CERTAINTY EQUIVALENCE IMPLIES DETECTABILITY [1]

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Abstract: Using recently introduced concepts of input-to-state stability and detectability for nonlinear systems, it is shown that any stabilizing, certainty equivalence control used within an adaptive control system, causes the familiar interconnection of a controlled process and associated output estimator to be *detectable* through the estimator's output error e_p, for every frozen value of the index or parameter vector p upon which both the estimator and controller dynamics depend.

Keywords: Adaptive, Nonlinear, Switching

1. INTRODUCTION

"Certainty equivalence" is a well known heuristic idea which advocates that in an adaptive context, the feedback control to an imprecisely modeled process should, at each instant of time, be designed on the basis of a current estimate of what the process model is, with the understanding that each such estimate is to be viewed as correct even though it may not be. On the surface justification for certainty equivalence seems self-evident: if process model estimates converge to the "true" process model, then a certainty equivalence based controller ought to converge to the nonadaptive controller which would have been implemented had there been no process uncertainty. The problem with this justification is that because of noise and unmodelled dynamics, process model estimates don't typically converge to the true process model – even in those instances where certainty equivalence controls can be shown to perform in a satisfactory manner. A more plausible justification stems from the fact that any {stabilizing} certainty equivalence control causes the familiar interconnection of a controlled process and associated output estimator to be *detectable* through the estimator's output error e_p, for every frozen value of the index or parameter vector p upon which both the estimator and controller dynamics depend. Detectability is key because adaptive controller tuning/switching algorithms are invariably designed to make e_p small – and so with detectability, smallness of e_p ensures smallness of the state of the controlled process and estimator interconnection. The fact that certainty equivalence implies detectability has been known for some time – this has been shown to be so whenever the process model is linear and the controller and estimator models are also linear for every frozen value of p (Morse, 1992). In the full-length version of this paper (Hespanha and Morse, 1998), use is made of recently introduced concepts of input-to-state stability (Sontag, 1989) and detectability (Morse, 1995; Sontag and Wang, 1996*a*) for nonlinear systems, to explain why the same implication is valid in a more general, nonlinear setting. This result is summarized in the sequel.

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Notation: In the sequel, prime denotes transpose and $||x||$ denotes the 2-norm of a vector x in a real, finite dimensional space $\mathcal{X}$. The external direct sum of two real linear spaces $\mathcal{X}$ and $\mathcal{Y}$, is written as $\mathcal{X} \oplus \mathcal{Y}$. We write $\mathcal{L}^1[t_1,\ t_2)$ for the space of all real, vector-valued functions f on $[t_1,\ t_2)$ for which $\int_{t_1}^{t_2} ||f(\tau)||d\tau < \infty$.

Historical Note

Reference (Åström, 1996) attributes the term "certainty equivalence" to H. A. Simon who, in turn, gives the following historical account of the term's origin (Simon, n.d.).

> I used the term "certainty equivalence" in my paper "Dynamic Programming under Uncertainty with a Quadratic Criterion Function," *Econometrica* 24:74-81 (1956). There is no indication in the paper as to whether I thought I was coining a new term or had borrowed one that I already found in use. Perhaps Theil used it in the reference cited in the bibliography – I don't know.
>
> Precisely what I meant by the phrase is stated in the third paragraph of my paper: "When the criterion function is quadratic, the planning problem for the case of uncertainty can be reduced to the problem for the case of certainty simply by replacing, in the computation of the optimal first-period action, the 'certain' future values of variables by their unconditional expectations. In this sense, the unconditional expected values of these variables may be regarded as a set of sufficient statistics for the entire joint probability distribution, or alternatively, as a set of 'certainty equivalents'." Stated otherwise, under the given assumptions, the higher moments of the distribution function are irrelevant to the optimal decision.
>
> This work was done as a part of a project on decision rules for factory scheduling, reported later in a book co-authored with Holt, Muth and Modigliani, and led to ideas on control theory quite similar to some developed soon thereafter, but I think quite independently, by servomechanism theorists who were trying to create the new optimal control theory in engineering. It soon became clear that the quadratic character of the criterion function was essential to the proof, and that it could not be effectively generalized to other functions (old Gauss knew what he was doing when he introduced least squares!). Dick Bellman and members of his group worked for some time seeking non-trivial generalizations, and found none.
>
> H. A. Simon, February, 1997

