Logic-Based Switching Algorithms in Control

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Abstract

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This thesis deals with the use of logic-based switching in the control of imprecisely modeled nonlinear systems. Each control system considered consists of a continuous-time dynamical process to be controlled, a family of candidate controllers, and an event-driven switching logic. The need for switching arises when no single candidate controller is capable, by itself, of guaranteeing good performance when connected with a poorly modeled process. In this thesis we develop provably correct switching strategies capable of determining in real-time which candidate controller should be put in feedback with a process so as to achieve a desired closed-loop performance. The resulting closed-loop systems are hybrid in the sense that in each case, continuous dynamics interact with event-driven logic. In the process of designing these switching algorithms, we develop several tools for the analysis and synthesis of hybrid systems. Some of these tools also find application in the design of nonadaptive logic-based switching controllers for systems like the "nonholonomic integrator", a model that, although locally null controllable, is not stabilizable by smooth, time-invariant control laws.

Poorly modeled sensors are also considered in this thesis. In our study of imprecisely modeled sensors we focus on the problem of positioning a robot using a pair of cameras acting as a position measuring device. We derive conditions that enable one to decide on the basis of images acquired by an imprecisely modeled two-camera system, whether or not a prescribed robot positioning task has been precisely accomplished. This line of research represents an initial step towards the design of feedback control systems capable of precise robot positioning using visual feedback, in spite of camera miscalibration.

Para
Maria José e António Hespanha, e
Maria José e Carlos Cordeiro Pereira

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Contents

Chapter 1

Introduction

The idea of integrating logic and continuous dynamics in the control of complex systems is certainly not new. Consider for example an industrial setting where a human operator periodically adjusts the set points of an array of PID controllers in order to account for changes in the environment. One can recognize the human operator as a component of the feedback loop that adjusts the continuous dynamics using logic-based decision rules.

Only in the last 10 years has the theoretical analysis of hybrid systems (i.e., systems with both continuous dynamics and discrete logic) been pursued in a systematic fashion. A fair amount of research has been done on the modeling of hybrid systems [1, 2, 3, 4, 5, 6, 7], but only more recently has the analysis and synthesis of hybrid control laws begun to emerge in the literature. One of the main goals of this thesis is to broaden the body of knowledge concerned with hybrid systems. Toward this end we develop tools to expedite the analysis and design of hybrid control laws.

The basic problem around which this thesis is centered is the control of poorly modeled nonlinear systems. Our paradigm of choice to undertake this problem consists of an architecture in which a high-level, logic-based supervisor orchestrates the switching between a family of candidate controllers so as to achieve some desired behavior for the closed-loop system. The need for switching arises from the fact that no single candidate controller would be capable, by itself, of guaranteeing good performance when connected with the poorly modeled process.

The study of poorly modeled processes has lead us naturally to considering poorly modeled sensors. Imprecision in sensor models has two quite distinct consequences. The first (and more often considered) is the possible loss of stability of the closed-loop system. In this respect, imprecision in the models of the sensors is fundamentally not different from imprecision in the model of the process to be controlled or in the models of the actuators. The second consequence is that even if asymptotic stability is not lost, the closed-loop system may converge to the "wrong" equilibrium point because of sensor modeling errors. Here "wrong" means that, at that equilibrium point, some desired asymptotic behavior is not achieved. In our terminology we say that the "task" is not accomplished.

In our study of imprecisely modeled sensors we focus on the problem of positioning a robot using a pair of cameras acting as a position measuring device. Imprecision in camera models is common because these models depend on a large number of parameters that are difficult to determine.

The remaining of this chapter summarizes the original contribution of this thesis and briefly puts it into perspective with previous research. More detailed discussion of each topic can be found in the body of the thesis.

1.1 Analysis of Hybrid Systems

The first part of the thesis develops tools for the analysis and design of hybrid control laws. These tools are briefly summarized below.

1.1.1 Invariant Sets

Given a Lipschitz continuous function f from a finite dimensional linear space $\mathcal{X}$ into itself, a subset $\mathcal{Z}$ of $\mathcal{X}$ is said to be *(positively) invariant* with respect to the differential equation

$$\dot{x} = f(x), \qquad t \geq t_0 \tag{1.1}$$

if for any $x_0 \in \mathcal{Z}$, the solution x to (1.1) with $x(t_0) = x_0$ remains in $\mathcal{Z}$ for all times $t \geq t_0$ for which the solution is defined [8]. In Chapter 2 the above definition of an invariant set is extended to hybrid systems. The use of invariance in a hybrid systems context is touched upon in [7].

In this thesis, a hybrid system Σ is characterized by four entities: A finite dimensional linear space $\mathcal{X}$ called the *continuous state-space*; an arbitrary set $\mathcal{S}$ called the *discrete state-space*; a family $\{f_s : s \in \mathcal{S}\}$ of locally Lipschitz continuous functions from $\mathcal{X}$ to $\mathcal{X}$ called the *family of vector fields*; and a function $\phi : \mathcal{X} \times \mathcal{S} \to \mathcal{S}$ called the *discrete transition function*. The hybrid system Σ is then defined by the ordinary differential equation

$$\dot{x} = f_\sigma(x), \qquad t \geq t_0 \tag{1.2}$$

together with the recursive equation

$$\sigma = \phi(x, \sigma^-), \qquad t \geq t_0 \tag{1.3}$$

where, for each $t > t_0$, $\sigma^-(t)$ denotes the limit from the left of $\sigma(\tau)$ as $\tau \uparrow t$ and $\sigma^-(t_0)$ is equal to some element of $\mathcal{S}$ that effectively initializes (1.3). By a solution to Σ on the interval $[t_0, T)$ is meant a pair of signals $\{x : [t_0, T) \to \mathcal{X},\ \sigma : [t_0, T) \to \mathcal{S}\}$ with x—the *continuous state*—continuous and piecewise differentiable and σ—the *discrete state*—piecewise constant and right-continuous at every point, such that x and σ satisfy (1.2)–(1.3) on the interval $[t_0, T)$. Other models for hybrid systems can be found in [1, 9, 2, 3, 4, 5, 6, 7]

A pair of sets $\{\mathcal{Z},\ \mathcal{J}\}$ with $\mathcal{Z} \subset \mathcal{X}$ and $\mathcal{J} \subset \mathcal{S}$ is *invariant with respect to* Σ if, for every $x_0 \in \mathcal{Z}$ and every $\sigma_0 \in \mathcal{S}$, any solution $\{x, \sigma\}$ to Σ with $x(t_0) = x_0$ and $\sigma(t_0) = \sigma_0$ remains in $\mathcal{Z} \times \mathcal{J}$ for all times $t \geq t_0$ for which the solution is defined. Lemma 2.1, the main result of Section 2.1, provides a procedure to check for invariance of a given pair of sets just by observing the direction of the vector fields in $\{f_j : j \in \mathcal{J}\}$. The usefulness of this result is that invariance can be established without explicit computation of the hybrid system's state trajectory.

1.1.2 Exponential Stabilization of Nonholonomic Integrators

As an application of the results in Section 2.1, a hybrid control law to exponentially stabilize a "nonholonomic integrator" is constructed and analyzed in Section 2.2. By a *nonholonomic integrator* [10] is meant the three-dimensional system

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_3 = x_1 u_2 - x_2 u_1$$

where $x := [x_1 \quad x_2 \quad x_3]' \in \mathbb{R}^3$ and $u := [u_1 \quad u_2]' \in \mathbb{R}^2$. It turns out that any kinematic completely nonholonomic system with three states and two control inputs can be converted to a nonholonomic integrator by a local coordinate transformation [11]. An interesting property of the nonholonomic integrator is that, although locally null controllable, it is not stabilizable by large classes of continuous-time, time-invariant, control laws [10, 12, 13, 14] (cf. Section 2.2 for details).

The nonholonomic integrator can be asymptotically stabilized using time-varying periodic controllers [15, 16, 17, 18, 19, 20, 21], stochastic control laws [19], and sliding modes control laws [22, 23]. The control proposed in this thesis falls into the class of hybrid control laws, namely those employing both continuous dynamics and discrete logic. Applications of this type of laws to nonholonomic systems can be found in [24, 25, 26, 27, 28, 29]. In the first two references global convergence to the origin is achieved in finite time; however, these controls may result in chattering in the presence of unmodeled dynamics. In [29] a time varying hybrid controller is used to asymptotically stabilize a general class of nonholonomic systems represented in power form. The reader is referred to [30] for an extensive survey of recent results concerned with the control of nonholonomic systems. This thesis proposes a time-invariant hybrid control law that guarantees global asymptotic stability with *exponentially* fast convergence to the origin of the state and input of the nonholonomic integrator. Generalizations of the proposed control law for some higher dimensional systems are also given. Exponentially stabilization of systems like the nonholonomic integrator was also achieved in [21] using nonsmooth, continuous, time-varying control laws.

1.1.3 Switching Among Linear Systems

Given a finite set of matrices $\mathcal{A} := \{A(p) : p \in \mathcal{P}\}$, consider the linear time-varying system

$$\dot{x} = A(\sigma)x \tag{1.4}$$

where σ denotes a piecewise constant "switching signal" taking values on $\mathcal{P}$. The following question has often been posed: "Under what conditions is the system (1.4) uniformly asymptotically stable for *every* piecewise constant switching signal σ?" [31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. In [35] it is shown that uniform asymptotic stability of (3.1) for every switching signal σ is equivalent to the existence of an induced norm $\|\cdot\|_*$ and a positive constant α such that

$$\|e^{At}\|_* \leq e^{-\alpha t}, \qquad \forall t \geq 0,\ \forall A \in \mathcal{A}$$

In [36] it is shown that uniform asymptotic stability of (3.1) for every switching signal σ is also equivalent to the existence of a common Lyapunov function (not necessarily quadratic) for the family of linear time-invariant systems $\{\dot{z} = A(p)z : p \in \mathcal{P}\}$. In [32, 38] are given simple algebraic conditions on the elements of $\mathcal{A}$ that are sufficient for the existence of a common quadratic Lyapunov function for the family of linear time-invariant systems $\{\dot{z} = A(p)z : p \in \mathcal{P}\}$, and therefore for the uniform asymptotic stability of (3.1) for every switching signal σ. For more on this topic see [33, 34] and references therein.

When systems like (1.4) arise in control problems, in general, the matrices in $\mathcal{A}$ have specific structures. Often, these matrices are obtained from the feedback connection of a fixed process with one of several controllers, and the switching signal σ determines which controller is in the feedback loop at each instant of time. One can then ask if, by appropriate choice of the realizations for the controllers, it is possible to make the system (1.4) uniformly asymptotically stable for every switching signal σ.

In Chapter 3 it is assumed given a strictly proper process transfer matrix $\mathrm{H}_{\mathbb{P}}$ and a family of controller transfer matrices $\mathcal{K}_{\mathbb{C}} = \{\mathrm{K}_{\mathbb{C}}(p) : p \in \mathcal{P}\}$ such that every element of $\mathcal{K}_{\mathbb{C}}$ asymptotically stabilizes $\mathrm{H}_{\mathbb{P}}$. The main result of the chapter is that there always exist realizations[1] $\{A_{\mathbb{C}}(p), B_{\mathbb{C}}(p), C_{\mathbb{C}}(p), D_{\mathbb{C}}(p)\}$ for each transfer matrix $\mathrm{K}_{\mathbb{C}}(p)$ in $\mathcal{K}_{\mathbb{C}}$ such that for an appropriate realization $\{A_{\mathbb{P}}, B_{\mathbb{P}}, C_{\mathbb{P}}\}$ for $\mathrm{N}_{\mathbb{P}}$ there exists a common quadratic Lyapunov function for the family of linear time-invariant systems

$$\{\dot{z} = A(p)z : p \in \mathcal{P}\}$$

where each

$$\dot{z} = A(p)z$$

denotes the feedback connection of the process realized by $\{A_{\mathbb{P}}, B_{\mathbb{P}}, C_{\mathbb{P}}\}$ with the controller realized by $\{A_{\mathbb{C}}(p), B_{\mathbb{C}}(p), C_{\mathbb{C}}(p), D_{\mathbb{C}}(p)\}$. This guarantees that (1.4) is uniformly asymptotically stable for *every* switching signal σ.

1.1.4 Scale-Independent Hysteresis Switching

"Scale-independence" is a property of certain switching algorithms used in an adaptive context which is key to proving an algorithm's correctness when operating in the face of noise and disturbance inputs [41]. The concept of *dwell-time switching*—exploited in [41] and elsewhere—has the advantage of being scale-independent. However, the existence of a prescribed dwell-time makes it impossible to rule out the possibility of finite escape in applications of dwell-time switching to the adaptive control of nonlinear systems [42]. On the other hand, the popular idea of *hysteresis switching* [43, 9] does not have this shortcoming. Unfortunately, hysteresis switching is not a scale-independent algorithm. The objective of Chapter 4 is to introduce a new form of chatter-free switching that does not employ a prescribed dwell-time and which is scale independent. We call this logic "scale-independent hysteresis switching" and we prove its correctness for applications to adaptive control [42].

In Chapter 4 we consider the dynamical system

$$\dot{x} = f_{\sigma}(x, t), \qquad x(0) = x_0 \tag{1.5}$$

where $\{f_p : p \in \mathcal{P}\}$ is an indexed family of locally Lipschitz functions taking values on a finite dimensional space $\mathcal{X}$ and defined on $\mathcal{X} \times [0, \infty)$, and σ is a piecewise constant *switching signal* taking values in $\mathcal{P}$. The switching signal σ is chosen so as to cause the *performance signals*

$$\pi_p := \Pi(p, x, t), \qquad p \in \mathcal{P} \tag{1.6}$$

to have certain desired properties. Here, Π is a *performance function* from $\mathcal{P} \times \mathcal{X} \times [0, \infty)$ to $\mathbb{R}$ that is continuous with respect to the second and third arguments for frozen values of the first.

