Towards the High Performance Control of Uncertain Processes via Supervision

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Abstract

This paper deals with the problem of switching between several LTI controllers. Some of these controllers achieve high performance, but are not very robust with respect to changes in the process, while others are robust but do not achieve high performance. It will be the job of a specially designed supervisor to switch into the feedback loop the controller that achieves the best performance, but is still robust enough to guarantee stability. In order to account for the possibility of sudden changes in the process dynamics the supervisor re-accesses it decision periodically and may eventually never settle on a certain controller. Nevertheless, the resulting hybrid system is stable.

1 Introduction

Consider a situation in which one is designing a control system for a LTI process whose mathematical model is well known. Using standard methods one can probably design a feedback control system that guarantees stability and "good" performance. However, if the process is subject to aging and/or component failure, the behavior of the real process deviates from the one of the nominal model. This will surely compromise performance and may eventually result in instability.

The goal of both robust and adaptive control is to address this type of problem but the results attained so far suffer from important setbacks. Many of the standard adaptive control algorithms still require strong and often unrealistic assumptions (known relative degree, minimum phase, etc.) and the experimental results usually show bad transient responses and poor noise/disturbance rejection. The most serious argument against robust control theory is that robustness of stability with respect changes in the process is usually high in terms of performance. In the scenario described, in order for robust control theory to guarantee stability in the event that aging and/or component failure would cause severe changes in the process dynamics, one would have to accept a lower performance even while the process is new and fully operational. This paper addresses the question of using supervision to get high performance and simultaneously, a degree of robustness of stability that could not be achieved by standard linear robust control theory.

The solution proposed in this paper consists in switching between several LTI controllers. Some of these controllers will achieve high performance, but are not very robust with respect to changes in the process, while others are robust but do not achieve high performance. It will be the job of a specially designed supervisor to decide what controller to put on the loop at every instant. This decision will be based on real-time measurements. The behavior one expects from such a system is that while the process is new and fully operational, the supervisor puts on the loop one of the high performance controllers, but, in case there is a failure, the supervisor should "detect" this and preserve stability by putting on the loop one of the very robust controllers. Note that if such behavior is achieved, monitoring the decisions taken by the supervisor can give valuable information about when to re-tune or replace components in the process.

The algorithm proposed in indirect in nature since the supervisor does monitor directly the performance of the controller that is currently in the loop. Instead, the supervisor estimates online the $\mathcal{H}_\infty$-norm of unmodeled dynamics and puts on the loop the controller that robustly stabilizes the process for the estimated amount of unmodeled dynamics. In order for the supervised system to exhibit good performance, the supervisor is "biased" towards the controllers that are known to achieve better performance.

The results presented have two important features:

(i) Stability is assured in the presence of noise (and/or disturbances). Furthermore the algorithm does not require the knowledge of hard bound for the noise nor tries to estimate them. Supervision algorithm that use a hard bound for the noise often exhibit low performance even in the absence of noise.

(ii) Stability of the supervised system is guaranteed even if switching never stops. This is particularly important if the process is not truly stationary and can periodically exhibit abrupt changes in its dynamics. It should be emphasized that need for online supervision can only be truly advocated when dealing with processes that are not truly stationary and therefore offline identification can not be used.

The rest of this paper is organized as follows. The problem addressed in this paper is carefully formulated in section 2. This same section contains a brief discussion of how this problem relates to others that can be found in the literature. Section 3 presents the formal description of the supervisor together with a brief discussion of some of its properties. The main result of the paper is stated in section 4, namely that the supervisor defined in section 3 guarantees closed loop stability. Finally, section 5 contains a brief discussion of the results achieved so far and some directions for future research.

2 Problem Formulation

Let Σ_P be a multivariable LTI process with strictly proper transfer matrix T_P of McMillan degree equal to n_P and minimal realization

$$\begin{aligned} \dot{x}_P &= A_P x_P + B_P u \\ y &= C_P x_P \end{aligned} \tag{1}$$

where u denotes the control input and y the controlled output. The feedback configuration that will be used in this paper is shown in Figure 1. In this figure, the circles represent positive summation points, κ_p the controller transfer matrix, r a reference signal, and w_1 and w_2 unknown but bounded disturbances and/or noise. We will say that κ_p stabilizes T_P if the feedback connection shown in Figure 1 is internally asymptotically stable [1].

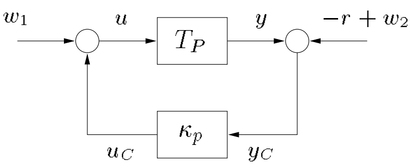

Figure 1: Feedback connection between T and κ.

To focus on the specific issue of improving performance using supervision, we will concentrate in a situation where the process transfer matrix T_P is not accurately known, but belongs to a class of admissible processes $\mathcal{C}_P$ centered around a strictly proper known nominal model ν_P. The class of admissible processes $\mathcal{C}_P$ is such that there exists a controller κ_1 that stabilizes every process in $\mathcal{C}_P$. In its most general (and less formal) form, the problem under consideration can be stated as follows

Problem 1. *Given a finite set of controller transfer matrices $\mathcal{K} = \{\kappa_i : i = 1, \ldots, N\}$, design a supervisor that orchestrates the switching between these controllers in such a way that:*

1. *the resulting hybrid system is stable; and*
2. *the hybrid system performs "better" than the simple interconnection of the process T_P with controller κ_1.*

Since the supervised system is a (nonlinear) hybrid system the concept of stability must be further qualified. In this paper we will demand stability in the BIBO sense.

What distinguishes Problem 1 from the one formulated in [2] is that we no longer require that every controller in $\mathcal{K}$ stabilize every process in $\mathcal{C}_P$. Under these milder assumptions it is natural that not to expect stability for every switching strategy. This problem also relates to the one formulated in [4] but in the present paper we introduce the simplifying assumption that there exists a controller that stabilizes every process in the class of admissible processes and the goal is not just to guarantee stability but also to improve performance by switching. In [5] a similar problem is considered where $\mathcal{K}$ is allowed to contain only model reference controllers either LTI or adaptive and performance is measured in terms of the maximum absolute value of the output error.

In the formulation of Problem 1, no assumptions were made regarding the set of controllers $\mathcal{K}$. We are interested in situations in which there is a particular hierarchy or ordering in the set $\mathcal{K}$. This ordering results from the trade-off between robustness of stability and performance of the closed-loop system. The following assumptions guarantee that this trade-off is well defined, i.e. that a decrease in the robustness is compensated by an improvement in the performance. We believe that controllers for which these assumptions hold arise naturally, for example, in the context of robust control.