2. OVERALL PROBLEM

Let $\mathbb{P}$ denote the model of a process with a control input u, a measured output y, and a piecewise-continuous disturbance/noise input w that cannot be measured. Suppose that u, y, and w take values in real, finite-dimensional vector spaces $\mathcal{U}$, $\mathcal{Y}$, and $\mathcal{W}$ respectively. Assume that $\mathbb{P}$ is an unknown member of some suitably defined family of dynamical systems $\mathcal{F}$ whose elements each have a real, finite-dimensional state space $\mathcal{X}_\mathbb{P}$ and a pair of defining equations of the form

$$\dot{x}_\mathbb{P} = A_\mathbb{P}(x_\mathbb{P}, w, u), \qquad y = C_\mathbb{P}(x_\mathbb{P}, w)$$

where $A_\mathbb{P}$ and $C_\mathbb{P}$ are at least locally Lipschitz continuous on $\mathcal{X}_\mathbb{P}\oplus\mathcal{W}\oplus\mathcal{U}$ and $\mathcal{X}_\mathbb{P}\oplus\mathcal{W}$ respectively. Assume, in addition, that $\mathcal{F}$ can be written as

$$\mathcal{F} = \bigcup_{p\in\mathcal{P}} \mathcal{F}_p$$

where $\mathcal{P}$ is either a finite set of indices or a compact subset of parameter values within a real, finite-dimensional, normed linear space. Here each $\mathcal{F}_p$ denotes a subfamily consisting of a given *nominal process model* $\mathbb{M}_p$ together with a collection of "perturbed versions" of $\mathbb{M}_p$. Typically one would require the allowable perturbations to be "small" enough so that for each possible process model $\mathbb{P}$ in $\mathcal{F}$, there is a nominal process model within the set $\mathcal{M} \triangleq \{\mathbb{M}_p : p \in \mathcal{P}\}$, which is "close" to $\mathbb{P}$ in some suitably defined sense. We shall not explicitly demand that this be so, since the main result of this paper does not require it.

The overall problem of interest is to devise a feedback control which regulates y about the value 0 and, in addition, causes y to tend to 0 in the event that w tends to zero. The main result of this paper, Theorem 1, applies to this problem and {with minor modification} to more general tracking problems such as set-point control where the references to be tracked are generated by exogenous, input-free, time-invariant dynamical systems. In the event that the class within which $\mathbb{P}$ resides – say $\mathcal{F}_{p^*}$ – were known and fixed, the problem might be dealt with using standard nonadaptive techniques. On the other hand, if $\mathbb{P}$ were to change slowly or intermittently from time to time, and the corresponding value of p^* over time were known or could be determined from measured data, then the problem might be approached using gain scheduling with p^* playing the role of a scheduling variable – but if the evolution of p^* could not be determined, then the problem would typically call for an adaptive solution. This paper is concerned with this situation.

3. MODEL-BASED CONTROL

Assume that one has chosen a family of off-the-shelf, candidate loop-controllers $\mathcal{C} \triangleq \{\mathbb{C}_p : p \in \mathcal{P}\}$, in such a way that for each $p \in \mathcal{P}$, $\mathbb{C}_p$ would "solve" the regulation problem, were $\mathbb{P}$ to be any element of $\mathcal{F}_p$. The idea then is to generate a *scheduling signal* σ taking values in $\mathcal{P}$, which causes the output y of the process in closed-loop with $\mathbb{C}_\sigma$, as shown in Figure 1, to be regulated about zero. We call $\mathbb{C}_\sigma$ a *multi-controller* and we require it to be a dynamical system with a real,