The algorithm used to generate σ considered in Chapter 4 is called a *scale-independent hysteresis switching logic* and can be regarded as a hybrid dynamical system $\mathbb{S}_{\mathbb{H}}$ whose input is x and whose state and output are both σ. To specify $\mathbb{S}_{\mathbb{H}}$ it is necessary to first pick a positive number $h > 0$ called a *hysteresis constant*. $\mathbb{S}_{\mathbb{H}}$'s internal logic is then defined by the computer diagram shown in Figure 1.1 where the π_p are defined by (1.6) and, at each time t,

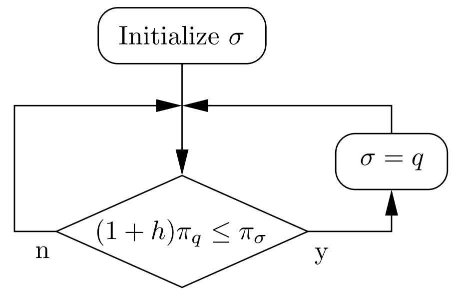

Figure 1.1: Computer Diagram of $\mathbb{S}_{\mathbb{H}}$.

$q := \arg\min_{p\in\mathcal{P}} \Pi(p,x,t)$. In interpreting this diagram it is to be understood that σ's value at each of its switching times $\bar{t}$ is equal to its limit from the right as $t \downarrow \bar{t}$. Thus if $\bar{t}_i$ and $\bar{t}_{i+1}$ are two consecutive switching times, then σ is constant on $[\bar{t}_i, \bar{t}_{i+1})$. The functioning of $\mathbb{S}_{\mathbb{H}}$ is roughly as follows. Suppose that at some time t_0, $\mathbb{S}_{\mathbb{H}}$ has just changed the value of σ to p. σ is then held fixed at this value unless and until there is a time $t_1 > t_0$ at which $(1+h)\pi_q \leq \pi_p$ for some $q \in \mathcal{P}$. If this occurs, σ is set equal to q and so on. This type of logic has numerous applications in adaptive and supervisory control [43, 9, 34, 44, 42, 45, 46, 47]. In Chapter 6 the scale-independent hysteresis switching logic is used in the supervisory control of nonlinear systems.

The main result of Chapter 4 is the scale-independent hysteresis switching theorem. This theorem states that under appropriate "open-loop" assumptions, switching will stop at some finite time. Being able to establish that switching stops in finite time is crucial to the stability analysis of adaptive and supervisory control algorithms using hysteresis switching. In Section 4.2 it is illustrated how the scale-independence property can be used to study the behavior of systems for which the "open-loop" assumptions required by the scale-independent hysteresis switching theorem are violated.

The switching logic described above is new. Its main advantage over the hysteresis switching logic considered in [43, 9] is that it is "scale-independent" in that its output σ remains unchanged if its performance function/input signal pair $\{\Pi, x\}$ is replaced by another pair $\{\bar{\Pi}, \bar{x}\}$ satisfying

$$\bar{\Pi}(p,\bar{x},t) = \vartheta\Pi(p,x,t), \qquad \forall p \in \mathcal{P},\ t \geq 0$$

where ϑ is a positive time function. This is because, for any fixed time t, (i) the value of p that minimizes $\Pi(p,x,t)$ is the same value of p that minimizes $\bar{\Pi}(p,\bar{x},t)$ and (ii) $(1+h)\Pi(q,x,t) \leq \Pi(p,x,t)$ is exactly equivalent to $(1+h)\bar{\Pi}(q,\bar{x},t) \leq \bar{\Pi}(p,\bar{x},t)$ for every $p,q \in \mathcal{P}$. Scale-independence often simplifies considerably the analysis of estimator-based supervisory control algorithms [41, 48, 42, 45, 46, 47].

1.2 Supervisory Control of Families of Nonlinear Systems

Consider a process $\mathbb{P}$ with a control input u, a measured output y, and a piecewise-continuous disturbance/noise input w that cannot be measured. The process $\mathbb{P}$ is assumed to be an

[1] A quadruple of matrices $\{A,B,C,D\}$ is called a *realization* for a transfer matrix T if $T(s) = C(sI-A)^{-1}B + D$ for every $s \in \mathbb{C}$. When the matrix D is equal to zero one often writes simply that $\{A,B,C\}$ is a realization for T.

unknown member of some suitably defined family of dynamical systems $\mathcal{F}$ that can be written as

$$\mathcal{F} = \bigcup_{p \in \mathcal{P}} \mathcal{F}_p \tag{1.7}$$

where $\mathcal{P}$ is a set of indices and each $\mathcal{F}_p$ denotes a subfamily consisting of a given *nominal process model* $\mathbb{M}_p$ together with a collection of "perturbed versions" of $\mathbb{M}_p$.

The overall problem of interest considered in Part II of this thesis is to devise a feedback control that regulates y about the value 0. In the event that the class within which $\mathbb{P}$ resides—say $\mathcal{F}_{p^*}$—were known and fixed, the problem might be dealt with using standard nonadaptive techniques. However, if p^* cannot be determined, then the problem typically calls for an adaptive solution. Chapters 5 and 6 are concerned with this situation.

Assume that one has chosen a family of off-the-shelf, candidate loop-controllers $\mathcal{C} := \{\mathbb{C}_p : p \in \mathcal{P}\}$, in such a way that for each $p \in \mathcal{P}$, $\mathbb{C}_p$ would "solve" the regulation problem, were $\mathbb{P}$ to be any element of $\mathcal{F}_p$. The idea then is to generate a *scheduling signal* σ taking values in $\mathcal{P}$, which causes the output y of the process $\mathbb{P}$ in closed-loop with $\mathbb{C}_\sigma$—as shown in Figure 1.2—to be regulated about zero. The algorithm used to generate σ is called a supervisor. Often

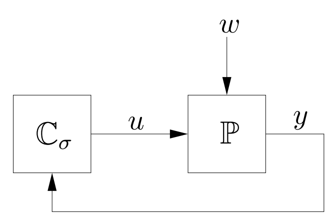

Figure 1.2: Process and Multi-Controller Feedback Loop

supervisors contain continuous dynamics and discrete logic and are therefore hybrid systems.

The use of logic and switching in the control of imprecisely modeled systems is not new in the control literature. In the pioneer work [49], it is shown that it is possible to build a switching controller capable of stabilizing *every* process that can be stabilized by some controller with order no larger than a given integer n. The controller proposed effectively "searches" for a stabilizing controller by switching among the elements of an ordered set of controllers that is dense on the set of all controllers of order up to n. Since then, this type of "ordered" or "pre-routed" search has been explored in many control algorithms proposed in the literature [50, 51, 52, 53, 54, 55, 56]. Switching among adaptive controllers has also appeared often in the literature: in [43, 9, 57] a hysteresis logic is used to switch among several adaptive controllers, each designed having in mind a specific set of assumptions on the process to be controlled; in [58, 59] "cyclic" switching is used to overcome the loss of stabilizability of the estimated process model; in [60] switching is used to overcome the loss of feedback "linearizability" of the estimated process model; and in [44, 61] switching is used to improve the performance of model reference adaptive control. For more on the use of logic-based switching in control see, for example, [34] and references therein.

The types of supervisors that seem to be the most promising[2] are those that utilize "estimators" and base controller selection on "certainty equivalence" [41, 48, 62, 63]. In an adaptive

[2] In [54] a nonestimator-based algorithm is compared with an estimator-based one [41, 48] in terms of simulation results. In the simulation presented, the transient of the estimator-based algorithm is superior but there are no theoretical results that support this observation.

context, certainty equivalence is a well known heuristic idea which advocates that, at each instant of time, one should design the feedback control to an imprecisely modeled process on the basis of the current estimate of what the process model is, with the understanding that each such estimate is to be viewed as correct even though it may not be. On the surface justification for certainty equivalence seems self-evident: if process model estimates converge to the "true" process model, then a certainty equivalence based controller ought to converge to the nonadaptive controller that would have been implemented had there been no process uncertainty. The problem with this justification is that, because of noise and unmodeled dynamics, process model estimates don't typically converge to the true process model—even in those instances where certainty equivalence controls can be shown to perform in a satisfactory manner. A more plausible justification stems from the fact that any (stabilizing) certainty equivalence control causes the familiar interconnection of a controlled process and associated output estimator to be *detectable* through the estimator's output error e_p, for every frozen value of the index or parameter vector p upon which both the estimator and controller dynamics depend. Detectability is key because adaptive controller tuning/switching algorithms are invariably designed to make e_p small—and so with detectability, smallness of e_p ensures smallness of the state of the controlled process and estimator interconnection.

The fact that certainty equivalence implies detectability has been known for some time—this has been shown to be so whenever the process model is linear and the controller and estimator models are also linear for every frozen value of p [64]. In Chapter 5 use is made of recently introduced concepts of input-to-state stability [65] and detectability [34, 66] for nonlinear systems, to explain why the same implication is valid in a more general, nonlinear setting.

Section 5.2 contains the formal definitions of input-to-state stability and detectability for nonlinear systems. Informally, a system is input-to-state stable whenever boundedness of its input guarantees that its state remains bounded. The definition of detectability for a nonlinear systems can be regarded as a generalization of input-to-state stability, in that a system is said to be detectable if boundedness of its input and output guarantees that its state remains bounded. The definition used in this thesis reduces to the familiar one in the event that the system is linear.

1.2.1 Certainty Equivalence Implies Detectability

To understand what certainty equivalence actually implies, let us assume that for each $p \in \mathcal{P}$, $\mathbb{C}_p$ has been chosen so that the system shown in Figure 1.3 is input-to-state stable. In this figure

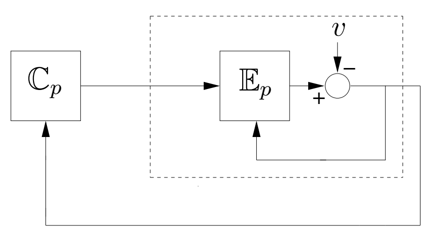

Figure 1.3: Feedback Interconnection

$\mathbb{E}_p$ denotes the pth *estimator*, i.e. it is any finite-dimensional, input-to-state stable dynamical system whose input is the pair $\{u,\ y\}$ and whose output is a signal y_p that would be an asymptotically correct estimate of y, if $\mathbb{M}_p$ were the actual process model and there were no measurement noise or disturbances. For $\mathbb{E}_p$ to have this property, it would have to exhibit (under the feedback interconnection $y := y_p$ and an appropriate initialization) the same input-output behavior between u and y_p as $\mathbb{M}_p$ does between its input and output. For linear systems such estimators are typically observers or identifiers [34].

Justification for requiring that, for each $p \in \mathcal{P}$, the system shown in Figure 1.3 be input-to-state stable[3] stems from the fact that the subsystem enclosed within the dashed box is input-output equivalent to $\mathbb{M}_p$ when $v \equiv 0$. Here v is to be regarded as a fictitious perturbing signal.

For each $p \in \mathcal{P}$, let Σ_p then denote the system shown in Figure 1.4 consisting of the interconnection of the process $\mathbb{P}$, the pth estimator $\mathbb{E}_p$, and the controller $\mathbb{C}_\sigma$ with σ held fixed at p. The main result of Chapter 5 is that for any detectable process model in $\mathcal{F}$ and any

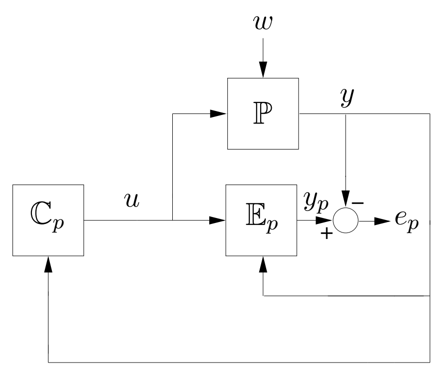

Figure 1.4: Σ_p

$p \in \mathcal{P}$, the system Σ_p is detectable.

In Section 5.5 the detectability of an "injected system" is also established. This injected system does not depend on the process model and is therefore especially useful when one tries to take unmodeled dynamics into account.

Recently several control algorithms motivated by the model validation paradigm [67, 68] have been proposed in the literature. The basic idea consists in starting by hypothesizing an a priori family of admissible process models $\mathcal{F}$—for example as in (1.7)—and then using data collected in real-time to "falsify" those elements of $\mathcal{F}$ that are incompatible with the measured data. In this way, the family $\mathcal{F}(t)$ of processes in which the actual process $\mathbb{P}$ is known to lie at time t, shrinks as time goes by, i.e.,

$$t_2 > t_1 \quad \Rightarrow \quad \mathcal{F}(t_2) \subset \mathcal{F}(t_1), \qquad\qquad \forall t_1, t_2$$

At each instant of time t, the closed-loop controller is then selected on the basis of the current family $\mathcal{F}(t)$ of admissible processes [69, 70, 71]. In [72, 73, 74] a different approach is proposed. In this work, one starts with a family of closed-loop controllers $\mathcal{C}$, at least one of which is guaranteed to achieve some desired level of performance when in feedback with the process $\mathbb{P}$. Data collected in real-time is then used to "falsify" those controllers in $\mathcal{C}$ that are unable to achieve the desired level of performance and, at each instant of time, a controller is selected from those that have not yet been falsified. In the implementations of many of these algorithms, it is possible to recognize structures like the one in Figure 1.4, where e_p typically corresponds

[3]These requirements are implicit in most standard adaptive algorithms.

to a "model validation error" or to a "performance specification error". This same structure can be found in multi-model $\mathcal{H}_\infty$ optimal control [75, 76]. The detectability property proved in this thesis can be used in the analysis of several of these control algorithms.

1.2.2 Supervision Under Exact Matching

In Chapter 6 it is demonstrated how the results in Chapter 5 can be used in the analysis of an adaptive system. To this effect, the analysis of a supervisory control system based on certainty equivalence is carried out in a fairly general setting. The main assumptions are that the disturbance/noise input w is identically zero and that one of the y_p is an asymptotically correct estimate of y. It is also assumed that $\mathcal{P}$ is a finite set. The switching logic used is the scale-independent hysteresis switching logic introduced in Chapter 4. Under the above assumptions, global boundedness and asymptotic convergence are deduced. For illustrative purposes, in Section 6.2, these results are applied to the control of three nonlinear systems.