Assumption 1 (Stability Embedding). There exists a sequence of classes of admissible processes $\{\mathcal{C}_i : i = 1, \ldots, N\}$ ordered by the inclusion relation

$$\mathcal{C}_P \equiv \mathcal{C}_1 \supset \mathcal{C}_2 \supset \cdots \supset \mathcal{C}_N \supset \{\nu_P\}$$

such that κ_i stabilizes every process in $\mathcal{C}_i$ $(i \in \{1, \ldots, N\})$.

Assumption 2 (Optimality). There exists a measure of the closed-loop performance $\phi : \mathcal{C}_P \times \mathcal{K} \to \mathbb{R}^+$ such that

$$\sup_{\nu \in \mathcal{C}_i} \phi(\nu, \kappa_i) < \sup_{\nu \in \mathcal{C}_i} \phi(\nu, \kappa_{i-1}), \quad i \in \{2, \ldots, N\} \tag{2}$$

Assumption 2 guarantees that the worst case performance of the controller κ_i is "better" than the worst case performance of the controller κ_{i-1} for any process in $\mathcal{C}_i$. If this was not the case, there would be no advantage in using controller κ_i over controller κ_{i-1} since the former is less robust than the latter and it would not result in performance improvement (at least in a worst case sense). In practice, Assumption 2 guarantees that the set of controllers is "optimal" in the sense that a decrease in robustness is compensated by an improvement of performance. This assumption together with Problem 1, motivates the following problem:

Problem 2 (Biased Supervision). *Given a finite set of controller transfer matrices $\mathcal{K} = \{\kappa_i : i = 1, \ldots, N\}$, design a supervisor that orchestrates the switching between these controllers in such a way that:*

1. *the resulting hybrid system is stable; and*
2. *the supervisor should keep σ as large as possible without compromising the previous requirement.*

Supervisors that solve the above are "biased" toward the controllers with high indices. In fact, although it is known that κ_1 stabilizes the process the supervisor will try to put into the loop controllers with higher indices in the hope that this will improve performance. The interesting feature of Problem 2 is that it is formulated without explicit reference to the measure of performance ϕ.

3 Controller Structure

The structure proposed for the hybrid controller is shown in Figure 2. In this figure, Σ_C is a dynamical

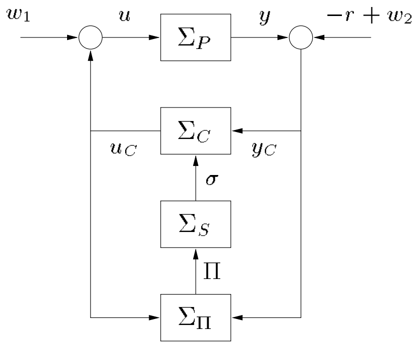

Figure 2: Supervisor structure.

system called a *multi-controller* indexed by switching signal $\sigma : \mathbb{R}^+ \to \{1, \ldots, N\}$. For any constant signal $\sigma = i \in \{1, \ldots, N\}$), Σ_C is linear and time invariant from the sensed process output y_C to the controller output u_C and its transfer matrix is equal to κ_i. The supervisor itself is the hybrid system that results from the interconnection between two subsystems: a *performance signal generator* Σ_Π and a *switching logic* Σ_S. The former is a dynamical system whose inputs are the controller output u_C and the sensed process output y_C and whose output is a vector of performance signals Π. Σ_S is typically a discrete event system that samples Π and generates a sequence of symbols, each one corresponding to a different controller. This sequence is then converted into the switching signal σ which is fed into Σ_C.

Often Σ_C and Σ_Π share similar subsystems and one can reduce the dimension of the overall controller by avoiding duplication of these subsystems. In terms of Figure 2 this means that other signals may flow from Σ_C to Σ_Π. In practice it may also happen that the switching logic Σ_S may have other inputs. For example the reference signal r, inputs provided by a human operator and/or generated by other processes which are independent of Σ_P. In the next sections we will describe these subsystems in detail.

3.1 Multi-controller

There are several ways to realize the multi-controller Σ_C and the reader is referred to [6] for a discussion on some of the alternative. The realization we propose in this paper is very similar to the one in [2]. The following lemma guarantees the existence of a family of systems that will be used to build Σ_C. This lemma is inspired in the so called Youla parameterization of stabilizing controller [7].

Lemma 1. *Given a transfer matrix ν_P and a set of controller transfer matrices $\mathcal{K} = \{\kappa_i : i = 1, \ldots, N\}$ such that for every $i \in \{1, \ldots, N\}$, κ_i stabilizes ν_P. One can compute a family of asymptotically stable transfer matrices $\mathcal{Q} = \{Q_i : i = 1, \ldots, N,\ Q_1 = 0\}$ and a LTI system Σ_J with inputs y_C and v, outputs u_C and e, and state space realization*

$$\begin{aligned} \dot{x}_J &= A_J x_J + D_J y_C + B_J v \\ u_C &= F_J x_J + v \\ e &= C_J x_J + y_C \end{aligned} \tag{3}$$

such that

1. *For every $i \in \{1, \ldots, N\}$, if one sets the input v of Σ_J to be the output of a system with transfer matrix Q_i and input e, the transfer matrix from y_C to u_C is equal to κ_i (see Figure 3).*

2. *Given a system Σ_P with minimal realization given by* (1) *and transfer matrix T_P such that κ_1 stabilizes T_P, the system Σ_I with inputs w_1, $w_2 - r$, and v and outputs y, u and e that results from the connection between Σ_J and Σ_P obtained by setting $u = u_C + w_1$ and $y_C = y - r + w_2$ (see Figure 3) is asymptotically stable with transfer matrix equal to*

$$\begin{bmatrix} X(\tilde{N} + \delta\tilde{X}) & X(\tilde{M} + \delta\tilde{Y}) - I & N + X\delta \\ I + Y(\tilde{N} + \delta\tilde{X}) & Y(\tilde{M} + \delta\tilde{Y}) & M + Y\delta \\ \tilde{N} + \delta\tilde{X} & \tilde{M} + \delta\tilde{Y} & \delta \end{bmatrix} \tag{4}$$

where $\delta \triangleq (-\tilde{N} + \tilde{M}T_P)(\tilde{X} - \tilde{Y}T_P)^{-1}$, N, M, $\tilde{N}$, $\tilde{M}$, X, Y, $\tilde{X}$, and $\tilde{Y}$ are asymptotically stable transfer matrix such that (i) M, $\tilde{M}$, X, and $\tilde{X}$ have causal inverse and (ii) $\nu = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and $\kappa_1 = YX^{-1} = \tilde{X}^{-1}\tilde{Y}$.

