detailed proof of Theorem 1 can be found in the appendix. At this point only an outline will be given. Figure 3 shows the projection of a typical trajectory on the $({x_3}^2, {x_1}^2 + {x_2}^2)$-space which may be useful to follow the proof.

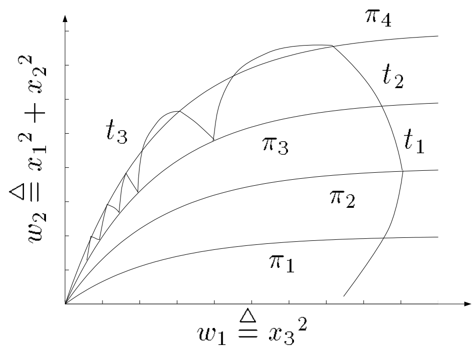

Figure 3: Projection of a typical trajectory on the $({x_3}^2, {x_1}^2 + {x_2}^2)$-space.

1. Convergence: Let $w_1 \triangleq {x_3}^2$ and $w_2 \triangleq {x_1}^2 + {x_2}^2$. While the state is in $\mathcal{R}_1$ {and therefore $\sigma(t) = 1$ or 2}, w_2 grows without bound and therefore at some finite time t_1 this region will be abandoned. Furthermore, observing the direction of the velocity vector over the boundary of $\mathcal{R}_1$, one concludes by Lemma 1 that the state will never get into $\mathcal{R}_1$ again. Pursuing the same reasoning, one concludes that after some finite time t_2 the state will be trapped into $\overline{\mathcal{R}}_3$. At this point the switching signal σ will always be equal to 2, 3, or 4 and therefore

$$\dot{w}_1 = -2\, w_1\,, \quad \forall t \geq t_1$$

which shows that $w_1 \to 0$ as fast as e^{-2t}. In order to show that w_2 also converges to zero another invariant region will be used. Analyzing the direction of the velocity vectors it is possible to conclude that once the state gets into $\overline{\mathcal{R}}_2 \cap \overline{\mathcal{R}}_3$ it will never leave this region, therefore two possibilities exist:

1. x gets into $\overline{\mathcal{R}}_2 \cap \overline{\mathcal{R}}_3$ at some finite time t_3 and never leaves this region. In this case

$$w_2(t) \leq \pi_4(w_1(t))\,, \quad \forall t \geq t_3$$

2. x remains in $\overline{\mathcal{R}}_3 \setminus \overline{\mathcal{R}}_2$ which means that σ is always equal to 3 and therefore

$$\dot{w}_2 = -2\, w_2\,, \quad \forall t \geq t_0$$

[1]According to Lemma 1, only the portion of the neighborhood outside $\mathcal{R}$ (therefore, excluding the boundary) needs to be considered.

In either case exponentially fast convergence to zero of x is achieved. As for the control signal, simple algebra shows that for $t \geq t_2$

$$\|u\| \leq 2\left(\sqrt{w_2} + \sqrt{\frac{w_1}{w_2}}\right) \leq 2\left(\sqrt{w_2} + \sqrt{\frac{w_1}{\pi_2(w_1)}}\right)$$

The hypothesis on π_2 guarantee that $\|u\|$ also converges to zero.

2. Stability: Let us start by assuming that x starts outside $\mathcal{R}_1$. It was argued above that in this case the state never gets into this region again, which means that σ will always be equal to 2, 3, or 4 and therefore

$$\dot{w}_1 = -2\,w_1 \quad \text{and} \quad \dot{w}_2 = \begin{cases} 2\,w_2 & \text{if } \sigma = 2 \\ -2\,w_2 & \text{otherwise} \end{cases}$$

This shows that w_1 is ever decreasing and that w_2 will be decreasing outside $\overline{\mathcal{R}}_2 \cap \overline{\mathcal{R}}_3$ and smaller than $\pi_4(w_1)$ once it gets inside $\overline{\mathcal{R}}_2 \cap \overline{\mathcal{R}}_3$, therefore

$$w_1(t) \leq w_1(t_0) \quad \text{and} \quad w_2(t) \leq \max\{w_2(t_0), \pi_4(w_1(t_0))\}, \qquad \forall t \geq t_0$$

Hence, by making $w(t_0)$ small enough one can keep the whole trajectory arbitrarily close to the origin. The only case that was not covered by the above argument was when $\sigma(t_0) = 1$, which can only happen when x starts in $\mathcal{R}_1$. But it was argued in the convergence proof that once x leaves this region it will never enter it again, therefore one could only have problems in the interval $[t_0, t_1)$ for which $\sigma = 1$. Since while σ is equal to 1 the system is linear, it is not surprising to discover that nothing can go wrong. A careful study of this situation can be found in the appendix. $\square$

Figures 4 and 5 show simulations with the control law defined by equations (2), (3), and (5).