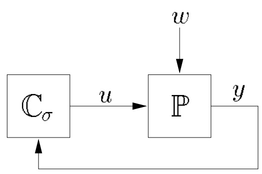

Fig. 1. Process and Multi-Controller Feedback Loop

finite dimensional state space $\mathcal{X}_\mathbb{C}$ and defining equations of the form

$$\dot{x}_\mathbb{C} = F_\sigma(x_\mathbb{C}, y), \qquad u = G_\sigma(x_\mathbb{C}, y) \tag{1}$$

where for each fixed $p \in \mathcal{P}$, the equations

$$\dot{\bar{x}}_\mathbb{C} = F_p(\bar{x}_\mathbb{C}, y), \qquad u_p = G_p(\bar{x}_\mathbb{C}, y)$$

model $\mathbb{C}_p$ and the F_p and G_p are locally Lipschitz continuous functions on $\mathcal{X}_\mathbb{C} \oplus \mathcal{Y}$. Thus we are implicitly assuming that all $\mathbb{C}_p$ have the same finite dimension. This is in fact a very mild assumption, since for $p \in \mathcal{P}$, $\mathbb{C}_p$ is not required to be "observable" through u_p. For example, if one initially had two controllers – say $\mathbb{C}_p$ and $\mathbb{C}_q$ – with the dimension of $\mathbb{C}_p$ less than that of $\mathbb{C}_q$, one could increase the dimension of $\mathbb{C}_p$ to match that of $\mathbb{C}_q$ by simply adjoining to $\mathbb{C}_p$ any stable subsystem Σ of dimension $\dim(\mathbb{C}_q) - \dim(\mathbb{C}_p)$, in such a way that Σ has no influence on $\mathbb{C}_p$'s input-output behavior. Lots of ways to do this can be found in the literature.

Depending on the way in which σ is varied {which might be in either a piecewise-continuous or even a piecewise-constant manner}, the algorithm which generates σ might be called either a "tuner" or a "supervisor." In the sequel we use the term supervisor and we restrict σ to be piecewise-constant since this case is a little easier to understand. Much of which follows however, is also valid for piecewise-continuously tuned σ.

The types of supervisors which seem to be the most promising are those which utilize "estimators" and base controller selection on "certainty equivalence." Certainty equivalence is a well known heuristic idea which advocates that in an adaptive context, the feedback control applied to an imprecisely modeled process should, at each instant of time, be designed on the basis of a current estimate of what the process is, with the understanding that each such estimate is to be viewed as correct even though it may not be. On the surface justification for certainty equivalence seems self-evident: if process model estimates converge to the actual process model, then a certainty equivalence controller ought to converge to the nonadaptive controller which would have been implemented had the correct process model been known in the first place. The problem with this justification is that process model estimates don't typically converge to the actual process model because of noise, disturbances and unmodelled dynamics. So why then does certainty equivalence prove to be especially useful in an adaptive context? To answer this question we need the notion of "input-to-state stability" and an appropriate definition of detectability for nonlinear systems. These ideas are discussed next.

4. INPUT-TO-STATE STABILITY AND DETECTABILITY

Let

$$\dot{x} = A(x, u), \qquad y = C(x, u) \tag{2}$$

be a finite dimensional dynamical system whose state, input, and output take values in real, finite dimensional spaces $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$, respectively. Suppose that A and C are at least locally Lipschitz continuous on $\mathcal{X} \oplus \mathcal{U}$ and that $\tilde{u}$ is an *equilibrium input* of A; i.e., a fixed vector in $\mathcal{U}$ satisfying $A(\tilde{x}, \tilde{u}) = 0$ for some $\tilde{x} \in \mathcal{X}$. The following definition extends to nonzero equilibrium inputs, the concept of "input-to-state stability" given in (Sontag and Wang, 1995).