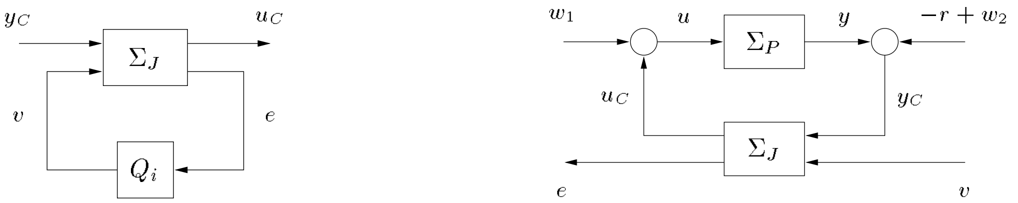

Figure 3: Feedback connections between Σ_J, Q_i, and Σ_P.

The proof of this lemma can be found in the appendix. In order to build Σ_C one has to go through the following steps:

1. Compute the realization for Σ_J and the corresponding set of transfer matrices $\mathcal{Q}$ as given by Lemma 1.
2. Compute a collection of n_Q dimensional asymptotically stable realizations for the transfer matrices in $\mathcal{Q}$. These realizations will be denoted by $\{(A_i, B_i, C_i) : i = 1, \ldots, N\}$.
3. Let Σ_Q be a dynamical system with inputs σ and e, output v and state variable x_Q such that

$$\begin{aligned} \dot{x}_Q &= A_{\sigma(t)} x_Q + B_{\sigma(t)} e \\ v &= C_{\sigma(t)} x_Q \end{aligned} \tag{5}$$

4. The proposed structure for Σ_C is shown in Figure 4.

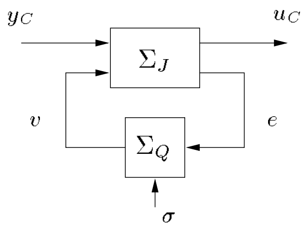

Figure 4: Proposed structure for Σ_C.

Defining[1]

$$n_1 \triangleq \tilde{X}\, w_1 + \tilde{Y}\, w_2\,, \qquad r_1 \triangleq -\tilde{Y}\, r\,,$$

[1] In the sequel the following standard notation will be used: given a transfer matrix $M(s) : \mathbb{C} \to \mathbb{C}^{m \times n}$ and a piecewise constant signal $u : \mathbb{R}^+ \to \mathbb{R}^n$ we will denote by (Mu) the output $y : \mathbb{R}^+ \to \mathbb{R}^m$ of a system with transfer matrix M, input u and zero initial conditions.

$$n_2 \triangleq \tilde{N}\, w_1 + \tilde{M}\, w_2\,, \qquad r_2 \triangleq -\tilde{M}\, r$$

by virtue of claim 2. of Lemma 1 we conclude that there exists a realization (A_I, B_I, C_I) for δ that is asymptotically stable such that

$$\begin{aligned} \dot{x}_I &= A_I x_I + B_I(v + r_1 + n_1) \\ e &= C_I x_I + r_2 + n_2 \end{aligned} \tag{6}$$

Furthermore, the signals y and u can be obtained by

$$y = r - w_2 + X e + N v \quad \text{and} \quad u = w_1 + Y e + M v \tag{7}$$

The advantage of this structure for Σ_C is that it reduces Problem 2 to the following

Problem 3. *Given a finite set of asymptotically stable realizations* $\{(A_i, B_i, C_i) : i = 1, \dots, N\}$*, design a supervisor such that:*

1. *the hybrid system defined by* (5), (6)*, and the supervisor is stable; and*
2. *the supervisor keeps* σ *as large as possible without compromising the previous requirement.*

Before proceeding let us introduce the following standard notation: given a function $H : \mathbb{C} \to \mathbb{C}^{n\times m}$ which is analytical in the closed right hand side complex plane, we define

$$\|H(s)\|_\infty = \sup_{\Re[s]\geq 0} \|H(s)\|$$

where $\|H(s)\|$ denotes the largest singular value of the matrix $H(s)$.

3.2 Supervisor

Let λ be any non negative constant such that $\delta(s-\lambda)$ and $Q_i(s-\lambda)$ $(i \in \{1,\dots,N\})$ are asymptotically stable transfer functions. The performance signal generator Σ_Π is a continuous time dynamical system whose inputs are e, v and r and whose outputs are the signals π_e and π_v given by

$$\begin{aligned} \dot{\pi}_e &= -2\lambda\pi_e + \|e(\tau) - r_2(\tau)\|^2, \qquad \pi_e(0) = 0 \\ \dot{\pi}_v &= -2\lambda\pi_v + \|v(\tau) + r_1(\tau)\|^2, \qquad \pi_v(0) = 0 \end{aligned} \tag{8}$$

where $r_1 \triangleq -\tilde{Y} r$ and $r_2 \triangleq -\tilde{M} r$. The switching logic Σ_S is a hybrid system whose inputs are π_e and π_v and whose output is the piecewise-constant switching signal σ whose value is held constant between successive event times. For the logic considered here event times are equally spaced by a pre-specified positive number τ_D called *dwell time*. To be more precise, event times are simply the set of times $\{k\,\tau_D : k \in \mathbb{N}\}$. For each $k \in \mathbb{N}$, σ's value on $[k\,\tau_D, (k+1)\tau_D)$ is defined by

$$\sigma = \max\Big\{ i \in \{1,\dots,N\} : \eta_i\, \pi_e(k\,\tau_D)^{1/2} \leq \gamma\, \pi_v(k\,\tau_D)^{1/2} + \Gamma \Big\} \tag{9}$$

where $\eta_i \triangleq \|Q_i(s-\lambda)\|_\infty$ $(i \in \{1,\dots,N\})$ and γ and Γ are given constants such that $\gamma \in (0,1)$ and $\Gamma \geq 0$. It will be seen latter that the key property of this supervisor is that for all $k \in \mathbb{N}$

$$\eta_i\, \pi_e(k\,\tau_D)^{1/2} \leq \gamma\, \pi_v(k\,\tau_D)^{1/2} + \Gamma \tag{10}$$

where $i = \sigma(k\,\tau_D)$.