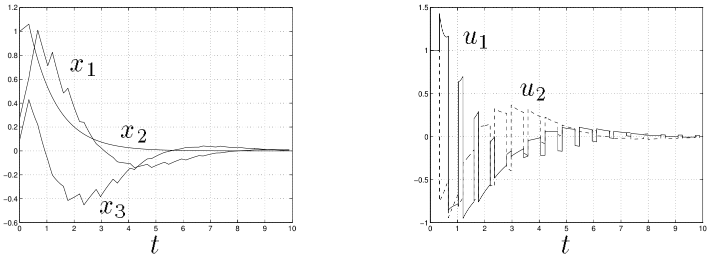

Figure 4: Simulation of the closed loop hybrid system.

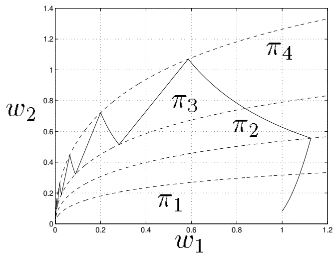

Figure 5: Projection of the trajectory shown in Figure 4 into the (w_1, w_2)-space.

4 Generalization to the Vector Case

The results presented before generalize to the following higher dimensional nonholonomic integrator

$$\dot{x}_1 = u_1 \,, \quad \dot{x}_2 = u_2 \,, \quad \dot{x}_3 = x_1' u_2 - x_2' u_1$$

where $x_1, x_2, u_1, u_2 \in \mathbb{R}^n$ and $x_3 \in \mathbb{R}$. In this case, the regions must be defined by

$$\begin{aligned}
\mathcal{R}_1 &= \left\{ x \in \mathbb{R}^3 : 0 \le x_1'\, x_1 + x_2'\, x_2 < \pi_2\left({x_3}^2\right) \right\}, \\
\mathcal{R}_2 &= \left\{ x \in \mathbb{R}^3 : \pi_1\left({x_3}^2\right) < x_1'\, x_1 + x_2'\, x_2 < \pi_4\left({x_3}^2\right) \right\}, \\
\mathcal{R}_3 &= \left\{ x \in \mathbb{R}^3 : \pi_3\left({x_3}^2\right) < x_1'\, x_1 + x_2'\, x_2 \right\}, \qquad \mathcal{R}_4 \;=\; \{0\}
\end{aligned}$$

and the control laws

$$g_1(x) = \begin{bmatrix} e_1 \\ e_1 \end{bmatrix}, \quad g_{2/3}(x) = \begin{bmatrix} \pm x_1 + \frac{x_3}{x_1'\, x_1 + x_2'\, x_2} x_2 \\ \pm x_2 - \frac{x_3}{x_1'\, x_1 + x_2'\, x_2} x_1 \end{bmatrix}, \quad g_4(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $e_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}'$. Another generalization to a higher dimensional nonholonomic integrator is

$$\dot{x}_1 = u_1 \,, \quad \dot{x}_2 = u_2 \,, \quad \dot{x}_3 = x_1\, u_2 - x_2\, u_1$$

where $x_1, u_1 \in \mathbb{R}$ and $x_2, x_3, u_2 \in \mathbb{R}^n$. In this case the regions must be defined by

$$\begin{aligned}
\mathcal{R}_1 &= \left\{ x \in \mathbb{R}^3 : 0 \le {x_1}^2 + x_2'\, x_2 < \pi_2\left(x_3'\, x_3\right) \right\}, \\
\mathcal{R}_2 &= \left\{ x \in \mathbb{R}^3 : \pi_1\left(x_3'\, x_3\right) < {x_1}^2 + x_2'\, x_2 < \pi_4\left(x_3'\, x_3\right) \right\}, \\
\mathcal{R}_3 &= \left\{ x \in \mathbb{R}^3 : \pi_3\left(x_3'\, x_3\right) < {x_1}^2 + x_2'\, x_2 \right\}, \quad \mathcal{R}_4 \;=\; \{0\}
\end{aligned}$$