Input-to-State Stability: The system defined by (2) {or just the state dynamic $\dot{x} = A(x, u)$, or even just A} is said to be *input-to-state stable about* $\tilde{u}$, if $A(\tilde{x}, \tilde{u}) = 0$ for some state $\tilde{x} \in \mathcal{X}$, and there exist continuous, positive definite, radially unbounded functions $V, X : \mathcal{X} \to [0, \infty)$, $U : \mathcal{U} \to [0, \infty)$ such that V is continuously differentiable and

$$\frac{\partial V(x - \tilde{x})}{\partial x} A(x, u) \leq -X(x - \tilde{x}) + U(u - \tilde{u}) \tag{3}$$

for all $(x, u) \in \mathcal{X} \oplus \mathcal{U}$. Note that for each equilibrium input $\tilde{u} \in \mathcal{U}$ about which (2) is input-to-state stable, there can be only one state $\tilde{x} \in \mathcal{X}$ at which $A(\tilde{x}, \tilde{u}) = 0$. We call $\tilde{x}$ the *equilibrium state induced by* $\tilde{u}$. In the event that (2) is input-to-state stable about each of its equilibrium inputs, we say that (2) is input-to-state stable. In the sequel we call any list of functions $\{V, X, U\}$ with the aforementioned properties, a *stability triple* for (2) about $\tilde{u}$. Thus (2) is input-to-state stable about $\tilde{u}$ just in case $\tilde{u}$ is an equilibrium input of (2) and (2) possesses a stability triple about $\tilde{u}$.

As defined, input-to-state stability implies that if u is bounded then so is x and if $u = \tilde{u}$, $t \geq 0$, then $x \to \tilde{x}$ as $t \to \infty$. A more precise statement of these implications, for the case $\tilde{u} = 0$, can be found in (Sontag and Wang, 1996*b*).

It is possible to define detectability in a number of different ways {See (Sontag and Wang, 1997) and references therein}. An especially useful characterization is in terms of an inequality much like (3). A definition along these lines, for systems without inputs, appears in (Morse, 1995). A generalization of this definition which is particularly well-suited to our application, appears in (Sontag and Wang, 1996*a*) and for nonzero equilibrium input-output pairs [2] is as follows.

Detectability: The system defined by (2) {or just the pair $(C,\ A)$} is said to be *detectable about an input-output pair* $\{\tilde{u}, \tilde{y}\}$ if $A(\tilde{x}, \tilde{u}) = 0$ and $\tilde{y} = C(\tilde{x}, \tilde{u})$ for some state $\tilde{x} \in \mathcal{X}$, and there exist continuous, positive definite, radially unbounded functions $V, X : \mathcal{X} \rightarrow [0, \infty)$, $U : \mathcal{U} \rightarrow [0, \infty)$, $Y : \mathcal{Y} \rightarrow [0, \infty)$ such that V is continuously differentiable and

$$\begin{aligned}\frac{\partial V(x-\tilde{x})}{\partial x} A(x,u) \leq -X(x-\tilde{x}) & \\ +Y\big(C(x,u)-\tilde{y}\big) & \\ +U(u-\tilde{u}) & \qquad (4)\end{aligned}$$

for all $(x, u) \in \mathcal{X} \oplus \mathcal{U}$. It can be shown that for each equilibrium input-output pair $\{\tilde{u},\ \tilde{y}\}$ about which (2) is detectable there is exactly one state $\tilde{x} \in \mathcal{X}$ at which $A(\tilde{x}, \tilde{u}) = 0$ and $\tilde{y} = C(\tilde{x}, \tilde{u})$. We call $\tilde{x}$ the *equilibrium state* induced by $\{\tilde{u},\ \tilde{y}\}$. In the sequel we call any list of functions $\{V,\ X,\ U,\ Y\}$ with the aforementioned properties, a *detectability quadruple* for (2) about $\{\tilde{u},\ \tilde{y}\}$. Thus (2) is detectable about $\{\tilde{u},\ \tilde{y}\}$ just in case $\{\tilde{u}, \tilde{y}\}$ is an equilibrium input-output pair and (2) possesses a detectability quadruple about $\{\tilde{u},\ \tilde{y}\}$. In case (4) holds without the term $U(u - \tilde{u})$, (2) is said to be *strongly detectable about the equilibrium output* [3] $\tilde{y}$ and the list of functions $\{V,\ X,\ Y\}$ is called a *strong detectability triple* about $\tilde{y}$. Clearly strong detectability implies detectability. It can be shown that for each equilibrium output $\tilde{y}$ about which (2) is strongly detectable there is exactly one state $\tilde{x} \in \mathcal{X}$ at which $A(\tilde{x}, \tilde{u}) = 0$ and $\tilde{y} = C(\tilde{x}, \tilde{u})$, for some $\tilde{u}$. We call $\tilde{x}$ the *equilibrium state* induced by the equilibrium output $\tilde{y}$.