1.3 Decidability of Robot Positioning Tasks Using Vision

Implicit in the formulation of the control problem considered in Chapters 5 and 6 is the assumption that the signal to be regulated can be directly measured. This does not happen when one wants to control the position of a robot employing a pair of video cameras acting as a position measuring device. Indeed, in this case the control objective—moving the robot to a specified position—is expressed in terms of the Cartesian position and orientation of the robot, but the measured signals are the image coordinates of observed robot "features". If the camera models are accurately known, one can compute a one-to-one correspondence between the robot's Cartesian position/orientation and the image coordinates of the observed robot features. Thus, the signal to be regulated can be accurately "reconstructed" from the measurements, and one falls into the problem addressed in Part II. However, if the cameras are imprecisely modeled, this cannot be done. In fact, in this case, it is not clear whether or not it is possible to determine if the control objective has been accomplished just by making use of the cameras' measurements. This observation is the starting point for the main question addressed in Part III of this thesis:

> *When is it possible to decide if a prescribed robot positioning task has been accomplished using images acquired by an imprecisely modeled stereo vision system?*

Feedback control systems employing video cameras as sensors have been studied in the robotics community for many years (cf. tutorial on visual servoing [77] and review [78]). Demonstrated applications of vision within a feedback loop—often referred to as visual servoing or, more generally, *vision-based control*—include automated driving [79], flexible manufacturing [80, 81], and tele-operation with large time delays [82] to name a few.

An especially interesting feature of vision-based control systems is that often *both* the process output (e.g., the position and orientation of the robot in its workspace) and the reference set-point (e.g., a set of desired positions and orientations) can be simultaneously observed through the *same* sensors (i.e., cameras). In prior work [83, 84, 85, 86, 87], it has been observed that because of this unusual architectural feature, it is sometimes possible to achieve *precise* positioning (in the absence of measurement noise), despite sensor/actuator and process model imprecision, just as in the case of a conventional set-point control system with a loop-integrator and fixed exogenous reference. But in contrast to a set-point control system where what to choose for an error is usually clear, in vision-based systems there are many choices, each with different attributes. Some of the observations just made are implicit in work extending back more than 15 years [88]. Some of these issues are touched upon in [89] and in the monograph [90].

1.3.1 Task Decidability

The aim of Chapter 7 is to give conditions that enable one to decide on the basis of images of point features observed by an imprecisely modeled two-camera vision system, whether or not a prescribed positioning task has been accomplished. By a positioning task is meant, roughly speaking, the objective of bringing the pose of a robot—i.e., its position and orientation—to a target in the robot's workspace. Both the pose of the robot under consideration and the target to which it is to be brought, are determined by a list of simultaneously observed point features $f_1, f_2, \ldots, f_n$ in the two cameras' joint field of view $\mathcal{V}$. A task is then formally defined to be an equation of the form $T(f) = 0$ where T is a function mapping lists of point features of the form $f = \{f_1, f_2, \ldots, f_n\}$ into the integer set $\{0, 1\}$[4]. Such a task is said to be accomplished if the equation $T(f) = 0$ is satisfied by the observed feature list of interest.

The images of observed point features appear in the two cameras' joint image space $\mathcal{Y}$ and are available for processing. The two-camera model that maps point features in $\mathcal{V}$ into $\mathcal{Y}$ is not presumed to be known with certainty. Rather the model is assumed to be an unspecified member of some known class of two-camera models $\mathcal{C}$. The only information available for deciding whether or not a given task has been accomplished is thus the task function T, the images of the observed point features in $\mathcal{Y}$, and the class $\mathcal{C}$. A given task is said to be decidable on $\mathcal{C}$ if the available information, namely T, $\mathcal{C}$ and the images of the point features, is sufficient to determine whether or not the task has been accomplished.

One way to formalize the decidability question is by means of task encodings. The concept of a task encoding is discussed briefly in the Doctoral Thesis [91], which contains further references, and is also studied in the Doctoral Thesis [92]. In the present thesis, an encoded task is simply an equation of the form $E_T(y) = 0$ where y is a list of the images of observed point features as they appear in $\mathcal{Y}$, and E_T is a function that maps such lists into the reals. The construction of E_T must be based only on the knowledge of T and $\mathcal{C}$, and *not* on the actual two-camera model in $\mathcal{C}$, which is not assumed known. The encoded task is accomplished if the equation $E_T(y) = 0$ is satisfied by the list of images of observed point features of interest. The task $T(f) = 0$ is said to be verifiable on $\mathcal{C}$ with the encoding $E_T(y) = 0$ if accomplishing the encoded task is equivalent to accomplishing the original task, no matter which model in $\mathcal{C}$ correctly describes the actual two-camera system. The original task $T(f) = 0$ is then decidable on $\mathcal{C}$ if there is an encoded task that verifies $T(f) = 0$ on $\mathcal{C}$. In Section 7.2.2 are given necessary and sufficient conditions for a task $T(f) = 0$ to be decidable on a general set of admissible two-camera models $\mathcal{C}$. These conditions are expressed without regard to any particular form of encoding.

Two well-known methods of encoding are presented in Section 7.2.3: "Cartesian-based" and "image-base" encodings. In Cartesian-based encodings, an estimate of the actual camera model is used to attempt to reconstruct the observed feature list. This method of encoding is based on the heuristic idea of "certainty equivalence." Image-based encodings are not constrained to have a reconstruction step and are therefore more general. A third method of encoding,

[4]The reason for requiring the codomain of such a task function to be $\{0, 1\}$—rather than the reals which is more common—is that this requirement leads to somewhat simpler notation. There is in fact no real loss of generality in so constraining such T's.

called "modified" Cartesian-based, is considered in Section 7.2.4. One of the motivations for the use of Cartesian-based encodings is that they utilize estimates of features taking values in Cartesian space, which is the natural space for specifying robot positioning tasks. The "modified" Cartesian-based encoding, introduced in [93], generalizes the idea of Cartesian-based encodings to achieve verifiability for a richer set of tasks.

Section 7.3 briefly discusses how the results in Chapter 7 can be used to design feedback control systems capable of precise robot positioning using visual feedback, in spite of camera miscalibration.

1.3.2 Decidability on Sets of Projective Camera Models

In Chapter 7, the two-camera models are assumed to be injective functions, but nothing more. However, it is well known that two-camera systems have a rich geometric structure. This has led to recent work in the vision literature [94] considering the following question:

> "What can be seen in three dimensions with an uncalibrated stereo rig?"

Roughly speaking, [94] shows that, in the absence of measurement noise, using a two-camera vision system that has been calibrated just using images of point features, it is possible to exactly reconstruct the positions of other point features "up to a projective transformation" on three-dimensional projective space. A two-camera system calibrated using only measured point correspondences is said to be weakly calibrated [95].

These findings suggest that for a weakly calibrated two-camera model class, there ought to be a close relationship between the decidability of a given task $T(f) = 0$ and the invariant properties of the task function T under projective transformations. Theorem 8.3, the main result of Chapter 8, states that a given task is decidable on a weakly calibrated two-camera class *if and only if* the task is a projective invariant. This result thus serves to underscore the observation made in [89, 96, 84, 97] that accurate metric information is not needed for the accomplishment of many types of positioning tasks with a stereo vision system.

The two-camera models considered in this chapter are pairs of projective camera models that map subsets of $\mathbb{P}^3$ containing $\mathcal{V}$, into $\mathbb{P}^2 \times \mathbb{P}^2$. Projective models of this type have been widely used in computer vision [98, 99] in part because they include as special cases, perspective, affine and orthographic camera models. By restricting our attention to projective models, we are able to provide a complete and concise characterization of decidable tasks in terms of projective invariance.

1.3.3 Projective Invariance

It is shown in Chapter 8 that projectively invariant tasks are of special importance when dealing with sets of projective camera models. In fact, projective invariance is a necessary condition for decidability on the set of uncalibrated two-camera models, and a necessary and sufficient condition for decidability on any set of weakly calibrated two-camera models. The objective of Chapter 9 is to study the properties of projectively invariant tasks.

In Section 9.1 it is defined an equivalence relation on the set of admissible features, which is called projective equivalence. It is shown that the set of all projectively invariant tasks can be completely characterized by the set of equivalence classes determined by projective equivalence. In fact, there is a one-to-one correspondence between projectively invariant tasks and sets of equivalence classes determined by projective equivalence.

Section 9.2 introduces a few invariants of projective equivalence. These invariants characterize the geometric structure of the features in each equivalence class determined by projective equivalence. In particular, they determine how the point features in each feature list are organized into lines and planes.

In Section 9.3 it is shown that an appropriately defined set of features in "normalized upper triangular form" is a set of canonical forms for projective equivalence. Thus, features in normalized upper triangular form characterize the set of equivalence classes determined by projective equivalence and can therefore be used to construct any projectively invariant task. This leads to the main result of the chapter, Theorem 9.4, which states that there is a one-to-one correspondence between tasks that are projectively invariant and sets of features in normalized upper triangular form.

Part I

Tools for the Analysis and Design of Hybrid Control Systems

Chapter 2

Invariant Sets for Hybrid Systems

Given a Lipschitz continuous function f from a finite dimensional linear space $\mathcal{X}$ into itself, a subset $\mathcal{Z}$ of $\mathcal{X}$ is said to be *(positively) invariant* with respect to the differential equation

$$\dot{x} = f(x), \qquad t \geq t_0 \tag{2.1}$$

if for any $x_0 \in \mathcal{Z}$, the solution x to (2.1) with $x(t_0) = x_0$ remains in $\mathcal{Z}$ for all times $t \geq t_0$ for which the solution is defined [8].

In this chapter the above definition of an invariant set is extended to systems in which continuous dynamics interact with discrete logic. Such systems are often called hybrid [1, 9, 2, 3, 4, 5, 6, 7]. Tests to determine if a set is invariant with respect to a hybrid system are derived. These tests do not require the computation of the hybrid system's state trajectory. The use of invariance in a hybrid systems context is touched upon in [7].

As an application of these results, a hybrid control law to exponentially stabilize a "nonholonomic integrator" is constructed and analyzed. By a *nonholonomic integrator* [10] is meant the three-dimensional system

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_3 = x_1 u_2 - x_2 u_1$$

where $x := [x_1 \quad x_2 \quad x_3]' \in \mathbb{R}^3$ and $u := [u_1 \quad u_2]' \in \mathbb{R}^2$. An interesting property of the nonholonomic integrator is that, although locally null controllable, it is not stabilizable by a time-invariant smooth control law [10].

The remainder of this chapter is organized as follows. Section 2.1 contains the formal definition of an invariant set with respect to a hybrid system and tests to check for invariance are derived. In Section 2.2 a hybrid control law for a nonholonomic integrator is proposed. The analysis of the resulting closed-loop hybrid system is carried out using the results derived in Section 2.1. Generalizations of the proposed control law for some higher dimensional systems are also given. Finally, Section 2.3 contains a brief discussion of the results presented and some directions for future research.

2.1 Invariant Sets

In this thesis, a hybrid system Σ is characterized by four entities: A finite dimensional linear space $\mathcal{X}$ called the *continuous state-space*; an arbitrary set $\mathcal{S}$ called the *discrete state-space*; a family $\{f_s : s \in \mathcal{S}\}$ of locally Lipschitz continuous functions from $\mathcal{X}$ to $\mathcal{X}$ called the *family of* *vector fields*; and a function $\phi : \mathcal{X} \times \mathcal{S} \to \mathcal{S}$ called the *discrete transition function*. The hybrid system Σ is then defined by the ordinary differential equation

$$\dot{x} = f_\sigma(x), \qquad t \geq t_0 \tag{2.2}$$

together with the recursive equation

$$\sigma = \phi(x, \sigma^-), \qquad t \geq t_0 \tag{2.3}$$

where, for each $t > t_0$, $\sigma^-(t)$ denotes the limit from the left of $\sigma(\tau)$ as $\tau \uparrow t$ and $\sigma^-(t_0)$ is equal to some element of $\mathcal{S}$ that effectively initializes (2.3). By a solution to Σ on the interval $[t_0, T)$ is meant a pair of signals $\{x : [t_0, T) \to \mathcal{X},\ \sigma : [t_0, T) \to \mathcal{S}\}$ with x—the *continuous state*—continuous and piecewise differentiable and σ—the *discrete state*—piecewise constant and right-continuous at every point, such that x and σ satisfy (2.2)–(2.3) on the interval $[t_0, T)$. Other models for hybrid systems can be found in [1, 9, 2, 3, 4, 5, 6, 7].

A pair of sets $\{\mathcal{Z},\ \mathcal{J}\}$ with $\mathcal{Z} \subset \mathcal{X}$ and $\mathcal{J} \subset \mathcal{S}$ is *invariant with respect to* Σ if, for every $x_0 \in \mathcal{Z}$ and every $\sigma_0 \in \mathcal{S}$, any solution $\{x, \sigma\}$ to Σ with $x(t_0) = x_0$ and $\sigma^-(t_0) = \sigma_0$ remains in $\mathcal{Z} \times \mathcal{J}$ for all times $t \geq t_0$ for which the solution is defined.

The following lemma provides a procedure to prove invariance of a given pair of sets by observing the values of the functions $f_j : \mathcal{X} \to \mathcal{X}$, $j \in \mathcal{S}$ at the boundary of $\mathcal{Z}$. The following terminology is used: Given a subset $\mathcal{Z}$ of $\mathcal{X}$, a vector[1] $\vec{v} = (x, v) \in \mathcal{X} \times \mathcal{X}$ at a point x on the boundary of $\mathcal{Z}$ is said to *point towards* $\mathcal{Z}$ if there exist positive constants h and r such that the cone with spherical base $C[\vec{v}, h, r]$ shown in Figure 2.1 and defined by

$$C[\vec{v}, h, r] := \left\{ z \in \mathcal{X} : \|x + \rho\, hv - z\| \leq \rho\, r,\ \rho \in [0, 1] \right\}$$

is contained in $\mathcal{Z}$. Here we are using the norm topology on $\mathcal{X} := \mathbb{R}^n$.