and the control laws

$$g_1(x) = \begin{bmatrix} 1 \\ e_1 \end{bmatrix}, \quad g_{2/3}(x) = \begin{bmatrix} \pm x_1 + \frac{x_2'\, x_3}{{x_1}^2 + x_2'\, x_2} \\ \pm x_2 - \frac{x_1}{{x_1}^2 + x_2'\, x_2}\, x_3 \end{bmatrix}, \quad g_4(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note that in either case, while $\sigma = 2$ or 3 one still has $\dot{w}_1 = -2w_1$ and $\dot{w}_2 = \pm 2w_2$, which is enough for the proofs to go through.

5 Conclusion

The main purpose of this paper is to show that logic-based switching can be used efficiently in the control of nonholonomic systems. The application of this type of control to the nonholonomic integrator suggests a methodology to stabilize other nonholonomic systems. The example presented shows that straight forward arguments can be used to prove stability and exponentially fast convergence of the state of the nonholonomic integrator to the origin. Similar controllers can be designed to stabilize other types of nonholonomic systems, in particular chained systems in the power form. A stability analysis for these systems can be made along the same lines as the one presented here, but it is not clear if exponential convergence can always be achieved.

The controller proposed has the drawback that the resulting control signals are discontinuous time functions, but it should be emphasized that this is not a general characteristic of logic-based switching controllers. Further effort is being made to develop alternative control laws without this problem. Another question that deserves attention is prompted by A. Teel's observation [29] that for hybrid systems like the one proposed in this paper, the classical solution to the continuous dynamics varies discontinuously with respect to continuous variations of the initial state, therefore leading to the *hidden possibility of indecision*.

Acknowledgment

The first author would like to thank Lingji Chen for helpful discussion which greatly contributed to this work.

A Appendix

Proof of Lemma 1. By contradiction assume that $x(t)$ leaves the region $\mathcal{R}$, then we can define $\bar{t} = \sup\{t \in [t_0, T) : x([t_0, t]) \in \mathcal{R}\}$. The continuity of $x(t)$ guarantees that $x(\bar{t})$ belongs to $\partial\mathcal{R}$. On the other hand, the piecewise continuity of $f(x, t)$ implies that there must exist a positive constant δ (without loss of generality smaller than δ_1 and δ_3) for which the solution $x(.)$ is continuously differentiable in $(\bar{t}, \bar{t} + \delta]$. Therefore for all $\epsilon \in [0, \delta]$

$$\begin{aligned} x(\bar{t}+\epsilon) &= x(\bar{t}) + \epsilon\dot{x}(\bar{t}+\xi(\epsilon)) \qquad\qquad \text{for some } \xi(\epsilon) \in (0, \epsilon) \\ &= x(\bar{t}) + \epsilon f(x(\bar{t}+\xi(\epsilon)), \bar{t}+\xi(\epsilon)) \end{aligned}$$

Since $x(.)$ is continuous, it is always possible to choose δ small enough such that $x(\bar{t} + \xi(\epsilon))$ always belongs to $B_{\delta_2}(x(\bar{t})) \setminus \mathcal{R}$. But by the definition of $\bar{t}$, $x(\bar{t} + \epsilon) \notin \mathcal{R}$ for small enough ϵ which contradicts condition 2. ∎

In the sequel, given an interval $\mathrm{I} \subset \mathbb{R}$ and $f : \mathrm{I} \to \mathbb{R}^n$, graph$(f)$ denotes the graph of f, i.e.

$$\mathrm{graph}(f) = \{(x_1, x_2) \in \mathbb{R}^{n+1} : x_2 = f(x_1), x_1 \in \mathrm{I}\}$$

The following proof has not been fixed yet

Proof of Theorem 1. The proof of this theorem will be divided into claims for readability.