Before proceeding to the analysis of the closed loop system we will discuss the motivating idea that led to the switching algorithm proposed in this paper. If one would set $\sigma = i \in \{1,\dots,N\}$, the system defined

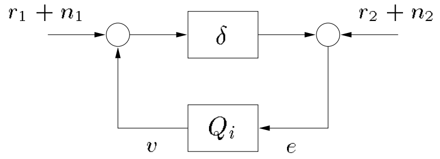

Figure 5: Closed loop system for fixed $\sigma = i$.

by equations (5) and (6) is shown in Figure 5 and its stability is guaranteed by a small gain argument as long as

$$\eta_i \, \|\delta(s-\lambda)\|_\infty \le \gamma \quad (< 1)$$

In fact, by using shifted $\mathcal{H}_\infty$-norms we can actually conclude that if both n_1 and n_2 are equal to zero, all signals converge to zero as fast as $e^{-\lambda t}$. If the value of $\epsilon \triangleq \|\delta(s-\lambda)\|_\infty$ was known, Problem 2 could be solved by picking σ to be the largest integer in $\{1, \ldots, N\}$ such that $\eta_i \, \epsilon \le \gamma < 1$. Since the $\mathcal{H}_\infty$-norm of δ is not known, the supervisor will try to estimate it using information collected in real time. The estimate for the $\mathcal{H}_\infty$-norm of δ steams from the well known identity [8]

$$\|\delta(s-\lambda)\|_\infty = \sup_{\int_0^\infty e^{2\lambda\tau}\|u(\tau)\|^2 \, d\tau < \infty} \left(\frac{\int_0^\infty e^{2\lambda\tau}\|(\delta u)(\tau)\|^2 \, d\tau}{\int_0^\infty e^{2\lambda\tau}\|u(\tau)\|^2 \, d\tau} \right)^{1/2}$$

The above formula motivates the following estimate for ϵ at time $t \ge 0$

$$\left(\frac{\int_0^t e^{2\lambda\tau}\|e(\tau) - r_2(\tau) - n_2(\tau)\|^2 \, d\tau}{\int_0^t e^{2\lambda\tau}\|v(\tau) + r_1(\tau) + n_1(\tau)\|^2 \, d\tau} \right)^{1/2}$$

This estimate has the problem that neither n_1 nor n_2 are available for measurement and therefore the best that can be done is to remove n_1 and n_2 from the above formula. This approximation is correct when both w_1 and w_2 are equal to zero and leads to the following estimate for ϵ

$$\hat{\epsilon}(t) \triangleq \left(\frac{\int_0^t e^{2\lambda\tau}\|e(\tau) - r_2(\tau)\|^2 \, d\tau}{\int_0^t e^{2\lambda\tau}\|v(\tau) + r_1(\tau)\|^2 \, d\tau} \right)^{1/2} = \left(\frac{\pi_e(t)}{\pi_v(t)} \right)^{1/2}$$

This shows that in case $\Gamma = 0$, inequality (10) guarantees that $\eta_i \, \hat{\epsilon}(k\,\tau_D) \le \gamma$ (< 1) and therefore this supervisor extends the concept of certainty equivalence. It should be emphasized that this argument is not enough to guarantee stability of the switched system since there is no guarantee that $\hat{\epsilon}$ is a correct estimate of ϵ, nevertheless in the next section it will be proven that this supervisor stabilized the closed loop system. It will also be shown that adding a positive constant Γ to the right hand side of (10) does not compromise stability and simulation results actually show that picking some $\Gamma > 0$ may improve the response of the overall system.

4 Main Result

The stability results presented in this paper are restricted to the case of 2 controllers ($N = 2$). In this case we can define Σ_Q to be

$$\begin{aligned} \dot{x}_Q &= A_2 \, x_Q + B_2 \, e \\ v' &= C_2 \, x_Q \\ v &= g_\sigma v' \end{aligned} \tag{11}$$

where (A_2, B_2, C_2) is a minimal realization of Q_2 and

$$g_i = \begin{cases} 0 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \end{cases}$$

The system to be analyzed is shown in Figure 6.

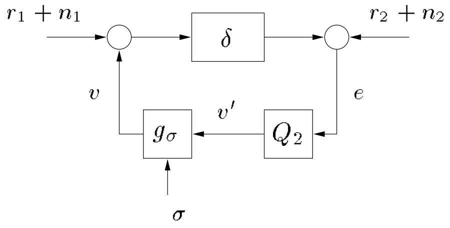

Figure 6: Closed loop system.

Theorem 1. *Let λ be a positive constant such that $\delta(s+\lambda)$ and $Q_2(s+\lambda)$ are asymptotically stable transfer matrices. For any bounded signals w_1, w_2, r and initial conditions $x_I(0)$ and $x_Q(0)$, any solution of the hybrid system defined by (6), (8), (9), and (11), is bounded. Furthermore, in case w_1, w_2, r, and Γ are all equal to 0, all signals converge to zero as fast $e^{-\lambda t}$.*

Before proceeding let us introduce some notation. Given a non zero constant λ and two time instants $t_0, t_1 \in \mathbb{R}^+$ such that $t_0 < t_1$. We will denote by $\mathcal{L}_{\lambda,[t_0,t_1)}$, the space of signals $x : [t_0, t_1) \to \mathbb{R}^{n\times m}$ for which the following integral exists and is finite

$$\int_{[t_0,t_1)} e^{-2\lambda(t_1-\tau)} \|x(\tau)\|^2 \, d\tau$$

The function $\|\cdot\|_{\lambda,[t_0,t_1)} : \mathcal{L}_{\lambda,[t_0,t_1)} \to \mathbb{R}^+$ defined by

$$\|x\|_{\lambda,[t_0,t_1)} \triangleq \left(\int_{[t_0,t_1)} e^{-2\lambda(t_1-\tau)} \|x(\tau)\|^2 \, d\tau \right)^{1/2}, \qquad x \in \mathcal{L}_{\lambda,[t_0,t_1)}$$

is a norm on $\mathcal{L}_{\lambda,[t_0,t_1)}$. Indeed, one can show that

(i) $\|x\|_{\lambda,[t_0,t_1)} \geq 0$ for all $x \in \mathcal{L}_{\lambda,[t_0,t_1)}$ with equality if and only if $x = 0$ almost everywhere in $[t_0, t_1)$,

(ii) $\|\alpha x\|_{\lambda,[t_0,t_1)} = \alpha \|x\|_{\lambda,[t_0,t_1)}$ for every $\alpha \in \mathbb{R}^+$, $x \in \mathcal{L}_{\lambda,[t_0,t_1)}$,

(iii) $\|x + y\|_{\lambda,[t_0,t_1)} \leq \|x\|_{\lambda,[t_0,t_1)} + \|y\|_{\lambda,[t_0,t_1)}$ for every $x, y \in \mathcal{L}_{\lambda,[t_0,t_1)}$ (c.f. [9, p. 548]).