The preceding definition of detectability reduces to the familiar one in the event that (2) is a linear system and $\tilde{u}$ and $\tilde{y}$ are both zero. In addition, the definition proves to capture the intuitive notion of detectability, namely that if u and y are bounded, then so is x and if u and y converge to $\tilde{u}$ and $\tilde{y}$ respectively then x converges to $\tilde{x}$. A more precise statement of this property can be found in (Sontag and Wang, 1997).

If $\{V,\ X,\ U\}$ is a stability triple for (2) about $\tilde{u}$, and Y is any scalar-valued, locally Lipschitz continuous, positive definite function, then $\{V,\ X,\ U,\ Y\}$ is a detectability quadruple for (2) about $\{\tilde{u},\ C(\tilde{x}, \tilde{u})\}$ where $\tilde{x}$ is the equilibrium state of (2) induced by $\tilde{u}$; thus input-to-state stability about $\tilde{u}$ implies detectability about $\{\tilde{u},\ C(\tilde{x},\ \tilde{u}\}$. Even more is true if the point at which u enters (2) is an *injection channel*; i.e., if $\mathcal{U} = \mathcal{Y}$ and there exists a locally Lipschitz continuous function $C^{-1} : \mathcal{X} \oplus \mathcal{Y} \rightarrow \mathcal{Y}$ such that $C(x, C^{-1}(x, u)) = u$ for all $u \in \mathcal{U}$ and all $x \in \mathcal{X}$.

Lemma 1. Suppose that the point at which u enters (2) is an injection channel and that (2) is input-to-state stable about an equilibrium input $\tilde{u}$. Then $\tilde{y} \stackrel{\Delta}{=} \tilde{u}$ is an equilibrium output of the system

$$\dot{x} = A(x, C(x,u)), \qquad y = C(x,u) \qquad (5)$$

and (5) is strongly detectably about $\tilde{y}$. Moreover the equilibrium state of (5) induced by its equilibrium output $\tilde{y}$ is the same as the equilibrium state of (2) induced by its equilibrium input $\tilde{u}$.

5. ESTIMATOR-BASED SUPERVISOR

An estimator-based supervisor consists of three subsystems, a "multi-estimator" $\mathbb{E}$, a performance signal generator $\mathbb{PS}$, and a switching logic $\mathbb{S}$.