1. Convergence

Claim 1. There exists a finite time $t_1 \geq t_0$ such that for all $t \geq t_1$

$$x(t) \in \mathbb{R}^3 \setminus \mathcal{S}_1 \quad (\Leftrightarrow\ w(t) \in \Omega \setminus \mathcal{R}_1)$$

This claim will be proven in two steps. First it will be shown that even if $x(t_0) \in \mathcal{S}_1$, there exists a time $t_1 \geq t_0$ such that $x(t_1) \notin \mathcal{S}_1$ and second that if $x(t_1) \notin \mathcal{S}_1$ then $x(t) \notin \mathcal{S}_1$ for all $t \geq t_1$. By contradiction assume that $x(t)$ remains in $\mathcal{S}_1$ for all $t \geq t_0$. Since $x(t_0) \in \mathcal{S}_1$ then $\sigma(t_0) = 1$ or 2. In the first case, i.e. $\sigma(t_0) = 1$, $\sigma(t)$ will always be equal to 1 (since $x(t)$ never leaves $\mathcal{S}_1$ by the contradiction hypothesis). Therefore $\dot{x}_1 = \dot{x}_2 = 1$, $\dot{x}_3 = x_1 - x_2$. It is then easy to check that for all $t \geq t_0$

$$w_2(t) = {x_1}^2(t) + {x_2}^2(t) = 2\left(t - t_0 + \frac{x_1(t_0) + x_2(t_0)}{2}\right)^2 + \frac{\big(x_1(t_0) - x_2(t_0)\big)^2}{2}$$

is unbounded which is a contradiction since in $\mathcal{R}_1$, $w_2(t) \leq \pi_2(w_1(t))$ and $\pi_2(.)$ is a bounded function. Consider now the case $\sigma(t_0) = 2$ (which corresponds that $x(t_0) \in \mathcal{S}_2$). If $\sigma(t)$ would ever switch to 1, by the previous reasoning one would get a contradiction, therefore we must have $\sigma(t) = 2$ for all $t \geq t_0$. This means that

$$\begin{array}{rcl} \dot{x}_1 &=& x_1 + x_2 x_3 \\ \dot{x}_2 &=& x_2 - x_1 x_3 \\ \dot{x}_3 &=& -({x_1}^2 + {x_2}^2)x_3 \end{array} \quad \text{and} \quad \begin{array}{rcl} \dot{w}_1 &=& -2w_1 w_2 \\ \dot{w}_2 &=& 2w_2 \end{array}$$

hence $w_2(t) = w_2(t_0)\, e^{2t}$ (> 0 for $x(t_0) \in \mathcal{S}_2$) is also unbounded which is a contradiction. To conclude that for $t \geq t_1$, $x(t)$ will remain outside $\mathcal{S}_1$ (i.e. inside the closed set $\mathbb{R}^3 \setminus \mathcal{S}_1$), one can use Lemma 1. Condition 2. is satisfied since when $x(\bar{t}) \in \partial\mathcal{S}_1$ ($= W^{-1}(\text{graph}(\pi_2))$), $\sigma(\bar{t}) = 2$ and it will remain constant until the next switch which must occur after some positive time interval (say $\delta_3 > 0$) and therefore for $t \in [\bar{t}, \bar{t} + \delta_3]$

$$\begin{array}{rcl} \dot{x}_1 &=& x_1 + x_2 x_3 \\ \dot{x}_2 &=& x_2 - x_1 x_3 \\ \dot{x}_3 &=& -({x_1}^2 + {x_2}^2)x_3 \end{array} \quad \text{and} \quad \begin{array}{rcll} \dot{w}_1 &=& -2w_1 w_2 & (\leq 0) \\ \dot{w}_2 &=& 2w_2 & (\geq 0) \end{array}$$

which shows that the velocity vector points to the outside (non strictly) of $\mathcal{S}_1$ at least until $\bar{t} + \delta_3$ (note that $\pi_2(.)$ is monotone nondecreasing).

Claim 2. There exists a finite time $t_2 \geq t_1$ such that for all $t \geq t_2$

$$x(t) \in \overline{\mathcal{S}}_3 \quad (\Leftrightarrow\ w(t) \in \overline{\mathcal{R}}_3)$$

This claim will also be proven in two steps. First it will be shown that even if $w(t_1) \notin \overline{\mathcal{R}}_3$, there exists a finite time $t_2 \geq t_1$ such that $w(t_2) \in \overline{\mathcal{R}}_3$ and second that if $w(t_2) \in \overline{\mathcal{R}}_3$ then $w(t) \in \overline{\mathcal{R}}_3$ for all $t \geq t_2$. By contradiction assume that $w(t) \notin \overline{\mathcal{R}}_3$ for all t in $[t_1, +\infty)$. Since we also know that during this interval $w(t) \notin \mathcal{R}_1$, then $\sigma(t) = 2$ for all t in $[t_1, +\infty)$ which means that

$$\begin{array}{rcl} \dot{w}_1 & = & -2w_1w_2 \\ \dot{w}_2 & = & 2w_2 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} w_1(t) & \leq & w_1(t_1) \\ w_2(t) & = & w_2(t_1)\, e^{2t} \quad (> 0 \text{ for } w(t_1) \notin \overline{\mathcal{R}}_3) \end{array}$$