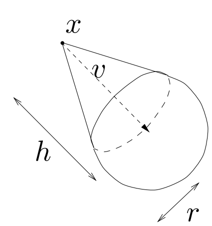

Figure 2.1: Cone with spherical base $C[\vec{v}, h, r]$

Lemma 2.1. *Consider the hybrid system Σ defined by (2.2)–(2.3) and a pair of sets $\{\mathcal{Z},\ \mathcal{J}\}$ with $\mathcal{Z} \subset \mathcal{X}$, $\mathcal{J} \subset \mathcal{S}$ such that*[2]

$$\phi(\mathcal{Z}, \mathcal{J}) \subset \mathcal{J} \tag{2.4}$$

The pair $\{\mathcal{Z},\ \mathcal{J}\}$ is invariant with respect to Σ if, for every $\bar{x}$ on the boundary of $\mathcal{Z}$, at least one of the following conditions holds:

1. *$\bar{x} \in \mathcal{Z}$ and $f_j(\bar{x}) = 0$ for every $j \in \phi(\{\bar{x}\}, \mathcal{J})$.*

[1] Here, a vector at a point $x \in \mathcal{X} := \mathbb{R}^n$ is a pair $\vec{v} = (x, v)$ where $v \in \mathcal{X}$. Geometrically, $\vec{v}$ can be regarded as the vector v translated so that its "tail" is at x rather than at the origin.

[2] With $\mathcal{A}$ and $\mathcal{B}$ sets, $\mathcal{C} \subset \mathcal{A}$, and $f : \mathcal{A} \to \mathcal{B}$, $f(\mathcal{C})$ denotes the f-image of $\mathcal{C}$ that is defined by $\{f(a) : a \in \mathcal{C}\}$.

2. *$\bar{x} \in \mathcal{Z}$ and $\vec{v}_j := \big(\bar{x}, f_j(\bar{x})\big)$ points towards $\mathcal{Z}$ for each $j \in \phi(\{\bar{x}\}, \mathcal{J})$.*

3. *$\bar{x} \notin \mathcal{Z}$ and there exists a neighborhood $\mathcal{N}_{\bar{x}}$ of $\bar{x}$ such that $\vec{v}_j := \big(\bar{x}, -f_j(\bar{x})\big)$ points towards $\mathcal{X} \setminus \mathcal{Z}$ for each $j \in \phi(\mathcal{N}_{\bar{x}} \cap \mathcal{Z}, \mathcal{J})$.*

Proof of Lemma 2.1. By contradiction assume that there exists a solution $\{x, \sigma\}$ to Σ on $[t_0, T)$ ($T \leq +\infty$), with $x(t_0) = x_0 \in \mathcal{Z}$ and $\sigma^-(t_0) = \sigma_0 \in \mathcal{J}$, such that

$$\bar{t} := \inf \big\{ t \in [t_0, T) : x(t) \notin \mathcal{Z} \text{ or } \sigma(t) \notin \mathcal{J} \big\} \tag{2.5}$$

is strictly smaller than T. The vector $\bar{x} := x(\bar{t})$ cannot be in the interior of the complement of $\mathcal{Z}$, otherwise, by continuity of x, there would be a time $t < \bar{t}$ for which $x(t) \notin \mathcal{Z}$, which contradicts (2.5). Suppose now that $\bar{x}$ is in the interior of $\mathcal{Z}$ and therefore, by continuity, that

$$x(t) \in \mathcal{Z}, \qquad \forall t \in [\bar{t}, \bar{t} + \delta_1) \tag{2.6}$$

for some $\delta_1 > 0$. Since $\sigma^-(\bar{t})$ is still in $\mathcal{J}$,

$$\sigma(\bar{t}) = \phi\big(\bar{x}, \sigma^-(\bar{t})\big) \in \phi(\{\bar{x}\}, \mathcal{J})$$

and therefore, because of its right-continuity, σ must remain in $\phi(\{\bar{x}\}, \mathcal{J})$ for some time after $\bar{t}$. This, (2.4), and (2.6) would contradict (2.5) and therefore $\bar{x}$ cannot belong to the interior $\mathcal{Z}$. Since $\bar{x}$ is not in the interior of $\mathcal{Z}$ nor in the interior of its complement, it must be on the boundary of $\mathcal{Z}$. We consider 3 cases separately:

Case 1: $\bar{x} \in \mathcal{Z}$ and $f_j(\bar{x}) = 0$ for every $j \in \phi(\{\bar{x}\}, \mathcal{J})$. Since $\sigma^-(\bar{t})$ is still in $\mathcal{J}$,

$$\sigma(\bar{t}) = \phi\big(\bar{x}, \sigma^-(\bar{t})\big) \in \phi(\{\bar{x}\}, \mathcal{J}) \tag{2.7}$$

and therefore, because of its right-continuity, σ must remain in $\phi(\{\bar{x}\}, \mathcal{J})$ on some interval $[\bar{t}, \bar{t} + \delta]$ of positive length. But then (2.2) has a unique solution $x(t) = \bar{x}$ for $t \in [\bar{t}, \bar{t} + \delta]$. Thus $x \in \mathcal{Z}$ and $\sigma \in \mathcal{J}$ on $[\bar{t}, \bar{t} + \delta]$, which contradicts (2.5).

Case 2: $\bar{x} \in \mathcal{Z}$ but $f_j(\bar{x}) \neq 0$ for some $j \in \phi(\{\bar{x}\}, \mathcal{J})$. Reasoning as in Case 1, one concludes that there must then be an interval $[\bar{t}, \bar{t} + \delta]$ of positive length in which σ remains constant and equal to some $\bar{\sigma} \in \phi(\{\bar{x}\}, \mathcal{J})$. Since $f_{\bar{\sigma}}$ is continuous and $\sigma = \bar{\sigma}$ on $[\bar{t}, \bar{t} + \delta]$, x must be continuously differentiable on the same interval. Therefore, by the Mean Value Theorem (cf. [100, Corollary 4.4, p. 67]),

$$\|x(\bar{t} + \epsilon) - x(\bar{t}) - \epsilon \dot{x}(\bar{t})\| \leq \epsilon \sup_{\tau \in [\bar{t}, \bar{t} + \epsilon]} \|\dot{x}(\tau) - \dot{x}(\bar{t})\|, \qquad \forall \epsilon \in [0, \delta]$$

which means that

$$\|x(\bar{t} + \epsilon) - \bar{x} - \epsilon f_{\bar{\sigma}}(\bar{x})\| \leq \epsilon \sup_{\tau \in [\bar{t}, \bar{t} + \epsilon]} \|f_{\bar{\sigma}}(x(\tau)) - f_{\bar{\sigma}}(x(\bar{t}))\|, \qquad \forall \epsilon \in [0, \delta] \tag{2.8}$$

Since $f_{\bar{\sigma}}$ is locally Lipschitz and x is continuously differentiable, the composition of these two functions is locally Lipschitz. Therefore there must exist a constant c such that

$$\|f_{\bar{\sigma}}(x(\tau)) - f_{\bar{\sigma}}(\bar{x})\| \leq c \|\tau - \bar{t}\|, \qquad \tau \in [\bar{t}, \bar{t} + \delta]$$

From this and (2.8) one concludes that

$$\|x(\bar{t} + \epsilon) - \bar{x} - \epsilon f_{\bar{\sigma}}(\bar{x})\| \leq c\, \epsilon^2, \qquad \forall \epsilon \in [0, \delta] \tag{2.9}$$

Now, since $\vec{v}_{\bar\sigma} := \big(\bar x, f_{\bar\sigma}(\bar x)\big)$ points towards $\mathcal{Z}$, there must exist positive constants h and r such that $C[\vec{v}_{\bar\sigma}, h, r] \subset \mathcal{Z}$. Rewriting (2.9) as

$$\|x(\bar t+\epsilon) - \bar x - \rho\, h f_{\bar\sigma}(\bar x)\| \le \rho\, hc\epsilon, \qquad \forall \epsilon \in [0, \delta]$$

where $\rho := \frac{\epsilon}{h}$, one then concludes that

$$\sigma(\bar t+\epsilon) \in \mathcal{J}, \qquad x(\bar t+\epsilon) \in C[\vec{v}_{\bar\sigma}, h, r] \subset \mathcal{Z}, \qquad 0 \le \epsilon \le \min\left\{\delta, h, \frac{r}{c\,h}\right\}$$

which contradicts (2.5).

Case 3: $\bar x \notin \mathcal{Z}$. By the hypothesis of the Lemma, there must then exist a neighborhood $\mathcal{N}_{\bar x}$ of $\bar x$ such that $\vec{v}_j := \big(\bar x, -f_j(\bar x)\big)$ points towards $\mathcal{X} \setminus \mathcal{Z}$ for each $j \in \phi(\mathcal{N}_{\bar x} \cap \mathcal{Z}, \mathcal{J})$. Since $x(t_0) \in \mathcal{Z}$ and $\bar x = x(\bar t) \notin \mathcal{Z}$, one must have $\bar t > t_0$ and therefore there must be an interval $[\bar t - \delta, \bar t) \subset [t_0, T)$ of positive length on which x remains in $\mathcal{Z}$. Because of the continuity of x and the right-continuity of σ, one can pick δ small enough so that x remains inside $\mathcal{N}_{\bar x}$ on $[\bar t-\delta, \bar t)$ and σ is equal to some constant $\bar\sigma \in \phi(\mathcal{N}_{\bar x} \cap \mathcal{Z}, \mathcal{J})$ on $[\bar t - \delta, \bar t)$. Since $f_{\bar\sigma}$ is continuous and $\sigma = \bar\sigma$ on $[\bar t - \delta, \bar t)$, x must be continuously differentiable on the same interval. Therefore, by the Mean Value Theorem,

$$\|x(\bar t-\epsilon) - x(\bar t) + \epsilon \dot x(\bar t)\| \le \epsilon \sup_{\tau\in[\bar t-\epsilon, \bar t)} \|\dot x(\tau) - \dot x(\bar t)\|, \qquad \forall \epsilon \in [0, \delta]$$

Proceeding as in Case 2 one can then conclude that there exists a constant c such that

$$\|x(\bar t-\epsilon) - \bar x + \epsilon f_{\bar\sigma}(\bar x)\| \le c\,\epsilon^2, \qquad \forall \epsilon \in [0, \delta] \tag{2.10}$$

Now, since $\vec{v}_{\bar\sigma} := \big(\bar x, -f_{\bar\sigma}(\bar x)\big)$ points towards $\mathcal{X} \setminus \mathcal{Z}$, there must exist positive constants h and r such that $C[\vec{v}_{\bar\sigma}, h, r] \subset \mathcal{X} \setminus \mathcal{Z}$. Rewriting (2.10) as

$$\|x(\bar t-\epsilon) - \bar x + \rho\, h f_{\bar\sigma}(\bar x)\| \le \rho\, hc\epsilon, \qquad \forall \epsilon \in [0, \delta]$$

where $\rho := \frac{\epsilon}{h}$, one then concludes that

$$x(\bar t-\epsilon) \in C[\vec{v}_{\bar\sigma}, h, r] \subset \mathcal{X} \setminus \mathcal{Z}, \qquad 0 \le \epsilon \le \min\left\{\delta, h, \frac{r}{c\,h}\right\}$$

which contradicts (2.5). ∎

2.2 Exponential Stabilization of Nonholonomic Integrators

Over the last decade there has been a great deal of research concerned with the problem of stabilizing systems that are locally null controllable but fail to meet Brockett's condition for smooth stabilizability [10]: *Given the system*

$$\dot x = f(x, u), \qquad x(t_0) = x_0, \qquad f(0,0) = 0 \tag{2.11}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *continuously differentiable. If* (2.11) *is smoothly stabilizable, i.e., there exists a continuously differentiable function* $g : \mathbb{R}^n \to \mathbb{R}^m$ *such that the origin is an asymptotically stable equilibrium point of* $\dot x = f(x, g(x))$, *with stability defined in the Lyapunov sense, then the image of* f *must contain an open neighborhood of the origin.*

In [12] it is noted that this condition extends to the class of time-invariant feedback laws that are only locally Lipschitz and recently it was shown in [13, 14] that Brockett's condition also extends to an even larger class that includes a wide variety of time-invariant discontinuous feedback laws[3], when one demands that the origin be an asymptotically stable equilibrium point for all Filippov's solutions [101] of the closed-loop system. A prototype example of a system that is not smoothly stabilizable is the so called "nonholonomic integrator" [10]:

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_3 = x_1 u_2 - x_2 u_1 \tag{2.12}$$

where $x := [x_1 \quad x_2 \quad x_3]' \in \mathbb{R}^3$ and $u := [u_1 \quad u_2]' \in \mathbb{R}^2$. Since the image of the map $[x' \;\; u']' \mapsto [u_1 \;\; u_2 \;\; x_1u_2 - x_2u_1]'$ does not contain the point $[0 \;\; 0 \;\; \epsilon]'$ for any $\epsilon \neq 0$, Brockett's condition implies that there is no time-invariant continuously differentiable control law that asymptotically stabilizes the origin. It turns out that any kinematic completely nonholonomic system with three states and two control inputs can be converted to a nonholonomic integrator by a local coordinate transformation [11].

The difficulties implied by Brockett's condition can be avoided using time-varying periodic controllers [15, 16, 17, 18, 19, 20, 21], stochastic control laws [19], and sliding modes control laws [22, 23]. The control proposed in this thesis falls into the class of hybrid control laws, namely those employing both continuous dynamics and discrete logic. Applications of this type of laws to nonholonomic systems can be found in [24, 25, 26, 27, 28, 29]. In the first two references global convergence to the origin is achieved in finite time; however, these controls may result in chattering in the presence of unmodeled dynamics. In [29] a time-varying hybrid controller is used to asymptotically stabilize a general class of nonholonomic systems represented in power form. The reader is referred to [30] for an extensive survey of recent results concerned with the control of nonholonomic systems. This thesis proposes a time-invariant hybrid control law that guarantees global asymptotic stability with *exponentially* fast convergence to the origin of the state of the nonholonomic integrator. Exponential stabilization of systems like the nonholonomic integrator was also achieved in [21] using nonsmooth, continuous, time-varying control laws.