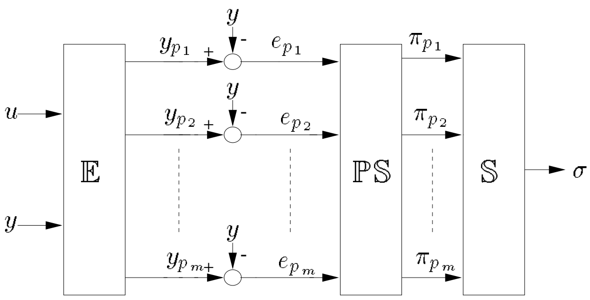

Fig. 2. Estimator-Based Supervisor

A multi-estimator is a parallel implementation of a family of "estimators," one for each $\mathbb{M}_p \in \mathcal{M}$. By a *estimator* $\mathbb{E}_{\mathbb{M}_p}$ for a given nominal process model $\mathbb{M}_p$, is meant any finite-dimensional, input-to-state stable dynamical system whose input is the pair $\{u,\ y\}$ and whose output is a signal y_p which would be an asymptotically correct estimate of y, if $\mathbb{M}_p$ were the actual process model and there were no measurement noise or disturbances. For $\mathbb{E}_{\mathbb{M}_p}$ to have this property, it would have to exhibit {under the feedback interconnection $y \stackrel{\Delta}{=} y_p$ and an appropriate initialization} the same input-output behavior between u and y_p as $\mathbb{M}_p$ does between its input and output. For linear systems such estimators would typically be observers or identifiers (Morse, 1995). Estimators can also be defined quite easily for certain types of nonlinear systems including those which are linearizable by output injection; in this category is any system

[2] A pair $\{\tilde{u},\ \tilde{y}\} \in \mathcal{U} \oplus \mathcal{Y}$ is an *equilibrium input-output pair* of (2) if for some state $\tilde{x}$, $A(\tilde{x}, \tilde{u}) = 0$ and $\tilde{y} = C(\tilde{x}, \tilde{u})$.
[3] A vector $\tilde{y} \in \mathcal{Y}$ is an *equilibrium output* of (2) if for some $\tilde{u} \in \mathcal{U}$, $\{\tilde{u}, \tilde{y}\}$ is an equilibrium input-output pair of (2).

whose state and measured output is one and the same.

By a *multi-estimator* $\mathbb{E}$ for a given family of nominal process models $\mathcal{M} = \{\mathbb{M}_p : p \in \mathcal{P}\}$ is meant a finite dimensional, input-to-state stable system of the form

$$\dot{x}_{\mathbb{E}} = A_{\mathbb{E}}(x_{\mathbb{E}}, u, y), \qquad y_p = C_p(x_{\mathbb{E}}),\ p \in \mathcal{P} \qquad (6)$$

where, for each fixed $p \in \mathcal{P}$,

$$\dot{x}_{\mathbb{E}} = A_{\mathbb{E}}(x_{\mathbb{E}}, u, y), \qquad y_p = C_p(x_{\mathbb{E}}) \qquad (7)$$

is an estimator for $\mathbb{M}_p$. $\mathbb{E}$ is thus a *parallel implementation* of a family of estimators defined by (7) for each $p \in \mathcal{P}$. In the sequel we write $\mathbb{E}_p$ for the pth such estimator and $\mathcal{X}_{\mathbb{E}}$ for the state space of $\mathbb{E}$. We require $A_{\mathbb{E}}$ and the C_p to be locally Lipschitz continuous on $\mathcal{X}_{\mathbb{E}} \oplus \mathcal{U} \oplus \mathcal{Y}$ and $\mathcal{X}_{\mathbb{E}}$ respectively.

Note that all of the estimators implemented by $\mathbb{E}$, *share* the same state $x_{\mathbb{E}}$. The problem of constructing a family of state-sharing estimators $\{\mathbb{E}_p : p \in \mathcal{P}\}$ from a given family of estimators $\{\mathbb{E}_{\mathbb{M}_p} : p \in \mathcal{P}\}$ proves to be quite easy if $\mathcal{P}$ is a finite set. If $\mathcal{P}$ is not finite, such a construction can still be carried out under certain conditions (Hespanha and Morse, 1998).

A *performance signal generator* $\mathbb{PS}$ is a dynamical system whose inputs are *output estimation errors* $e_p \triangleq y_p - y,\ p \in \mathcal{P}$ and whose outputs are *performance signals* $\pi_p,\ p \in \mathcal{P}$. For each $p \in \mathcal{P}$, $\pi_p(t)$ is intended to be a suitably defined measure of the size of the e_p.