But then $w_2(t)$ would be unbounded which is a contradiction since in $\Omega \setminus \overline{\mathcal{R}}_3$, $w_2(t) < \pi_3(w_1(t)) \leq \pi_3(w_1(t_1))$. To conclude that for $t \geq t_2$, $w(t)$ will remain in $\overline{\mathcal{R}}_3$, one can use Lemma 1. Condition 2. is satisfied since when $w(\bar{t}) \in \partial\mathcal{R}_3$ $(= \text{graph}(\pi_3))$, $\sigma(\bar{t}) = 2$ or 4 and it will remain constant until the next switch which must occur after some positive time interval after (say $\delta_3 > 0$). In this case, for all $t \in [\bar{t}, \bar{t} + \delta_3]$

$$\dot{w}_1 = -2w_1w_2 \ (\leq 0) \quad \text{and} \quad \dot{w}_2 = 2w_2 \ (\geq 0)$$

and therefore, since $\pi_3(.)$ is monotone nondecreasing, the velocity vector indeed points to the inside (non strictly) of $\overline{\mathcal{R}}_3$ at least until $\bar{t} + \delta_3$.

Claim 3. $w_1(t)$ converges to zero as $t \to +\infty$. For $t \geq t_1$, $w_2(t) \geq \pi_2(w_1(t))$ therefore

$$\dot{w}_1(t) \leq -2\pi_2(w_1(t))w_1(t) \qquad \forall t \geq t_1$$

This shows that $w_1(t)$ (≥ 0) is monotone non increasing hence $\lim_{t\to\infty} w_1(t)$ must exist. Since $\pi_2(.)$ is positive definite, the only equilibrium point of $w_1(t)$ is zero therefore the limit must be zero.

Claim 4. There exists a finite time $t_3 \geq t_2$ such that for all $t \geq t_3$

$$x(t) \in \overline{\mathcal{S}_2 \cap \mathcal{S}_3} \quad (\Leftrightarrow \ w(t) \in \overline{\mathcal{R}_2 \cap \mathcal{R}_3})$$

This claim will also be proven in two steps. First it will be shown that even if $w(t_2) \notin \overline{\mathcal{R}_2 \cap \mathcal{R}_3}$, there exists a finite time $t_3 \geq t_2$ such that $w(t_3) \in \overline{\mathcal{R}_2 \cap \mathcal{R}_3}$ and second that if $w(t_3) \in \overline{\mathcal{R}_2 \cap \mathcal{R}_3}$ then $w(t) \in \overline{\mathcal{R}_2 \cap \mathcal{R}_3}$ for all $t \geq t_2$. Since $w_1(t) \to 0$ as $t \to +\infty$ one can assume without loss of generality that t_2 is large enough such that $w_1(t) \leq c$ for all $t \geq t_2$ and therefore

$$\frac{\partial \pi_4}{\partial w_1}(w_1(t)) < \frac{1}{w_1(t)} \qquad \forall t \geq t_2$$

By contradiction assume that $w(t) \notin \overline{\mathcal{R}_2 \cap \mathcal{R}_3}$ for all t in $[t_2, +\infty)$. Since we also know that during this interval $w(t) \in \overline{\mathcal{R}}_2$, then $\sigma(t) = 3$ for all t in $[t_2, +\infty)$ which means that

$$\begin{array}{rcl} \dot{w}_1 & = & -2w_1w_2 \\ \dot{w}_2 & = & -2w_2 \end{array} \quad \Rightarrow \quad \frac{\partial w_2}{\partial w_1} = \frac{1}{w_1} \quad \Rightarrow \quad w_2(t) = w_2(t_2) + \log\left(\frac{w_1(t)}{w_1(t_2)}\right)$$

But since $w_1(t)$ converges to zero, this would imply that after some time $w_2(t) < 0$ which is a contradiction. To conclude that for $t \geq t_3$, $w(t)$ will remain in $\overline{\mathcal{R}_2 \cap \mathcal{R}_3}$, one can use Lemma 1. To check that condition 2 is satisfied note that $\partial(\mathcal{R}_2 \cap \mathcal{R}_3) = \text{graph}(\pi_3) \cup \text{graph}(\pi_4)$.