2.2.1 Switching Controller

Consider again the nonholonomic integrator (2.12). No matter what control law is used, whenever x_1 and x_2 are both zero, $\dot{x}_3$ will also be zero and x_3 will remain constant. Furthermore, whenever x_1 and x_2 are "small", only "large" control signals will be able to produce significant changes in x_3. A plausible strategy to make the origin an attractor of the close-loop system is to keep the state away from the axes $x_1 = x_2 = 0$ while x_3 is large and, as x_3 decreases, to let x_1 and x_2 became small. Several control laws can achieve the aforementioned type of behavior. The one presented in this thesis has the virtue of being easy to analyze, not only in terms of stability, but also in terms of speed of convergence. The control law proposed is constructed as follows:

1. Pick four continuous, monotone nondecreasing functions

$$\pi_j : [0, +\infty) \to \mathbb{R}, \qquad j \in \mathcal{S} := \{1, 2, 3, 4\}$$

with the following properties:

(i) $\pi_j(0) = 0$ for each $j \in \mathcal{S}$, and $0 < \pi_1(w) < \pi_2(w) < \pi_3(w) < \pi_4(w)$ for every $w > 0$.

(ii) π_1 and π_2 are bounded.

[3]The reader is referred to [13, 14] for the precise classes of discontinuous feedback laws considered.

(iii) π_1 is such that if $w \to 0$ exponentially fast[4] then $\frac{w}{\pi_1(w)} \to 0$ exponentially fast.

(iv) π_4 is smooth on some non-empty interval $(0, c]$, and

$$\pi_4'(w) < \frac{\pi_4(w)}{w}, \qquad w \in (0, c] \tag{2.13}$$

Moreover, if $w \to 0$ exponentially fast then $\pi_4(w) \to 0$ exponentially fast.

2. Partition $\mathbb{R}^3$ into 4 overlapping regions

$$\begin{aligned}
\mathcal{R}_1 &:= \left\{x \in \mathbb{R}^3 : 0 \le {x_1}^2 + {x_2}^2 < \pi_2\left({x_3}^2\right)\right\}, \\
\mathcal{R}_2 &:= \left\{x \in \mathbb{R}^3 : \pi_1\left({x_3}^2\right) < {x_1}^2 + {x_2}^2 < \pi_4\left({x_3}^2\right)\right\}, \\
\mathcal{R}_3 &:= \left\{x \in \mathbb{R}^3 : \pi_3\left({x_3}^2\right) < {x_1}^2 + {x_2}^2\right\}, \\
\mathcal{R}_4 &:= \{0\}
\end{aligned}$$

3. Define the control law

$$u = g_\sigma(x), \qquad t \ge t_0 \tag{2.14}$$

where σ is a piecewise constant, continuous from the right at every point switching signal taking values on $\mathcal{S}$ and, for each $x \in \mathbb{R}^3$,

$$g_1(x) := \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad g_2(x) := \begin{bmatrix} x_1 + \frac{x_2 x_3}{{x_1}^2 + {x_2}^2} \\ x_2 - \frac{x_1 x_3}{{x_1}^2 + {x_2}^2} \end{bmatrix}, \quad g_3(x) := \begin{bmatrix} -x_1 + \frac{x_2 x_3}{{x_1}^2 + {x_2}^2} \\ -x_2 - \frac{x_1 x_3}{{x_1}^2 + {x_2}^2} \end{bmatrix}, \quad g_4(x) := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.15}$$

The signal σ is determined recursively by

$$\sigma = \phi(x, \sigma^-), \qquad t \ge t_0 \tag{2.16}$$

where, for each $t > t_0$, $\sigma^-(t)$ is equal to the limit from the left of $\sigma(\tau)$ as $\tau \uparrow t$, $\sigma^-(t_0)$ is equal to some element of $\mathcal{S}$ that effectively initializes (2.16), and $\phi : \mathbb{R}^3 \times \mathcal{S} \to \mathcal{S}$ is the transition function defined by

$$\phi(x, j) = \begin{cases} j & \text{if } x \in \mathcal{R}_j \\ \max\{i \in \mathcal{S} : x \in \mathcal{R}_i\} & \text{if } x \notin \mathcal{R}_j \end{cases} \qquad x \in \mathbb{R}^3,\ j \in \mathcal{S} \tag{2.17}$$

Example 2.2. A typical choice for the functions π_j is $\pi_1(w) = (1 - e^{-\sqrt{w}})$, $\pi_2 = 2\,\pi_1$, $\pi_3 = 3\,\pi_1$, and $\pi_4 = 4\,\pi_1$. Figure 2.2 shows the projection of the corresponding regions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ into the $({x_3}^2, {x_1}^2 + {x_2}^2)$-space.

The type of control proposed is similar to that in [2] and can be viewed as an extension of the hysteresis switching algorithm considered in [9]. Its appeal comes from the fact that it naturally excludes the possibility of infinitely fast chattering and therefore does not require the concept of generalized solution in Filippov's sense [22, 23].

[4] A signal $x : [0, \infty) \to \mathbb{R}^n$ is said to converge to zero exponentially fast if there exist positive constants a, λ such that $\|x(t)\| \le a e^{-\lambda t}$ for every $t \ge 0$.

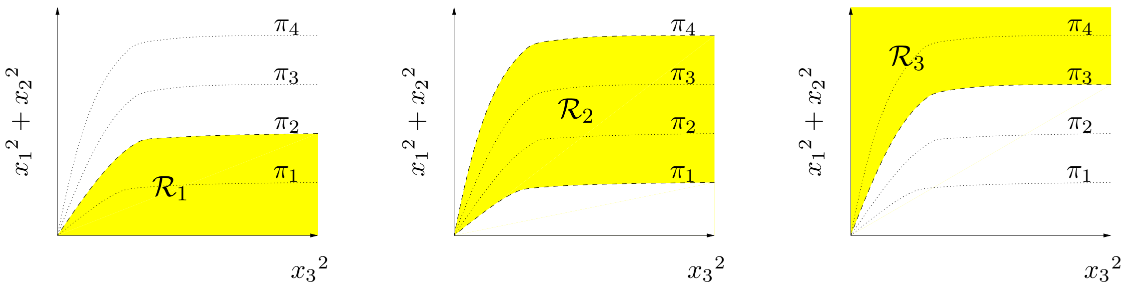

Figure 2.2: Projection of the regions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ into the $({x_3}^2, {x_1}^2 + {x_2}^2)$-space.

2.2.2 Main Result

The aim of this section is to study the closed-loop hybrid dynamical system described in the previous section. The relevant equations are

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_3 = x_1 u_2 - x_2 u_1, \qquad u = g_\sigma(x), \qquad \sigma = \phi(x, \sigma^-),$$

where $x := [x_1 \quad x_2 \quad x_3]' \in \mathbb{R}^3$, $u := [u_1 \quad u_2]' \in \mathbb{R}^2$, and $g_\sigma(x)$, $\phi(x,\sigma)$ are defined by equations (2.15) and (2.17), respectively. Although the closed-loop system is not globally Lipschitz, global existence of solutions can be easily justified. Indeed, defining $w_1 := {x_3}^2$ and $w_2 := {x_1}^2 + {x_2}^2$, simple algebra shows that

$$\dot{w}_1 \leq 2\, w_1 + w_2, \qquad \dot{w}_2 \leq 2\, w_2 + 2$$

Since the bounds for the right-hand sides of the above equations are globally Lipschitz with respect to w_1 and w_2, these variables and their derivatives must be bounded on any finite interval. Moreover, the distance between two points in the (w_1, w_2)-space where consecutive switchings can occur is always nonzero. The boundedness of $\dot{w}_1$ and $\dot{w}_2$ thus guarantees that the time interval between consecutive discontinuities of σ is always positive, i.e., σ is piecewise constant. Now the system of differential equations given by (2.12) and (2.14) can be written as

$$\dot{x} = f_{\sigma(t)}(x) \tag{2.18}$$

with each f_j, $j \in \mathcal{S}$, locally Lipschitz. Since it has been established that σ is piecewise constant, the right-hand side of (2.18) is locally Lipschitz with respect to x and piecewise continuous with respect to t. This together with the fact that x is bounded on any finite interval (because the same is true for w_1 and w_2) guarantees that the solution exists globally and is unique. The fact that the regions $\mathcal{R}_i$, $i \in \mathcal{S}$ used to define the transition function ϕ are open subsets of $\mathbb{R}^3$ guarantees that σ is indeed continuous from the right at every point.

The above argument excludes the possibility of infinitely fast chattering in the sense that the interval between consecutive discontinuities of σ is bounded below by a positive constant on any finite interval. Latter we are going to see that the interval between consecutive switchings is bounded away from zero even as time goes to infinity (cf. Remark 2.5 in Section 2.2.3).

The usual definition of Lyapunov stability extends in a natural way to hybrid systems: the origin is a *Lyapunov stable equilibrium point* of the hybrid system Σ defined by (2.2)–(2.3) if

(i) $f_i(0) = 0$ for each $i \in \mathcal{S}$ such that $\phi(0, i) = i$, and

(ii) for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $x_0 \in \mathcal{X}$ and every $\sigma_0 \in \mathcal{S}$ with $\|x_0\| < \delta$, any solution $\{x, \sigma\}$ to Σ with $x(t_0) = x_0$ and $\sigma^-(t_0) = \sigma_0$, exists globally and $\|x(t)\| < \epsilon$ for $t \geq t_0$.

Moreover, if for any initial conditions, the continuous state x converges to the origin, then the origin is said to be *globally asymptotically stable*. The main result of this section is the following theorem.

Theorem 2.3. *Let Σ denote the hybrid system defined by* (2.12), (2.14), *and* (2.16).

1. *The origin is a globally asymptotically stable equilibrium point of Σ.*
2. *The continuous state x of Σ and the control signal u converge to zero exponentially fast along any solution to Σ.*

Figure 2.3 shows a simulation of the closed-loop system Σ defined by equations (2.12), (2.14), and (2.16), with

$$\pi_1(w) = .5(1 - e^{-\sqrt{w}}), \quad \pi_2(w) = 1.7\pi_1(w), \quad \pi_3(w) = 2.5\pi_1(w), \quad \pi_4(w) = 4\pi_1(w), \quad w \geq 0 \tag{2.19}$$

As expected, x and u converge to zero exponentially fast. For the trajectory shown in these

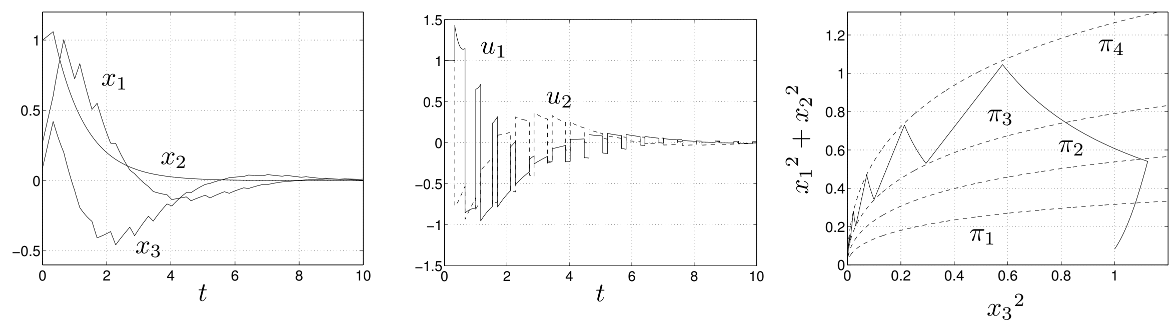

Figure 2.3: Simulation of the closed-loop hybrid system Σ: x versus time, u versus time, and projection of x into the $({x_3}^2, {x_1}^2 + {x_2}^2)$-space.

plots, not only is chattering precluded on any finite time interval, but it is also true that the interval between consecutive switchings is bounded away from zero as time goes to infinity. It turns out that this is true for *any* trajectory of this hybrid system (cf. Remark 2.5 in Section 2.2.3).

To test the robustness of the controller proposed with respect to modeling errors, the hybrid controller defined by equations (2.14) and (2.16) was also applied to the system

$$\dot{x}_1 = v_1, \qquad \dot{x}_2 = v_2, \qquad \dot{x}_3 = x_1 v_2 - x_2 v_1, \tag{2.20}$$

$$\dot{v}_1 = -10v_1 + 9.5u_1, \qquad \dot{v}_2 = -10v_2 + 10.5u_2, \tag{2.21}$$

This system consists of a nonholonomic integrator (2.20) in cascade with first order low-pass filters (2.21) with DC gains close but not equal to 1. Equations (2.21) could model, for example, simple actuator dynamics. Figure 2.4 shows a simulation of the closed-loop hybrid system defined by equations (2.20)–(2.21), (2.14), and (2.16). In this simulation one can see that the "actuator dynamics"(2.21) do not compromise the exponentially fast convergence to the origin nor do they introduce chattering.

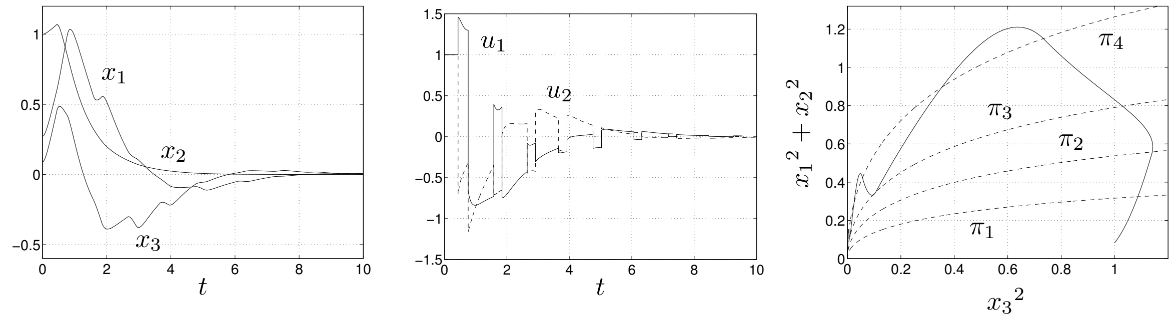

Figure 2.4: Simulation of the closed-loop hybrid system with modeling errors: x versus time, u versus time, and projection of x into the $(x_3{}^2, x_1{}^2 + x_2{}^2)$-space.