The third subsystem of an estimator-based supervisor is a switching logic $\mathbb{S}$. The role of $\mathbb{S}$ is to generate σ. Although there are many different ways to define $\mathbb{S}$, in each case the underlying strategy for generating σ is more or less that same: From time to time set σ equal to that value of $p \in \mathcal{P}$ for which π_p is the smallest. The motivation for this idea is obvious: the nominal process model whose associated performance signal is the smallest, "best" approximates what the process is and thus the candidate controller designed on the basis of that model ought to be able to do the best job of controlling the process. The origin of this idea is the concept of certainty equivalence which we've discussed above.

6. THE IMPLICATION OF CERTAINTY EQUIVALENCE

To understand what certainty equivalence actually implies, let us assume that for each $p \in \mathcal{P}$, $\mathbb{C}_p$ has been chosen so that the system shown in Figure 3 has $\tilde{v} \triangleq 0$ as an equilibrium input and that about this input the system is at least input-to-state stable. Suppose in addition, that $\bar{y}_p = 0$ at the equilibrium state of this system induced by $\tilde{v}$.

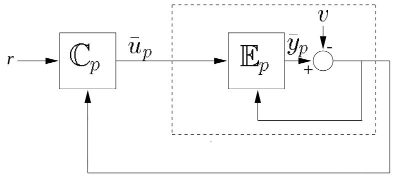

Fig. 3. Feedback Interconnection

By this we mean that for each $p \in \mathcal{P}$, the interconnected system

$$\left.\begin{aligned} \dot{\bar{x}}_{\mathbb{E}} &= A_{\mathbb{E}}(\bar{x}_{\mathbb{E}}, \bar{u}_p, \bar{y}_p - v) & \bar{y}_p &= C_p(\bar{x}_{\mathbb{E}}) \\ \dot{\bar{x}}_{\mathbb{C}} &= F_p(\bar{x}_{\mathbb{C}}, \bar{y}_p - v) & \bar{u}_p &= G_p(\bar{x}_{\mathbb{C}}, \bar{y}_p - v) \end{aligned}\right\} \qquad (8)$$

with input v, is input-to-state stable about an equilibrium input $\tilde{v} = 0$, and moreover that at the equilibrium state induced by this input, $\bar{y}_p = 0$. Justification for placing these requirements on $\mathbb{C}_p$ stems first from the fact that the subsystem enclosed within the dashed box in Figure 3 is input-output equivalent to $\mathbb{M}_p$ if $v \equiv 0$ and second from the control objective which is to regulate y about zero. Here v is to be regarded as a fictitious perturbing signal. Later v will be identified with e_p.

For each $p \in \mathcal{P}$, let Σ_p then denote the system shown in Figure 4 consisting of the interconnection of the process model $\mathbb{P}$, the estimator $\mathbb{E}_p$, and the controller $\mathbb{C}_\sigma$ with σ held fixed at p.

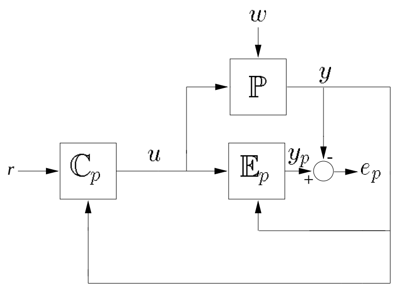

Fig. 4. Σ_p

Thus Σ_p is a dynamical system with input w and output

$$e_p = y_p - y \qquad (9)$$

defined by the process model

$$\dot{x}_{\mathbb{P}} = A_{\mathbb{P}}(x_{\mathbb{P}}, w, u) \qquad y = C_{\mathbb{P}}(x_{\mathbb{P}}, w) \qquad (10)$$

together with the pth estimator-controller equations

$$\left.\begin{aligned} \dot{x}_{\mathbb{E}} &= A_{\mathbb{E}}(x_{\mathbb{E}}, u, y) & y_p &= C_p(x_{\mathbb{E}}) \\ \dot{x}_{\mathbb{C}} &= F_p(x_{\mathbb{C}}, y) & u &= G_p(x_{\mathbb{C}}, y) \end{aligned}\right\} \qquad (11)$$

The main result of this paper is as follows.