Note that for $\lambda = 0$, $t_0 = 0$, and $t_1 = +\infty$ we have the usual $\mathcal{L}_2$ norm. Lemma 2 in the appendix gives bounds on the state and output of LTI systems as a function of the $\|\cdot\|_{\lambda,[t_0,t_1)}$-norm of its input and the norm of its initial conditions.

Proof of Theorem 1. Let us consider an arbitrary interval $[t_0, t_1)$ ($t_1 \in (t_0, +\infty]$ such that $\sigma(t) = 1$ for all $t \in [t_0, t_1)$. Defining $x \triangleq [x_I' \; x_Q']'$, in the the interval $[t_0, t_1)$ we have that

$$\dot{x} = \begin{bmatrix} A_I & 0 \\ B_2 C_I & A_2 \end{bmatrix} x + \begin{bmatrix} B_I & 0 \\ 0 & B_2 \end{bmatrix} n$$

where $n \triangleq [r_1' + n_1' \; r_2' + n_2']'$. Since A_I and A_2 are stability matrices, if one could prove that the norm of the state at the beginning of such an interval is always bounded, it would follow that the norm of the state is bounded for all times. In the sequel, we will prove x is bounded for all time t for which $\sigma(t) = 2$ and therefore (by continuity) it is bounded at the beginning of any interval for which $\sigma = 1$.

Consider two consecutive event times τ and $\tau + \tau_D$ such that $\sigma(t) = 2$ for all $t \in [\tau, \tau + \tau_D)$. The supervisor logic guarantees that

$$\eta_2 \, \pi_e(\tau)^{1/2} \leq \gamma \, \pi_v(\tau)^{1/2} + \Gamma \quad \Leftrightarrow \quad \eta_2 \, \|e - r_2\|_{\lambda,[0,\tau)} \leq \gamma \, \|v + r_1\|_{\lambda,[0,\tau)} + \Gamma$$

and therefore

$$\begin{aligned} \eta_2 \, \|e\|_{\lambda,[0,\tau)} &\leq \eta_2 \, \|e - r_2\|_{\lambda,[0,\tau)} + \eta_2 \, \|r_2\|_{\lambda,[0,\tau)} \\ &\leq \gamma \, \|v + r_1\|_{\lambda,[0,\tau)} + \eta_2 \, \|r_2\|_{\lambda,[0,\tau)} + \Gamma \\ &\leq \gamma \, \|v\|_{\lambda,[0,\tau)} + \gamma \, \|r_1\|_{\lambda,[0,\tau)} + \eta_2 \, \|r_2\|_{\lambda,[0,\tau)} + \Gamma \end{aligned} \tag{12}$$

On the other hand, since $|g_i| \leq 1$ $(i = 1, 2)$

$$\|v\|_{\lambda,[0,\tau)} \leq \|v'\|_{\lambda,[0,\tau)}$$

and since $\eta_2 = \|Q_2(s - \lambda)\|_\infty$, by virtue of Lemma 2,

$$\|v\|_{\lambda,[0,\tau)} \leq \|v'\|_{\lambda,[0,\tau)} \leq \eta_2 \|e\|_{\lambda,[0,\tau)} + \chi_\eta e^{-\lambda\tau} \|x_Q(0)\| \tag{13}$$

where

$$\chi_\eta \triangleq \left(\int_{[0,+\infty)} \|C_2 e^{(A_2 + \lambda I)t}\|^2 \, dt \right)^{1/2}$$

Combining equations (12) and (13) yields

$$\|v + r_1 + n_1\|_{\lambda,[0,\tau)} \leq \omega_Q(\tau) \quad \text{and} \quad \|e\|_{\lambda,[0,\tau)} \leq \frac{1}{\eta_2} \omega_I(\tau)$$

where

$$\begin{aligned} \omega_Q(\tau) &\triangleq \|n_1\|_{\lambda,[0,\tau)} + \frac{1}{1-\gamma} \|r_1\|_{\lambda,[0,\tau)} + \frac{\eta_2}{1-\gamma} \|r_2\|_{\lambda,[0,\tau)} + \frac{\Gamma}{1-\gamma} + \frac{\chi_\eta}{1-\gamma} e^{-\lambda\tau} \|x_Q(0)\| \\ \omega_I(\tau) &\triangleq \frac{\gamma}{1-\gamma} \|r_1\|_{\lambda,[0,\tau)} + \frac{\eta_2}{1-\gamma} \|r_2\|_{\lambda,[0,\tau)} + \frac{\Gamma}{1-\gamma} + \frac{\gamma \chi_\eta}{1-\gamma} e^{-\lambda\tau} \|x_Q(0)\| \end{aligned}$$

Once again, by virtue of Lemma 2,

$$\begin{aligned} \|x_Q(\tau)\| &\leq \frac{\xi_Q}{\eta_2} \, \omega_I(\tau) + \chi_Q \, e^{-\lambda\tau} \|x_Q(0)\| \\ \|x_I(\tau)\| &\leq \xi_I \, \omega_Q(\tau) + \chi_I \, e^{-\lambda\tau} \|x_I(0)\| \end{aligned}$$

where

$$\xi_Q \triangleq \left(\int_{[0,+\infty)} \left\| e^{(A_2 + \lambda I)t} B_2 \right\|^2 dt \right)^{1/2} \qquad \chi_Q \triangleq \sup_{t \geq 0} \|e^{(A_2 + \lambda I)t}\|$$

$$\xi_I \triangleq \left(\int_{[0,+\infty)} \left\| e^{(A_I + \lambda I)t} B_I \right\|^2 dt \right)^{1/2} \qquad \chi_I \triangleq \sup_{t \geq 0} \|e^{(A_I + \lambda I)t}\|$$