2.2.3 Proof of Theorem 2.3

Consider the sets $\mathcal{J} := \{2, 3, 4\}$,

$$\mathcal{Z}_1 := \left\{x \in \mathbb{R}^3 : x_3{}^2 \leq c_1,\ \pi_1(x_3{}^2) < x_1{}^2 + x_2{}^2 \leq c_2\right\} \cup \{0\}, \tag{2.22}$$

$$\mathcal{Z}_2 := \left\{x \in \mathbb{R}^3 : \pi_3(x_3{}^2) \leq x_1{}^2 + x_2{}^2\right\} \tag{2.23}$$

$$\mathcal{Z}_3 := \left\{x \in \mathbb{R}^3 : x_3{}^2 \leq c,\ \pi_3(x_3{}^2) \leq x_1{}^2 + x_2{}^2 \leq \pi_4(x_3{}^2)\right\} \tag{2.24}$$

with c_1 an arbitrary constant, c_2 a constant larger than $\pi_4(c_1)$, and c as in (2.13). Defining $w := \Pi(x)$, with $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $[x_1\ \ x_2\ \ x_3]' \mapsto [x_3{}^2\ \ x_1{}^2 + x_2{}^2]'$, when σ takes values on $\mathcal{J}$, the evolution of w is completely determined by the hybrid system Σ_w defined[5] by

$$\dot{w} = f_\sigma(w), \qquad \sigma = \varphi(w, \sigma^-) \tag{2.25}$$

where, for every $w \in \mathbb{R}^2$ and every $j \in \{2, 3, 4\}$,

$$f_j(w) := \begin{cases} [-2|w_1| \ \ +2w_2]' & w_2 \geq 0,\ \ j = 2 \\ [-2|w_1| \ \ -2w_2]' & w_2 \geq 0,\ \ j \in \{3, 4\} \\ [-2|w_1| \ \ 0]' & w_2 < 0 \end{cases}$$

$$\varphi(w, j) := \begin{cases} j & w \in \Pi(\mathcal{R}_j) \text{ or } w_1 < 0 \text{ or } w_2 < 0 \\ \max\{i \in \mathcal{S} : w \in \Pi(\mathcal{R}_i)\} & w \notin \Pi(\mathcal{R}_j) \text{ and } w_1 \geq 0 \text{ and } w_2 \geq 0 \end{cases}$$

Moreover, defining

$$\mathcal{W}_1 := \left\{w : w_1 \in (0, c_1],\ \pi_1(w_1) < w_2 \leq c_2\right\} \cup \{w \in \mathbb{R}^2 : w_1 \leq 0,\ w_2 \leq c_2\},$$
$$\mathcal{W}_2 := \left\{w : w_1 \geq 0,\ w_2 \geq \pi_3(w_1)\right\} \cup \{w \in \mathbb{R}^2 : w_1 \leq 0\}$$
$$\mathcal{W}_3 := \left\{w : w_1 \in [0, c],\ \pi_3(w_1) \leq w_2 \leq \pi_4(w_1)\right\},$$

a vector $x \in \mathbb{R}^3$ belongs to $\mathcal{Z}_i$ if $w := \Pi(x)$ belongs to $\mathcal{W}_i$. Therefore, each pair $\{\mathcal{Z}_i,\ \mathcal{J}\}$ is invariant with respect to the hybrid system Σ defined by (2.12), (2.14), and (2.16), if the pair $\{\mathcal{W}_i,\ \mathcal{J}\}$ is invariant with respect to the hybrid system Σ_w defined by (2.25).

Consider a point $\bar{w} := [\bar{w}_1\ \ \bar{w}_2]'$ with $\bar{w}_1 = 0$, $\bar{w}_2 \leq 0$. Such a point belongs to $\mathcal{W}_1$, in fact it is on one of the vertical boundaries of this set (cf. Figure 2.5). For any $j \in \mathcal{J}$, $f_j(\bar{w}) = 0$. On the

[5] The values of the vector fields f_s and the discrete transition function φ, when either $w_1 < 0$ or $w_2 < 0$, are arbitrary. The definitions given simplify somewhat the analysis of Σ_w.

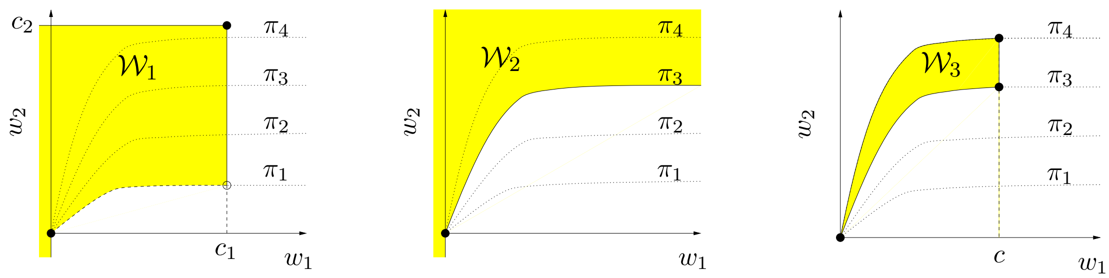

Figure 2.5: Sets $\mathcal{W}_1$, $\mathcal{W}_2$, and $\mathcal{W}_3$.

other hand if the point $\bar{w} := [\bar{w}_1 \;\; \bar{w}_2]'$ is on the "horizontal" boundary of $\mathcal{W}_1$, i.e., $\bar{w}_1 \leq c_1$ and $\bar{w}_2 = c_2$. Then for $j \in \varphi(\bar{w}, \mathcal{J}) = \{3\}$, the second component of $f_j(\bar{w})$ is negative and therefore $\vec{v_j} := \big(\bar{w}, f_j(\bar{w})\big)$ points towards $\mathcal{W}_1$. Suppose now that the point $\bar{w} := [\bar{w}_1 \;\; \bar{w}_2]'$ is on the "vertical" boundary of $\mathcal{W}_1$ defined by $\bar{w}_1 = c_1$, $\bar{w}_2 \in (\pi_1(c_1), c_2]$. For any $j \in \varphi(\bar{w}, \mathcal{J}) = \{2, 3\}$, the first component of $f_j(\bar{w})$ is negative and therefore $\vec{v_j} := \big(\bar{w}, f_j(\bar{w})\big)$ points towards $\mathcal{W}_1$. Suppose finally that the point $\bar{w} := [\bar{w}_1 \;\; \bar{w}_2]'$ is on the "curved" boundary of $\mathcal{W}_1$, i.e., that $\bar{w}_1 \in (0, c_1]$ and $\bar{w}_2 = \pi_1(\bar{w}_1)$. In this case $\bar{w}$ does not belong to $\mathcal{W}_1$. Let $\mathcal{N}_{\bar{w}}$ be a neighborhood of $\bar{w}$ that does not intersect $\Pi(\mathcal{R}_3)$. For $j \in \varphi(\mathcal{N}_{\bar{w}} \cap \mathcal{W}_1, \mathcal{J}) = \{2\}$, the first and second components of $f_j(\bar{w})$ are negative and positive, respectively. Because of this and the fact that π_1 is monotone nondecreasing, one concludes that $\vec{v_j} := \big(\bar{w}, -f_j(\bar{w})\big)$ points towards $\mathbb{R}^2 \setminus \mathcal{W}_1$. Thus, for every point on the boundary of $\mathcal{Z} := \mathcal{W}_1$, either Condition 1, Condition 2, or Condition 3 of Lemma 2.1 holds. Since $\varphi(\mathcal{W}_1, \mathcal{J}) = \mathcal{J}$, Lemma 2.1 allows one to conclude that $\{\mathcal{W}_1, \; \mathcal{J}\}$ is invariant with respect to Σ_w and therefore that $\{\mathcal{Z}_1, \; \mathcal{J}\}$ is invariant with respect to Σ.

Consider now a point $\bar{w} := [\bar{w}_1 \;\; \bar{w}_2]'$ with $\bar{w}_1 = 0$ and $\bar{w}_2 \leq 0$. Such a point belongs to $\mathcal{W}_2$, in fact it is on the "vertical" boundary of this set. For any $j \in \mathcal{J}$, $f_j(\bar{w}) = 0$. Suppose now that the point $\bar{w} := [\bar{w}_1 \;\; \bar{w}_2]'$ is on the "curved" boundary of $\mathcal{W}_1$, i.e., that $\bar{w}_1 > 0$ and $\bar{w}_2 = \pi_3(\bar{w}_1)$. For $j \in \varphi(\{\bar{w}\}, \mathcal{J}) = \{2\}$, the first and second components of $f_j(\bar{w})$ are negative and positive, respectively. Because of this and the fact that π_3 is monotone nondecreasing, one concludes that $\vec{v_j} := \big(\bar{w}, f_j(\bar{w})\big)$ points towards $\mathcal{W}_2$. Thus, for every point on the boundary of $\mathcal{Z} := \mathcal{W}_2$, either Condition 1 or Condition 2 of Lemma 2.1 holds. Since $\varphi(\mathcal{W}_2, \mathcal{J}) = \mathcal{J}$, Lemma 2.1 allows one to conclude that $\{\mathcal{W}_2, \; \mathcal{J}\}$ is invariant with respect to Σ_w and therefore that $\{\mathcal{Z}_2, \; \mathcal{J}\}$ is invariant with respect to Σ. Similar arguments, applied also to the pair $\{\mathcal{Z}_3, \; \mathcal{J}\}$, allow one to prove the following:

Lemma 2.4. *Each of the pairs* $\{\mathcal{Z}_i, \; \mathcal{J}\}$ *with* $i \in \{1, 2, 3\}$*, is invariant with respect to the hybrid system* Σ *defined by* (2.12), (2.14)*, and* (2.16)*.*

It was argued in section 2.2.2 that for every initialization, the system Σ defined by (2.12), (2.14), and (2.16) has a unique solution that exists globally. In the sequel let $\{x, \sigma\}$ denote such a solution defined on the interval $[t_0, \infty)$.

Lyapunov Stability. To prove that the origin is a Lyapunov stable equilibrium point of Σ it is enough to show that by making $\|x(t_0)\|$ small enough it is possible to guarantee that $x(t)$ remains in a ball around the origin of arbitrarily small radius for all $t \geq t_0$. We consider separately two cases:

$\sigma(t_0) \in \mathcal{J} := \{2, 3, 4\}$: Because of (2.16) and (2.17), $x(t_0)$ must belong to $\mathcal{R}_{\sigma(t_0)}$, which, since $\sigma(t_0)$ is equal to 2, 3, or 4, implies that either $\pi_1(x_3(t_0)^2) < x_1(t_0)^2 + x_2(t_0)^2$ or $x(t_0) = 0$.

Therefore $x(t_0)$ belongs to the set $\mathcal{Z}_1$ defined by (2.22) with $c_1 = \|x(t_0)\|^2$ and $c_2 = \|x(t_0)\|^2 + \pi_4(\|x(t_0)\|^2)$. Since the pair $\{\mathcal{Z}_1,\ \mathcal{J}\}$ is invariant with respect to Σ one concludes that $x(t) \in \mathcal{Z}_1$ for every $t \geq t_0$. From this and (2.22) one concludes that

$$\|x(t)\|^2 \leq c_1 + c_2 = 2\|x(t_0)\|^2 + \pi_4(\|x(t_0)\|^2), \qquad \forall t \geq t_0 \tag{2.26}$$

$\sigma(t_0) = 1$: While $\sigma = 1$, $u = g_1(x)$ and therefore

$$x_i(t) = x_i(t_0) + t - t_0, \qquad i \in \{1, 2\} \tag{2.27}$$

$$x_3(t) = x_3(t_0) + \big(x_1(t_0) - x_2(t_0)\big)(t - t_0) \tag{2.28}$$

In case $x_1(t_0) = x_2(t_0) < 0$ and $x_3(t_0) = 0$ then x becomes zero in finite time and

$$\|x(t)\| \leq \|x(t_0)\|, \qquad \forall t \geq t_0 \tag{2.29}$$

Otherwise, ${x_1}^2 + {x_2}^2$ grows quadratically with t and, since in $\mathcal{R}_1$ the signal ${x_1}^2 + {x_2}^2$ must be bounded, one concludes that x leaves $\mathcal{R}_1$ at some finite time t_1. At this time, σ will switch from 1 to 2. By intersecting the trajectory given by (2.27)–(2.28) with the boundary of $\mathcal{R}_1$ it is straightforward to conclude that

$$\|x(t)\|^2 \leq k\|x(t_0)\|^2 + \pi_2(k\|x(t_0)\|^2), \qquad t \in [t_0, t_1] \tag{2.30}$$

with $k := \big(2 + 8\sup_{w\geq 0}\pi_2(w)\big)$, and that $x(t_1)$ belongs to the set $\mathcal{Z}_1$ defined by (2.22) with $c_1 = k\|x(t_0)\|^2$ and $c_2 = 2\pi_4(k\|x(t_0)\|^2)$. Since the pair $\{\mathcal{Z}_1,\ \mathcal{J}\}$ is invariant with respect to Σ, one concludes that $x(t) \in \mathcal{Z}_1$ for every $t \geq t_1$. From this, (2.22), and (2.30) one concludes that

$$\|x(t)\|^2 \leq k\|x(t_0)\|^2 + 2\pi_4(k\|x(t_0)\|^2), \qquad t \geq t_0 \tag{2.31}$$

Note that this inequality holds even in the case when $x_1(t_0) = x_2(t_0) < 0$ and $x_3(t_0) = 0$ (cf. (2.29)).