Theorem 1. (Certainty Equivalence Stabilization). Let $\mathbb{P}$ be a process model in $\mathcal{F}$ which is detectable about each of its equilibrium input-output pairs. Let $p \in \mathcal{P}$ be fixed. The system Σ_p shown in Figure 4 and defined by (9)-(11) is detectable about any equilibrium input-output pair $\{\tilde{w},\ \tilde{e}_p\}$ it has, for which $\tilde{e}_p = 0$.

The implication of Theorem 1 is clear. For each process model $\mathbb{P} \in \mathcal{F}$ which is detectable about all of its equilibrium inputs, and for each $p \in \mathcal{P}$, the input-to-state stabilization of the system in Figure 3 by $\mathbb{C}_p$ causes the system Σ_p shown in Figure 4 to be detectable about each equilibrium input-output pair $\{\tilde{w},\ \tilde{e}_p\}$ it has {if any}, for which $\tilde{e}_p = 0$. This is what is meant by the phrase *certainty equivalence implies detectability.*

With the preceding in mind, recall the underlying decision making strategy of an estimator-based supervisor: From time to time select for σ, that value $q \in \mathcal{P}$ such that the performance signal π_q is the smallest among the $\pi_p,\ p \in \mathcal{P}$. Justification for this strategy is now clear: By choosing σ to maintain smallness of π_σ and consequently e_σ, the supervisor is also maintaining smallness of the composite state of the interconnection of $\mathbb{P}$, $\mathbb{C}_\sigma$, and $\mathbb{E}$ because of detectability through e_σ for each fixed value of σ.

7. CONCLUDING REMARKS

Throughout this paper we have made use of "linear" estimation errors of the form $e_p = y_p - y$. Given the nonlinear nature of the processes under consideration, it is natural to ask if the results in this paper would still hold if one were to utilize more general "nonlinear" errors. Towards this end, let us agree to call a locally Lipschitz continuous binary operation $\vee : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$, a *generalized sum* if, for all $y,\ \bar{y} \in \mathcal{Y}$, $y \vee \bar{y} = 0 \iff y + \bar{y} = 0$ and there is a locally Lipschitz continuous binary operation $\wedge : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ such that $y = -(-y \wedge \bar{y}) \vee \bar{y}$ for all $y, \bar{y} \in \mathcal{Y}$. For example, if $\vee$ were defined so that $y \vee \bar{y} \triangleq y^3 + \bar{y}^3$, then the definition of $\wedge$ would have to be $y \wedge \bar{y} \triangleq (y + \bar{y}^3)^{\frac{1}{3}}$. If $\vee$ were an associative operation, then $\wedge$ and $\vee$ would be the same.

Now consider a supervisory control system in which for $p \in \mathcal{P}$, e_p is of the form $e_p \triangleq -y \vee y_p$ where $\vee$ is a suitably defined generalized sum, possibly depending on p. Suppose that $\wedge$ is a corresponding binary operation such that $y = -(-y \wedge \bar{y}) \vee \bar{y}$ for all $y, \bar{y} \in \mathcal{Y}$. It is a simple matter to prove that if the e_p are so defined, and if the signals coming out of the summing junctions in Figures 3 and 4 are interpreted as $-v \wedge \bar{y}_p$ and e_p respectively, then Theorem 1 will still be true.

The main objective of this paper has been to make precise and justify a statement applicable many adaptive algorithms, namely that certainty equivalence implies detectability. In the full-length version of this paper (Hespanha and Morse, 1998) it is explained by example, how this implication might be used in analyzing an adaptive system. Theorem 1 is very general. It applies to off-the-shelf, nonadaptive controllers such as back-stepping designed controllers (Kanellakopoulos *et al.*, 1992; Krstić *et al.*, 1995), as well as to those specifically crafted to adaptive applications. The theorem's validity does *not* require the process model to be exactly matched by one of the nominal models in $\mathcal{M}$. In fact, all Theorem 1 requires of the process model is that it be detectable about each of its equilibrium input-output pairs.

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