The above bounds are only valid at the beginning of the interval, in order to derive a bound that holds until the next event time we will use Lemma 3. In the interval $[\tau, \tau + \tau_D)$ we have that

$$\dot{x} = A \, x + B \, n$$

where $n \triangleq [r_1' + n_1' \; r_2' + n_2']'$,

$$A \triangleq \begin{bmatrix} A_I & B_I C_2 \\ B_2 C_I & A_2 \end{bmatrix} \quad \text{and} \quad B \triangleq \begin{bmatrix} B_I & 0 \\ 0 & B_2 \end{bmatrix}$$

Therefore, by virtue of Lemma 3, for all $t \in [\tau, \tau + \tau_D)$

$$\begin{aligned} \|x(t)\| &\leq e^{\|A\|\tau_D} \|x(\tau)\| + \frac{\|B\|}{\|A\|} e^{\|A\|\tau_D} \sup_{\tau \in [\tau, \tau+\tau_D]} \|n(\tau)\| \\ &\leq e^{\|A\|\tau_D} \left(\frac{\xi_Q}{\eta_2} \omega_I(\tau) + \xi_I \, \omega_Q(\tau) + (\chi_Q + \chi_I) \, e^{-\lambda\tau} \|x(0)\| \right) \\ &\quad + \frac{\|B\|}{\|A\|} e^{\|A\|\tau_D} \sup_{\tau \in [\tau, \tau+\tau_D]} \|n(\tau)\| \end{aligned} \tag{14}$$

Since w_1, w_2, and r are bounded signals, ω_Q, ω_I and n are also bounded and therefore (14) gives a uniform bound on the norm of the state for every time t for which $\sigma(t) = 2$. Furthermore, in case w_1, w_2, r, and $\Gamma = 0$, the signals ω_Q, ω_I and n converge to zero as fast $e^{-\lambda t}$ and therefore so does $\|x\|$. ■

5 Concluding Remarks

The trade-off between robustness of stability and performance seems to be inevitable when dealing with uncertain systems. If one does not want to leave the world of LTI systems, the decision regarding the amount of uncertainty that the system will be able to endure, must be made offline. This paper presents an heuristic-based algorithm that makes this decision online. This algorithm has the benefit of being tractable from the analysis point of view and displaying good behavior in simulations. The result presented are still preliminary. Current research involves extending the analysis to more than 2 controllers and to show theoretically that the supervisor proposed exhibits better performance that a single LTI controller.

A Appendix

Proof of Lemma 1. Let (A_ν, B_ν, C_ν) be a stabilizable and detectable realization of ν_P, $(A_\kappa, B_\kappa, C_\kappa)$ a stabilizable and detectable realization of κ_1, and K_ν, and K_κ matrices such that $A_\nu + K_\nu C_\nu$ and $A_\kappa + K_\kappa C_\kappa$ are asymptotically stable. Defining

$$\begin{bmatrix} M & Y \\ N & X \end{bmatrix} = \begin{bmatrix} C_\kappa & 0 \\ 0 & C_\nu \end{bmatrix} \left(sI - \begin{bmatrix} A_\kappa & B_\kappa C_\nu \\ B_\nu C_\kappa & A_\nu \end{bmatrix} \right)^{-1} \begin{bmatrix} -K_\kappa & B_\kappa \\ B & -K \end{bmatrix} + I$$

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \begin{bmatrix} C_\kappa & 0 \\ 0 & C_\nu \end{bmatrix} \left(sI - \begin{bmatrix} A_\kappa + K_\kappa C_\kappa & 0 \\ 0 & A_\nu + K_\nu C_\nu \end{bmatrix} \right)^{-1} \begin{bmatrix} K_\kappa & -B_\kappa \\ -B_\nu & K_\nu \end{bmatrix} + I$$

one can prove [10, pp. 7-8] that the 8 asymptotically stable transfer matrices N, M, $\tilde{N}$, $\tilde{M}$, X, Y, $\tilde{X}$, and $\tilde{Y}$ are such that (i) M, $\tilde{M}$, X, and $\tilde{X}$ have causal inverse, (ii) $\nu_P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and $\kappa_1 = YX^{-1} = \tilde{X}^{-1}\tilde{Y}$, and (iii)

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I \quad \text{and} \quad \begin{bmatrix} M & Y \\ N & X \end{bmatrix} \begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = I \tag{15}$$

Let Σ_J be any system such that

$$\begin{bmatrix} u_C \\ e \end{bmatrix} = \begin{bmatrix} I - \tilde{X} & \tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} u_C \\ y_C \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} \tag{16}$$

One possible realization for Σ_J is given by (3) where

$$\begin{aligned} A_J &= \begin{bmatrix} A_\kappa + K_\kappa C_\kappa & 0 \\ 0 & A_\nu + K_\nu C_\nu \end{bmatrix} + B_J F_J, & B_J &= \begin{bmatrix} -K_\kappa \\ -B_\nu \end{bmatrix}, \\ D_J &= \begin{bmatrix} B_\kappa \\ K_\nu \end{bmatrix}, \qquad C_J = \begin{bmatrix} 0 & C_\nu \end{bmatrix}, & F_J &= \begin{bmatrix} C_\kappa & 0 \end{bmatrix} \end{aligned} \tag{17}$$

Since all the controllers in $\mathcal{K}$ stabilize ν_P, it is a consequences of the Youla parameterization that there exists a computable set of asymptotically stable transfer matrices $\mathcal{Q} = \{Q_i : i = 1, \dots, N \text{ and } Q_1 = 0\}$ such that

$$\kappa_i = (\tilde{X} + Q_i \tilde{N})^{-1} (\tilde{Y} + Q_i \tilde{M}) \quad \forall i \in \{1, \dots, N\}$$

Claim 1. can be proven by setting $e = Q_i v$ in (16). After some simple algebraic manipulation one gets

$$u_C = (\tilde{X} + Q_i \tilde{N})^{-1} (\tilde{Y} + Q_i \tilde{M}) y_C \quad \Rightarrow \quad u_C = \kappa_i \, y_C$$