When $w(\bar{t}) \in \text{graph}(\pi_3)$ it was already shown that the velocity vector points to the inside of $\overline{\mathcal{R}}_3 \subset \overline{\mathcal{R}_2 \cap \mathcal{R}_3}$. If on the other hand $w(\bar{t}) \in \text{graph}(\pi_4) \setminus \text{graph}(\pi_3)$ then $\sigma(\bar{t}) = 3$ and it will remain constant until the next switch which must occur after some positive time interval after (say $\delta_3 > 0$). In this case, for all $t \in [\bar{t}, \bar{t} + \delta_3]$

$$\dot{w}_1 = -2w_1 w_2 \; (< 0) \quad \text{and} \quad \dot{w}_2 = -2w_2 \; (< 0)$$

and therefore the velocity vector has a slope equal to $1/w_1(t)$. But for any $t \geq t_2$

$$\frac{\partial \pi_4}{\partial w_1}(w_1(t)) < \frac{1}{w_1(t)}$$

therefore the velocity vector points to the inside of $\overline{\mathcal{R}}_2 \subset \overline{\mathcal{R}_2 \cap \mathcal{R}_3}$, at least until $\bar{t} + \delta_3$.

Claim 5. $w_2(t)$ also converges to zero as $t \to +\infty$. To conclude the prove note that for all $t \geq t_3$, $w(t) \in \overline{\mathcal{R}_2 \cap \mathcal{R}_3}$ therefore $\pi_3(w_1(t)) \leq w_2(t) \leq \pi_4(w_1(t))$. Since $\pi_3(.)$ and $\pi_4(.)$ are continuous at zero and $w_1(t) \to 0$ as $t \to +\infty$ then $w_2(t)$ must converge to 0 as $t \to \infty$.

2. Stability

Claim 1. For all $\epsilon > 0$ there exists a $\delta_1 > 0$ such that for any set of initial conditions $x(t_0) = x_0$ and $\sigma^-(t_0) = \sigma_0$ such that $\|x(t_0)\| < \delta_1$ then $\|x(t)\| < \epsilon$ for all $t \in [t_0, \bar{t}_1]$ where

$$\bar{t}_1 \triangleq \sup\{t \geq t_0 : \sigma(t) = 1\}$$

If $\sigma(t_0) \neq 1$ then the claim is trivially true since $\bar{t}_1 = t_0$. Indeed, for $\sigma(t)$ to switch back to 1 it would have to pass through $\mathcal{R}_1 \cap \mathcal{R}_2$ with $\sigma(t) = 2$, but then $\dot{w}_2 = 2w_2 \geq 0$ which means that $w_2(t)$ increases and cannot get below $\pi_1(w_1(t))$. If $\sigma(t_0) = 1$ then $\bar{t}_1$ corresponds to the time at which $w(t)$ leaves $\mathcal{R}_1$. By the definition of $\bar{t}_1$, for all t in $[t_0, \bar{t}_1)$

$$\begin{aligned} x_1(t) &= x_1(t_0) + (t - t_0) \\ x_2(t) &= x_2(t_0) + (t - t_0) \\ x_3(t) &= x_3(t_0) + \{x_1(t_0) - x_2(t_0)\}(t - t_0) \end{aligned}$$

Due to the continuity of $w(t)$ and the closure of $\mathbb{R}^3 \setminus \mathcal{R}_1$, $w(\bar{t}_1) \in \partial\mathcal{R}_1 = \text{graph}(\pi_2)$, therefore

$$\pi_2(w_1(\bar{t}_1)) = w_2(\bar{t}_1) = 2\left(\bar{t}_1 - t_0 + \frac{x_1(t_0) + x_2(t_0)}{2}\right)^2 + \frac{\{x_1(t_0) - x_2(t_0)\}^2}{2}$$

Since $\pi_2(.)$ is bounded, there exists a $k > 0$ such that $\pi_2(w_1) \leq k^2$ for all $w_1 \geq 0$, therefore is easy to verify that

$${x_1}^2(\bar{t}_1) + {x_2}^2(\bar{t}_1) \leq k^2 \quad \Rightarrow \quad \bar{t}_1 - t_0 \leq \frac{k}{\sqrt{2}} - \frac{x_1(t_0) + x_2(t_0)}{2}$$