Finally, both from (2.26) and from (2.31), it is clear that by making $\|x(t_0)\|$ small enough it is possible to guarantee that $x(t)$ remains in a ball around the origin of arbitrarily small radius for all $t \geq t_0$. This proofs that the origin is a Lyapunov stable equilibrium point of Σ. ∎

Exponential Convergence. It was shown above that there exists a finite time t_1 after which x and σ enter the sets $\mathcal{Z}_1$ and $\mathcal{J}$, respectively[6]. Since $\sigma(t) \in \mathcal{J}$ for $t \geq t_1$, defining $w := \Pi(x)$ one concludes that

$$\dot{w}_1 = -2\,w_1, \qquad \forall t \geq t_1 \tag{2.32}$$

which means that $w_1 \to 0$ as fast as e^{-2t}. For $t \geq t_1$, while $x(t)$ is outside $\mathcal{Z}_2$, $\sigma(t) = 2$ and therefore $\dot{w}_2 = 2w_2$. Thus, after some finite time $t_2 \geq t_1$, w_2 becomes larger or equal to $\pi_3(w_1(t_1))$. Since π_3 is a monotone nondecreasing function and, because of (2.32), w_1 is also monotone nondecreasing,

$$w_2(t_2) \geq \pi_3(w_1(t_1)) \geq \pi_3(w_1(t_2))$$

[6]When $\sigma(t_0) \in \mathcal{J}$, one can just take $t_1 = t_0$.

Therefore $x(t_2)$ is inside the set $\mathcal{Z}_2$ defined by (2.23). Since $\sigma(t_2) \in \mathcal{J}$ and the pair $\{\mathcal{Z}_2,\ \mathcal{J}\}$ is invariant with respect to Σ, $x(t)$ remains in $\mathcal{Z}_2$ for $t \geq t_2$. Since it has been established that w_1 converges to zero, without loss of generality one can assume that t_2 is large enough so that $w_1(t_2) \leq c$. After t_2 two cases are possible:

Case 1: x gets into the set $\mathcal{Z}_3$ at some finite time t_3. Since $\sigma(t_3) \in \mathcal{J}$ and the pair $\{\mathcal{Z}_3,\ \mathcal{J}\}$ is invariant with respect to Σ, $x(t)$ remains in $\mathcal{Z}_3$ for $t \geq t_2$ and therefore

$$w_2(t) \leq \pi_4(w_1(t)), \qquad \forall t \geq t_3$$

Since w_1 converges to zero exponentially fast, because of the properties of π_4, w_2 also converges to zero exponentially fast.

Case 2: x remains in $\mathcal{Z}_2 \setminus \mathcal{Z}_3$. In this region, and for $w_2 \leq c$, σ can only be equal to 3 and therefore

$$\dot{w}_2 = -2\,w_2, \qquad \forall t \geq t_3$$

Also in this case w_2 converges to zero exponentially fast.

In either case, since $\|x\| = \sqrt{w_1 + w_2}$, exponentially fast convergence to zero of x is achieved. As for the control signal, simple algebra shows that for $t \geq t_1$

$$\|u\| \leq 2\left(\sqrt{w_2} + \sqrt{\frac{w_1}{w_2}}\right) \leq 2\left(\sqrt{w_2} + \sqrt{\frac{w_1}{\pi_1(w_1)}}\right) \tag{2.33}$$

Since w_1 and w_2 converge to zero exponentially fast, because of (2.33) and the properties of π_1, $\|u\|$ also converges to zero exponentially fast. ■

Remark 2.5. It was seen above that if $\{x, \sigma\}$ gets into $\mathcal{Z}_3 \times \mathcal{J}$ at some finite time t_3 then σ may switch forever between 2 and 3 (Case 1 in the proof above with $x(t_3) \neq 0$). Suppose this happens and let $\bar{t} \geq t_3$ denote an arbitrary time instant at which σ switches from 2 to 3. Then one must have

$$w_2(\bar{t}) = \pi_4\big(w_1(\bar{t})\big) \tag{2.34}$$

and σ can only switch back to 2 after some time interval Δt for which

$$w_2(\bar{t} + \Delta t) = \pi_3\big(w_1(\bar{t} + \Delta t)\big). \tag{2.35}$$

Since $\dot{w}_2 = -2\,w_2$ on the interval $[\bar{t}, \bar{t} + \Delta t)$,

$$w_2(\bar{t} + \Delta t) = w_2(\bar{t})e^{-2\Delta t}$$

From this, (2.34), and (2.35), one concludes that

$$\Delta t = \frac{1}{2}\log\frac{\pi_4\big(w_1(\bar{t})\big)}{\pi_3\big(w_1(\bar{t} + \Delta t)\big)} \tag{2.36}$$

But w_1 is decreasing for all $t \geq t_3$ and π_3 is monotone nondecreasing, thus $\pi_3\big(w_1(\bar{t} + \Delta t)\big) \leq \pi_3\big(w_1(\bar{t})\big)$. From this and (2.36) one concludes that

$$\Delta t \geq \frac{1}{2}\log\frac{\pi_4\big(w_1(\bar{t})\big)}{\pi_3\big(w_1(\bar{t})\big)}$$

Thus, for the π_j defined by (2.19), any time interval for which σ remains constant equal to 3 is bounded below by $\frac{1}{2}\log\frac{4}{2.5}$. A lower bound on any time interval for which σ remains constant equal to 2 can be computed in a similar fashion. One thus concludes that, not only is chattering precluded on any finite time interval, but also that the interval between consecutive switchings is bounded away from zero as time goes to infinity.

2.2.4 Generalization to the Vector Case

The results presented above generalize to the following higher dimensional nonholonomic integrator

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_3 = x_1' u_2 - x_2' u_1$$

where $x_1, x_2, u_1, u_2 \in \mathbb{R}^n$ and $x_3 \in \mathbb{R}$. In this case, the regions are defined by

$$\begin{aligned}
\mathcal{R}_1 &:= \left\{ x \in \mathbb{R}^{2n+1} : 0 \leq x_1'\, x_1 + x_2'\, x_2 < \pi_2 \left({x_3}^2 \right) \right\}, \\
\mathcal{R}_2 &:= \left\{ x \in \mathbb{R}^{2n+1} : \pi_1 \left({x_3}^2 \right) < x_1'\, x_1 + x_2'\, x_2 < \pi_4 \left({x_3}^2 \right) \right\}, \\
\mathcal{R}_3 &:= \left\{ x \in \mathbb{R}^{2n+1} : \pi_3 \left({x_3}^2 \right) < x_1'\, x_1 + x_2'\, x_2 \right\}, \\
\mathcal{R}_4 &:= \{0\}
\end{aligned}$$

and the control laws are

$$g_1(x) := \begin{bmatrix} e_1 \\ e_1 \end{bmatrix}, \qquad g_{2/3}(x) := \begin{bmatrix} \pm x_1 + \frac{x_3}{x_1'\, x_1 + x_2'\, x_2} x_2 \\ \pm x_2 - \frac{x_3}{x_1'\, x_1 + x_2'\, x_2} x_1 \end{bmatrix}, \qquad g_4(x) := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where e_1 denotes the first column of the $n \times n$ identity matrix. Another generalization to a higher dimensional nonholonomic integrator is

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_3 = x_1\, u_2 - x_2\, u_1$$

where $x_1, u_1 \in \mathbb{R}$ and $x_2, x_3, u_2 \in \mathbb{R}^n$. In this case the regions are defined by

$$\begin{aligned}
\mathcal{R}_1 &:= \left\{ x \in \mathbb{R}^{2n+1} : 0 \leq {x_1}^2 + x_2'\, x_2 < \pi_2 \left(x_3'\, x_3 \right) \right\}, \\
\mathcal{R}_2 &:= \left\{ x \in \mathbb{R}^{2n+1} : \pi_1 \left(x_3'\, x_3 \right) < {x_1}^2 + x_2'\, x_2 < \pi_4 \left(x_3'\, x_3 \right) \right\}, \\
\mathcal{R}_3 &:= \left\{ x \in \mathbb{R}^{2n+1} : \pi_3 \left(x_3'\, x_3 \right) < {x_1}^2 + x_2'\, x_2 \right\}, \\
\mathcal{R}_4 &:= \{0\}
\end{aligned}$$

and the control laws are

$$g_1(x) := \begin{bmatrix} 1 \\ e_1 \end{bmatrix}, \qquad g_{2/3}(x) := \begin{bmatrix} \pm x_1 + \frac{x_2'\, x_3}{{x_1}^2 + x_2'\, x_2} \\ \pm x_2 - \frac{x_1}{{x_1}^2 + x_2'\, x_2}\, x_3 \end{bmatrix}, \qquad g_4(x) := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2.3 Concluding Remarks

In this chapter the notion of a (positively) invariant set was extended to hybrid systems. Tests to determine if a set is invariant with respect to a hybrid system were given. These tests do not require the computation of the hybrid system's state trajectory.

A hybrid control law employing switching and logic to exponentially stabilize a nonholonomic integrator was proposed and analyzed. Arguments based on set invariance were used to prove Lyapunov stability and exponentially fast convergence of the state of the nonholonomic integrator to the origin. Simulation experiments show that simple "actuator dynamics" do not compromise the exponentially fast convergence nor do they introduce chattering.

The performance of the closed-loop system, in terms of speed of convergence and magnitude of the control signals, seems to be at least as good as the one obtained with time-varying controllers that achieve exponential convergence (e.g., [21]). Definite advantages/drawbacks of time-varying controllers over hybrid control laws can only be investigated in concrete applications.

Further effort is being made to design similar control laws for other types of nonholonomic systems. A question that also deserves attention is prompted by A. Teel's observation [102] that for hybrid systems like the one proposed in this thesis, the classical solution to the continuous dynamics varies discontinuously with respect to continuous variations of the initial state, therefore leading to the *hidden possibility of indecision*.

Chapter 3

Switching Between Stabilizing Controllers

Given a finite set of matrices $\mathcal{A} := \{A(p) : p \in \mathcal{P}\}$, consider the linear time-varying system

$$\dot{x} = A(\sigma)x \tag{3.1}$$

where σ denotes a piecewise constant "switching signal" taking values on $\mathcal{P}$. The following question has often been posed: "Under what conditions is the system (3.1) uniformly asymptotically stable for *every* piecewise constant switching signal σ?" [31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. In [35] it is shown that uniform asymptotic stability of (3.1) for every switching signal σ is equivalent to the existence of an induced norm $\|\cdot\|_*$ and a positive constant α such that

$$\|e^{At}\|_* \leq e^{-\alpha t}, \qquad \forall t \geq 0,\ \forall A \in \mathcal{A}$$

In [36] it is shown that uniform asymptotic stability of (3.1) for every switching signal σ is also equivalent to the existence of a common Lyapunov function (not necessarily quadratic) for the family of linear time-invariant systems $\{\dot{z} = A(p)z : p \in \mathcal{P}\}$. In [32, 38] are given simple algebraic conditions on the elements of $\mathcal{A}$ that are sufficient for the existence of a common quadratic Lyapunov function for the family of linear time-invariant systems $\{\dot{z} = A(p)z : p \in \mathcal{P}\}$, and therefore for the uniform asymptotic stability of (3.1) for every switching signal σ. For more on this topic see [33, 34] and references therein.

When systems like (3.1) arise in control problems, in general, the matrices in $\mathcal{A}$ have specific structures. Often, these matrices are obtained from the feedback connection of a fixed process with one of several controllers, and the switching signal σ determines which controller is in the feedback loop at each instant of time. One can then pose the question if, by appropriate choice of the realizations for the controllers, it is possible to make the system (3.1) uniformly asymptotically stable for every switching signal σ. Independent, unpublished work on this topic was reported in [103].

In this chapter it is assumed given a strictly proper process transfer matrix $\mathrm{H}_{\mathbb{P}}$ and a family of controller transfer matrices $\mathcal{K}_{\mathbb{C}} = \{\mathrm{K}_{\mathbb{C}}(p) : p \in \mathcal{P}\}$ such that every element of $\mathcal{K}_{\mathbb{C}}$ asymptotically stabilizes $\mathrm{H}_{\mathbb{P}}$. It is then shown that there always exist realizations[1]

[1] A quadruple of matrices $\{A, B, C, D\}$ is called a *realization* for a transfer matrix T if $T(s) = C(sI - A)^{-1}B + D$ for every $s \in \mathbb{C}$. When the matrix D is equal to zero one often writes simply that $\{A, B, C\}$ is a realization for T.

$\{A_{\mathbb{C}}(p), B_{\mathbb{C}}(p), C_{\mathbb{C}}(p), D_{\mathbb{C}}(p)\}$ for each transfer matrix $\mathrm{K}_{\mathbb{C}}(p)$ in $\mathcal{K}_{\mathbb{C}}$ such that for an appropriate realization $\{A_{\mathbb{P}}, B_{\mathbb{P}}, C_{\mathbb{P}}\}$ for $\mathrm{N}_{\mathbb{P}}$ there exists a common quadratic Lyapunov function for the family of linear time-invariant systems $\{\dot{z} = A(p)z : p \in \mathcal{P}\}$, where each

$$\dot{z} = A(p)z, \qquad\qquad p \in \mathcal{P}$$

denotes the feedback connection of the process realized by $\{A_{\mathbb{P}}, B_{\mathbb{P}}, C_{\mathbb{P}}\}$ with the controller realized by $\{A_{\mathbb{C}}(p), B_{\mathbb{C}}(p), C_{\mathbb{C}}(p), D_{\mathbb{C}}(p)\}$. This guarantees that (3.1) is uniformly asymptotically stable for every switching signal σ.

The problem addressed in this chapter is precisely formulated in Section 3.1. In Section 3.2 it is solved the simpler problem of finding realizations for a family of asymptotically stable transfer matrices such that all the realizations share a common quadratic Lyapunov function. Finally, in Section 3.3, the problem formulated in Section 3.1 is solved. The solution proposed makes use of the results derived in Section 3.2.