As for claim 2, the state space realization of Σ_I is

$$\begin{aligned} \begin{bmatrix} \dot{x}_P \\ \dot{x}_J \end{bmatrix} &= \begin{bmatrix} A_P & B_P F_J \\ D_J C_P & A_J \end{bmatrix} \begin{bmatrix} x_P \\ x_J \end{bmatrix} + \begin{bmatrix} B_P & 0 & B_P \\ 0 & D_J & B_J \end{bmatrix} \begin{bmatrix} w_1 \\ -r + w_2 \\ v \end{bmatrix} \\ \begin{bmatrix} y \\ u \\ e \end{bmatrix} &= \begin{bmatrix} C_P & 0 \\ 0 & F_J \\ C_P & C_J \end{bmatrix} \begin{bmatrix} x_P \\ x_J \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ -r + w_2 \\ v \end{bmatrix} \end{aligned}$$

Using the definitions in (17), we have

$$\begin{bmatrix} A_P & B_P F_J \\ D_J C_P & A_J \end{bmatrix} = \left[\begin{array}{cc|c} A_P & B_P C_\kappa & 0 \\ B_\kappa C_P & A_\kappa & 0 \\ \hline K_\nu C_P & -B_\nu C_\kappa & A_\nu + K_\nu C_\nu \end{array} \right]$$

Since κ_1 stabilizes T_P and $A_\nu + K_\nu C_\nu$ is asymptotically stable, the above matrix is asymptotically stable. As for the transfer matrix of Σ_I, replacing $y = T_P u$, $u_C = u - w_1$, and $y_C = y - r + w_2$ in (16) one gets

$$u = (I - \tilde{X})(u - w_1) + \tilde{Y}(y - r + w_2) + w_1 + v \tag{18}$$

$$e = -\tilde{N}(u - w_1) + \tilde{M}\,(y - r + w_2) \tag{19}$$

These equations imply that

$$\begin{aligned} e &= (-\tilde{N} + \tilde{M} T_P)\, u + \tilde{N}\, w_1 + \tilde{M}\,(-r + w_2) \\ u &= (\tilde{X} - \tilde{Y} T_P)^{-1} \left(\tilde{X}\, w_1 + \tilde{Y}\,(-r + w_2) + v \right) \end{aligned}$$

Since $\kappa_1 = \tilde{X}^{-1}\tilde{Y}$ stabilizes T_P, the transfer matrix $(\tilde{X} - \tilde{Y} T_P)^{-1}$ is asymptotically stable and therefore so is $\delta \triangleq (-\tilde{N} + \tilde{M} T_P)(\tilde{X} - \tilde{Y} T_P)^{-1}$. Expressing e in terms of the transfer matrix δ, one gets $e = (\tilde{N} + \delta \tilde{X}) w_1 + (\tilde{M} + \delta \tilde{Y})\,(-r + w_2) + \delta\, v$. On the other hand (18) is equivalent to $\tilde{X}\, u - \tilde{Y}\, y = \tilde{X}\, w_1 + \tilde{Y}\,(-r + w_2) + v$ and multiplying this expression from the left by N and M and using the double Bezout identities (15), one gets respectively

$$\begin{aligned} X\tilde{N}\, u + (I - X\tilde{M})\, y &= X\tilde{N}\, w_1 + (-I + X\tilde{M})\,(-r + w_2) + N\, v \\ \text{and} \quad (I + Y\tilde{N})\, u - Y\tilde{M}\, y &= (I + Y\tilde{N})\, w_1 + Y\tilde{M}\,(-r + w_2) + M\, v \end{aligned}$$

which is equivalent to

$$y = X\big(-\tilde{N}(u - w_1) + \tilde{M}\,(y - r + w_2)\big) + r - w_2 + N\, v$$

$$\text{and} \quad u = Y\big(-\tilde{N}(u-w_1) + \tilde{M}\,(y-r+w_2)\big) + w_1 + M\,v$$

Combining this equalities with (19) one concludes that $y = X\,e + r - w_2 + N\,v$ and $u = Y\,e + w_1 + M\,v$ and therefore

$$\begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} X\tilde{N} + X\delta\tilde{X} & X\tilde{M} + X\delta\tilde{Y} - I & N + X\delta \\ I + Y\tilde{N} + Y\delta\tilde{X} & Y\tilde{M} + Y\delta\tilde{Y} & M + Y\delta \\ \tilde{N} + \delta\tilde{X} & \tilde{M} + \delta\tilde{Y} & \delta \end{bmatrix} \begin{bmatrix} w_1 \\ -r + w_2 \\ v \end{bmatrix} \qquad \blacksquare$$

Lemma 2. *Consider a triple of matrices (A, B, C) such that $A + \lambda I$ is an asymptotically stable matrix and let*

$$\xi \triangleq \left(\int_{[0,+\infty)} \left\| e^{(A+\lambda I)t} B \right\|^2 dt \right)^{1/2} \qquad \chi_\xi \triangleq \sup_{t \geq 0} \| e^{(A+\lambda I)t} \|$$

$$\eta \triangleq \left\| C\,(sI - A - \lambda I)^{-1}\,B \right\|_\infty \qquad \chi_\eta \triangleq \left(\int_{[0,+\infty)} \| C e^{(A+\lambda I)t} \|^2\, dt \right)^{1/2}$$

For all $t_0, t \in \mathbb{R}^+$ such that $t_0 < t$

$$\|x(t)\| \leq \xi\, \|u\|_{\lambda,[t_0,t)} + \chi_\xi\, e^{-\lambda(t-t_0)} \|x(t_0)\| \tag{20}$$

$$\|y\|_{\lambda,[t_0,t)} \leq \eta\, \|u\|_{\lambda,[t_0,t)} + \chi_\eta\, e^{-\lambda(t-t_0)} \|x(t_0)\| \tag{21}$$

along any solution of

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Proof of Lemma 2. For every $t \geq t_0$,

$$\begin{aligned} x(t) &= e^{A(t-t_0)} x(t_0) + \int_{[t_0,t)} e^{A(t-\tau)} B u(\tau)\, d\tau \\ &= e^{(A+\lambda I)(t-t_0)} e^{-\lambda(t-t_0)} x(t_0) + \int_{[t_0,t)} e^{(A+\lambda I)(t-\tau)} B\, e^{-\lambda(t-\tau)} u(\tau)\, d\tau \end{aligned}$$

therefore

$$\|x(t)\| \leq \|e^{(A+\lambda I)(t-t_0)}\|\, e^{-\lambda(t-t_0)} \|x(t_0)\| + \int_{[t_0,t)} \|e^{(A+\lambda I)(t-\tau)} B\|\, e^{-\lambda(t-\tau)} \|u(\tau)\|\, d\tau$$