Assuming that $\|x(t_0)\| \leq \delta$ this implies that $\bar{t}_1 - t_0 \leq k/\sqrt{2} + \delta$ and therefore

$$w_1(\bar{t}_1) = [x_3(t_0) + \{x_1(t_0) - x_2(t_0)\}(\bar{t}_1 - t_0)]^2 \leq \delta^2 \left(1 + \sqrt{2}\,k + 2\delta\right)^2$$

this shows that $w_1(\bar{t}_1)$ can be made arbitrarily small by choosing $\|x(t_0)\|$ small enough. Since $\pi_2(.)$ is continuous at zero one concludes that for all $\epsilon > 0$ there exists a $\delta_1 > 0$ such that if $\|x(t_0)\| < \delta_1$ then $\pi_2(w_1(\bar{t}_1)) < \epsilon^2/81$. And without loss of generality assume that $\delta_1 < \epsilon/9 < 1/2$, then

$$\pi_2(w_1(\bar{t}_1)) = {x_1}^2(\bar{t}_1) + {x_2}^2(\bar{t}_1) < \frac{\epsilon^2}{81} \Rightarrow |x_1(\bar{t}_1)| < \frac{\epsilon}{9} \Rightarrow \bar{t}_1 - t_0 < \frac{\epsilon}{9} + |x_1(t_0)| < \frac{2}{9}\epsilon$$

But then for all $t \in [t_0, \bar{t}_1]$

$$\begin{array}{lllll} |x_1(t)| & < & |x_1(t_0)| + |t - t_0| & < & \epsilon/3 \\ |x_2(t)| & < & |x_2(t_0)| + |t - t_0| & < & \epsilon/3 \\ |x_3(t)| & < & |x_3(t_0)| + (|x_1(t_0)| + |x_2(t_0)|)|t - t_0| & < & \epsilon/3 \end{array} \quad \Rightarrow \quad \|x(t)\| < \epsilon$$

Claim 2. For all $\epsilon > 0$ there exists a $\delta_2 > 0$ such that for any set of initial conditions $x(t_0) = x_0$ and $\sigma^-(t_0) = \sigma_0$ such that $\|x(t_0)\| < \delta_2$ then $|w_1(t)| < \epsilon$ for all $t \geq t_0$.

By claim 1., given $\epsilon > 0$ there exists a $\delta_1 > 0$ such that if $\|x(t_0)\| < \delta_1$ then $|w_1(t)| \leq \|w(t)\| < \epsilon$ for all $t \in [t_0, \bar{t}_1]$ but for $t > \bar{t}_1$ $\sigma(t)$ will never be equal to 1 again therefore $\dot{w}_1 = -2w_1w_2$ (≤ 0) which means that $w_1(t) \leq w_1(\bar{t}_1) < \epsilon$ also for $t \geq \bar{t}_1$.

Claim 3. For all $\epsilon > 0$ there exists a $\delta_3 > 0$ such that for any set of initial conditions $x(t_0) = x_0$ and $\sigma^-(t_0) = \sigma_0$ such that $\|x(t_0)\| < \delta_3$ then $|w_2(t)| < \epsilon$ for all $t \geq t_0$.

If $w(\bar{t}_1) \in \mathcal{R}_3 \setminus \mathcal{R}_2$, while $w_2(t) > \pi_4(w_1(t))$, $\sigma(t) = 3$ and $\dot{w}_2 = -2w_2 \leq 0$, therefore $w_2(t) \leq w_2(\bar{t}_1)$. On the other hand, assuming that $w_1(\bar{t}_1) < c$, once $w(t)$ gets into $\overline{\mathcal{R}}_2$ it will never leave this region (see claim 4. of Theorem 1) therefore $w_2(t)$ becomes smaller or equal then $\pi_4(w_1(t))$. This means that for all $t \geq \bar{t}_1$

$$w_2(t) \leq \max\{w_2(\bar{t}_1), \pi_4(w_1(t))\} \leq \max\{w_2(\bar{t}_1), \pi_4(w_1(\bar{t}_1))\}$$

But by claim 1. one can make $w(t) = (w_1(t), w_2(t))$ arbitrarily small for $t \in [t_0, \bar{t}_1]$ therefore since $\pi_4(.)$ is continuous at zero, $w_2(t)$ can also be made arbitrarily small for $t \geq \bar{t}_1$. ∎

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