Using the Schwartz Inequality [9, p. 548] one concludes that

$$\begin{aligned} \|x(t)\| &\leq \|e^{(A+\lambda I)(t-t_0)}\|\, e^{-\lambda(t-t_0)} \|x(t_0)\| + \left(\int_{[t_0,t)} \|e^{(A+\lambda I)(t-\tau)} B\|^2 \right)^{1/2} \|u\|_{\lambda,[t_0,t)} \\ &\leq \|e^{(A+\lambda I)(t-t_0)}\|\, e^{-\lambda(t-t_0)} \|x(t_0)\| + \left(\int_{[0,t-t_0)} \|e^{(A+\lambda I)\mu} B\|^2 \right)^{1/2} \|u\|_{\lambda,[t_0,t)} \end{aligned}$$

which proves inequality (20). On the other hand for all $\mu \in [t_0, t)$,

$$y(\mu) = C e^{(A+\lambda I)(\mu - t_0)} e^{-\lambda(\mu - t_0)} x(t_0) + \int_{[t_0,\mu)} C e^{(A+\lambda I)(\mu-\tau)} B\, e^{-\lambda(\mu-\tau)} u(\tau)\, d\tau$$

Therefore

$$
\begin{aligned}
\|y\|_{\lambda,[t_0,t)} \leq & \left\| Ce^{(A+\lambda I)(\mu-t_0)} e^{-\lambda(\mu-t_0)} x(t_0) \right\|_{\lambda,[t_0,t)} \\
& + \left\| \int_{[t_0,\mu)} Ce^{(A+\lambda I)(\mu-\tau)} B\, e^{-\lambda(\mu-\tau)} u(\tau)\, d\tau \right\|_{\lambda,[t_0,t)} \\
= & \left(\int_{[t_0,t)} e^{-2\lambda(t-\mu)} \|Ce^{(A+\lambda I)(\mu-t_0)}\|^2 e^{-2\lambda(\mu-t_0)} \|x(t_0)\|^2\, d\mu \right)^{1/2} \\
& + \left(\int_{[t_0,t)} e^{-2\lambda(t-\mu)} \left\| \int_{[t_0,\mu)} Ce^{(A+\lambda I)(\mu-\tau)} B\, e^{-\lambda(\mu-\tau)} u(\tau)\, d\tau \right\|^2 d\mu \right)^{1/2} \\
= & \left(\int_{[t_0,t)} \|Ce^{(A+\lambda I)(\mu-t_0)}\|^2\, d\mu \right)^{1/2} e^{-\lambda(t-t_0)} \|x(t_0)\| \\
& + \left(e^{-2\lambda t} \int_{[t_0,t)} \left\| \int_{[t_0,\mu)} Ce^{(A+\lambda I)(\mu-\tau)} B\, e^{\lambda\tau} u(\tau)\, d\tau \right\|^2 d\mu \right)^{1/2}
\end{aligned}
$$

But it can be shown [8, p. 25] that

$$
\int_{[t_0,t)} \left\| \int_{[t_0,\mu)} Ce^{(A+\lambda I)(\mu-\tau)} B\, e^{\lambda\tau} u(\tau)\, d\tau \right\|^2 d\mu \leq \eta^2 \int_{[t_0,t)} e^{2\lambda\tau} \|u(\tau)\|^2\, d\tau
$$

and therefore inequality (21) holds. ∎

Lemma 3. *Consider a triple of matrices (A, B, C). For all $t_0, t \in \mathrm{I\!R}^+$ such that $t_0 < t$*

$$
\|x(t)\| \leq e^{\|A\|(t-t_0)} \|x(t_0)\| + \frac{\|B\|}{\|A\|} e^{\|A\|(t-t_0)} \sup_{\tau\in[t_0,t]} \|u(\tau)\| \tag{22}
$$

along any solution of

$$
\dot{x} = Ax + Bu
$$

Proof of Lemma 3. For every $t \geq t_0$,

$$
x(t) = x(t_0) + \int_{t_0}^{t} \Big(Ax(\tau) + Bu(\tau) \Big)\, d\tau
$$

and therefore

$$
\|x(t)\| \leq \|x(t_0)\| + \int_{t_0}^{t} \Big(\|A\| \|x(\tau)\| + \|B\|\, \|u(\tau)\| \Big)\, d\tau
$$

Using the Bellman-Gronwall Lemma [11] on the previous inequality, one concludes that

$$
\|x(t)\| \leq e^{\|A\|(t-t_0)} \|x(t_0)\| + \int_{t_0}^{t} e^{\|A\|(t-\tau)} \|B\|\, \|u(\tau)\|\, d\tau
$$

which guarantees that inequality (22) holds. ∎

References

[1] C. A. Desoer and W. S. Chan. The feedback interconnection of lumped linear time-invariant systems. *J. Franklin Inst.*, 300:335–351, Nov. 1975.

[2] J. P. Hespanha and A. S. Morse. Supervisory control of families of stabilizing controllers. Technical Report 9502, Lab. for Control Science & Eng., Yale University, Dec. 1995.

[3] A. S. Morse. Supervisory control of linear set-point controllers - part 1: exact matching. Technical report, Lab. for Control Science & Eng., Yale University, Mar. 1993.

[4] A. S. Morse. Supervisory control of linear set-point controllers - part 2: robustness. Submitted for publication, Nov. 1994.

[5] J. Balakrishnan. *Control Systems Design Using Multiple Models, Switching, and Tuning*. PhD thesis, Yale University, 1995.

[6] A. S. Morse. Control using logic-based switching. In A. Isidori, editor, *Trends in Control: A European Perspective*, pp. 69–113. Springer-Verlag, 1995.

[7] D. C. Youla, H. A. Jabr, and J. J. Bongiorno. Modern wiener-hopf design of optimal controllers - part ii. the multivariable case. *IEEE Trans. Automat. Contr.*, 21:319–338, 1976.

[8] C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*. Electrical Science Series. Academic Press, 1975.

[9] A. W. Naylor and G. R. Sell. *Linear Operator Theory in Engineering and Science*. Number 40 in Applied Mathematical Sciences. Springer-Verlag, 1982.

[10] T. T. Tay, J. B. Moore, and R. Horowitz. Indirect adaptive techniques for fixed controller performance enhancement. Personal communication, Dec. 1988.

[11] E. D. Sontag. *Mathematical Control Theory*, vol. 6 of *Texts in Applied Mathematics*. Springer-Verlag, 1990.