3.1 Problem Formulation

Let $\mathbb{P}$ be a multivariable LTI process with strictly proper transfer matrix $\mathrm{N}_{\mathbb{P}}$, control input u, and output y. The feedback configuration used in this chapter is shown in Figure 3.1. In this

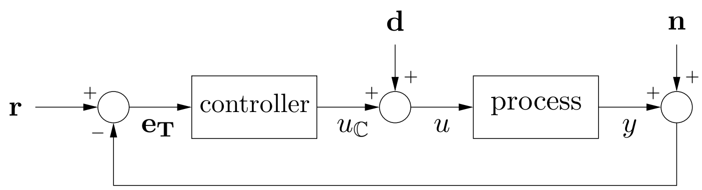

Figure 3.1: Feedback configuration

figure $\mathbf{r}$ denotes a bounded reference signal, $\mathbf{d}$ an unknown but bounded input disturbance, and $\mathbf{n}$ unknown but bounded measurement noise. A controller transfer matrix $\mathrm{K}_{\mathbb{C}}$ is said to *stabilize* $\mathrm{N}_{\mathbb{P}}$ if for any minimal realization $\{A_{\mathbb{P}}, B_{\mathbb{P}}, C_{\mathbb{P}}\}$ of $\mathrm{N}_{\mathbb{P}}$ and any minimal realization $\{A_{\mathbb{C}}, B_{\mathbb{C}}, C_{\mathbb{C}}, D_{\mathbb{C}}\}$ of $\mathrm{K}_{\mathbb{C}}$ the feedback connection shown in Figure 3.1 is asymptotically stable, i.e., all the poles of the matrix

$$\begin{bmatrix} A_{\mathbb{P}} - B_{\mathbb{P}} D_{\mathbb{C}} C_{\mathbb{P}} & B_{\mathbb{P}} C_{\mathbb{C}} \\ -B_{\mathbb{C}} C_{\mathbb{P}} & A_{\mathbb{C}} \end{bmatrix} \tag{3.2}$$

have negative real part. In case all the poles of (3.2) have real part smaller than $-\lambda$ for some $\lambda \geq 0$, $\mathrm{K}_{\mathbb{C}}$ is said to *stabilize* $\mathrm{N}_{\mathbb{P}}$ *with stability margin* λ.

Consider now a finite set of controller transfer matrices $\mathcal{K}_{\mathbb{C}} = \{\mathrm{K}_{\mathbb{C}}(p) : p \in \mathcal{P}\}$ each stabilizing $\mathrm{N}_{\mathbb{P}}$. The problem under consideration is to determine a $n_{\mathbb{P}}$-dimensional stabilizable and detectable realization $\{A_{\mathbb{P}}, B_{\mathbb{P}}, C_{\mathbb{P}}\}$ for $\mathrm{N}_{\mathbb{P}}$, $n_{\mathbb{C}}$-dimensional stabilizable and detectable realizations $\{A_{\mathbb{C}}(p), B_{\mathbb{C}}(p), C_{\mathbb{C}}(p), D_{\mathbb{C}}(p)\}$ for each $\mathrm{K}_{\mathbb{C}}(p)$ in $\mathcal{K}_{\mathbb{C}}$, and two symmetric positive definite matrices $P, Q \in \mathbb{R}^{(n_{\mathbb{P}}+n_{\mathbb{C}})\times(n_{\mathbb{P}}+n_{\mathbb{C}})}$ such that

$$Q\,\bar{A}(p) + \bar{A}(p)'\,Q \leq -P\,, \qquad\qquad p \in \mathcal{P} \tag{3.3}$$

where

$$\bar{A}(p) := \begin{bmatrix} A_{\mathbb{P}} - B_{\mathbb{P}}\, D_{\mathbb{C}}(p)\, C_{\mathbb{P}} & B_{\mathbb{P}}\, C_{\mathbb{C}}(p) \\ -B_{\mathbb{C}}(p)\, C_{\mathbb{P}} & A_{\mathbb{C}}(p) \end{bmatrix} \tag{3.4}$$

The motivation for this problem is the following: Let $\mathcal{S}$ denote the set of all piecewise constant "switching signals" taking values on $\mathcal{P}$ and, for each $\sigma \in \mathcal{S}$, let $\mathbb{C}(\sigma)$ denote the "multi-controller"

$$\dot{x}_{\mathbb{C}} = A_{\mathbb{C}}(\sigma)x_{\mathbb{C}} + B_{\mathbb{C}}(\sigma)\mathbf{e_T}\,, \qquad\qquad u_{\mathbb{C}} = C_{\mathbb{C}}(\sigma)x_{\mathbb{C}} + D_{\mathbb{C}}(\sigma)\mathbf{e_T}$$

For each constant switching signal $\sigma = p \in \mathcal{P}$, $\mathbb{C}(\sigma)$ is a linear time invariant system with transfer matrix equal to $\mathrm{K}_{\mathbb{C}}(p)$. In this sense, $\mathbb{C}(\sigma)$ "switches" between the controller transfer matrices in $\mathcal{K}_{\mathbb{C}}$ based on the current value of σ. Consider now the feedback connection between $\mathbb{C}(\sigma)$ and $\mathbb{P}$ shown in Figure 3.2. This feedback connection can be realized by the following

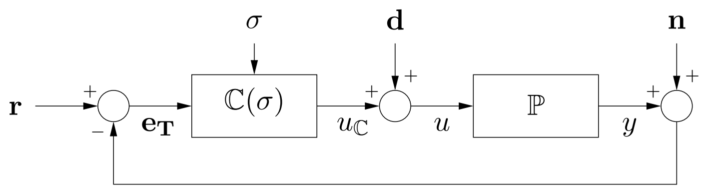

Figure 3.2: Feedback connection between $\mathbb{P}$ and $\mathbb{C}(\sigma)$.

dynamical system

$$\dot{x} = \bar{A}(\sigma)x + \bar{B}(\sigma)\begin{bmatrix}\mathbf{d}\\ \mathbf{r}-\mathbf{n}\end{bmatrix} \qquad\qquad y = \bar{C}x \tag{3.5}$$

with $x := [x'_{\mathbb{P}} \;\; x'_{\mathbb{C}}]'$ and, for each $p \in \mathcal{P}$, $\bar{A}(p)$ defined by (3.4), and

$$\bar{B}(p) := \begin{bmatrix} B_{\mathbb{P}} & 0 \\ 0 & B_{\mathbb{C}}(p)\end{bmatrix} \qquad\qquad \bar{C} := \begin{bmatrix} C_{\mathbb{P}} & 0\end{bmatrix}$$

Since each transfer matrix in $\mathcal{K}_{\mathbb{C}}$ stabilizes $\mathrm{N}_{\mathbb{P}}$, (3.5) is asymptotically stable for constant $\sigma = p \in \mathcal{P}$. But, in general, this is not enough to guarantee that for every $\sigma \in \mathcal{S}$ and every bounded signals $\mathbf{d}$, $\mathbf{r}$, and $\mathbf{n}$, the state of (3.5) remains bounded. However, because of (3.3), for every $\sigma \in \mathcal{S}$, $V(z) := z'Qz$ is a Lyapunov function for the system

$$\dot{z} = \bar{A}(\sigma)z \tag{3.6}$$

which is therefore exponentially stable (cf. [104, Corollary 4.2]). Thus, denoting by $\Phi(t, t_0; \sigma)$ with $\sigma \in \mathcal{S}$, the state transition matrix of (3.6), there exist constants c and λ such that

$$\|\Phi(t, t_0; \sigma)\| \le c\,e^{-\lambda(t-t_0)} \qquad\qquad \forall t, t_0 \ge 0 \tag{3.7}$$

Moreover, since V does not depend on σ, one can choose the constants c and λ such that (3.7) holds for every $\sigma \in \mathcal{S}$. From this and the variation of constants formula applied to (3.5) one concludes that for every $\sigma \in \mathcal{S}$ and every $t \ge 0$,

$$\begin{aligned}\|x(t)\| &\le \|\Phi(t, 0; \sigma)\|\,\|x(0)\| + \beta(\|\mathbf{d}\|_\infty + \|\mathbf{r}-\mathbf{n}\|_\infty)\int_0^t \|\Phi(t, \tau; \sigma)\|d\tau \\ &\le c\,e^{-\lambda t}\|x(0)\| + \frac{c\,\beta}{\lambda}(\|\mathbf{d}\|_\infty + \|\mathbf{r}-\mathbf{n}\|_\infty)\end{aligned}$$

where $\beta := \max_{p\in\mathcal{P}} \|\bar{B}(p)\|$ and $\|\cdot\|_\infty$ denotes the $\mathcal{L}_\infty$ norm. The following was proved:

Lemma 3.1. *Assume that there exist positive definite matrices Q and P for which* (3.3) *holds. For every switching signal $\sigma \in \mathcal{S}$ and every bounded piecewise continuous exogenous signals* $\mathbf{r}$*,* $\mathbf{n}$*, and* $\mathbf{d}$*, the state x of* (3.5) *is bounded. Moreover, if $\mathbf{r} = \mathbf{d} = \mathbf{n} = 0$, x decays to zero exponentially fast with a rate of decay that is independent of σ.*

Note that the exponential stability of (3.6) guarantees that the system (3.5) remains stable under small perturbations to the dynamics of the system. For a detailed discussion of this issue see, for example, [104, Section 4.5].

3.2 Realizations for Stable Transfer Matrices

This section addresses a simpler problem than the one formulated above. Consider a finite family of asymptotically stable transfer matrices $\mathcal{A} = \{\mathrm{S}(p) : p \in \mathcal{P}\}$ whose poles have real part smaller then $-\lambda$ for some $\lambda \geq 0$. It is shown below how to compute a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$, and stabilizable and detectable n-dimensional realizations $\{A(p), B(p), C(p), D(p)\}$ for each $\mathrm{S}(p) \in \mathcal{A}$ such that

$$QA(p) + A(p)'Q < -2\lambda Q\,, \qquad\qquad p \in \mathcal{P} \tag{3.8}$$

Let n be the McMillan degree of the transfer matrix in $\mathcal{A}$ with largest McMillan degree and, for each $p \in \mathcal{P}$, let $\{\tilde{A}(p), \tilde{B}(p), \tilde{C}(p), \tilde{D}(p)\}$ be any n-dimensional realization of $\mathrm{S}(p)$ with $\tilde{A}(p)+\lambda I$ asymptotically stable. For each $p \in \mathcal{P}$, because of the asymptotic stability of $\tilde{A}(p) + \lambda I$, the Lyapunov equation

$$Q(p)\big(\tilde{A}(p) + \lambda I\big) + \big(\tilde{A}(p) + \lambda I\big)'Q(p) = -I \tag{3.9}$$

must have a symmetric positive definite solution $Q(p)$. For each $p \in \mathcal{P}$, let

$$A(p) := Q(p)^{\frac{1}{2}}\tilde{A}(p)Q(p)^{-\frac{1}{2}}, \quad B(p) := Q(p)^{\frac{1}{2}}\tilde{B}(p), \quad C(p) := \tilde{C}(p)Q(p)^{-\frac{1}{2}}, \quad D(p) := \tilde{D}(p) \tag{3.10}$$

Since $\{A(p), B(p), C(p), D(p)\}$ is obtained from $\{\tilde{A}(p), \tilde{B}(p), \tilde{C}(p), \tilde{D}(p)\}$ by a similarity transformation, $\{A(p), B(p), C(p), D(p)\}$ is also a realization of $\mathrm{S}(p)$. Moreover, from (3.9) and (3.10) one concludes that

$$Q(p)^{\frac{1}{2}}\big(A(p) + A(p)'\big)Q(p)^{\frac{1}{2}} = -I - 2\lambda Q(p)$$

Left and right multiplication of the above equality by $Q(p)^{-\frac{1}{2}}$ yields

$$A(p) + A(p)' = -Q(p)^{-1} - 2\lambda I$$

and therefore one concludes that (3.8) holds with $Q = I$. The following was proved:

Lemma 3.2. *Given any finite family of asymptotically stable transfer matrices $\mathcal{A} = \{\mathrm{S}(p) : p \in \mathcal{P}\}$ whose poles have real part smaller then $-\lambda$ for some $\lambda \geq 0$, there exists an integer n, a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$, and stabilizable and detectable n-dimensional realizations $\{A(p), B(p), C(p), D(p)\}$ for each $\mathrm{S}(p) \in \mathcal{A}$ such that* (3.8) *holds.*

3.3 Realizations for Controller Transfer Matrices

Returning to the problem formulated in Section 3.1, let λ be a nonnegative constant such that each $\mathrm{K}_{\mathbb{C}}(p)$ in $\mathcal{K}_{\mathbb{C}}$ stabilizes $\mathrm{N}_{\mathbb{P}}$ with stability margin λ, and let $\mathrm{K}_{\mathbb{C}}$ be a controller transfer matrix that stabilizes $\mathrm{N}_{\mathbb{P}}$ with stability margin λ. For example, one can take $\mathrm{K}_{\mathbb{C}}$ to be one of the elements of $\mathcal{K}_{\mathbb{C}}$. Because $\mathrm{K}_{\mathbb{C}}$ stabilizes $\mathrm{N}_{\mathbb{P}}$ with stability margin λ, it is known[2] that there exist matrices A_E, B_E, C_E, D_E, F_E, and G_E with appropriate dimensions such that $A_E+\lambda I$ is a stability matrix, and $\{A_E+D_EC_E, B_E, C_E\}$ and $\{A_E-B_EF_E, D_E-B_EG_E, F_E, G_E\}$ are stabilizable and detectable realizations of $\mathrm{N}_{\mathbb{P}}$ and $\mathrm{K}_{\mathbb{C}}$, respectively, with stability margin[3] λ.

For each $p\in\mathcal{P}$, let

$$\mathrm{S}(p) := \big(-\mathrm{Y}_{\mathbb{C}}+\mathrm{X}_{\mathbb{C}}\mathrm{K}_{\mathbb{C}}(p)\big)\big(\mathrm{X}_{\mathbb{P}}+\mathrm{Y}_{\mathbb{P}}\mathrm{K}_{\mathbb{C}}(p)\big)^{-1} \tag{3.11}$$

where

$$\begin{bmatrix}\mathrm{X}_{\mathbb{C}} & -\mathrm{Y}_{\mathbb{C}}\\ \mathrm{Y}_{\mathbb{P}} & \mathrm{X}_{\mathbb{P}}\end{bmatrix} := \begin{bmatrix}F_E\\ C_E\end{bmatrix}(sI-A_E)^{-1}\begin{bmatrix}B_E & -D_E\end{bmatrix}+\begin{bmatrix}I & -G_E\\ 0 & I\end{bmatrix} \tag{3.12}$$

Straightforward algebra shows that the transfer function on the right-hand side of (3.11) is equal to the transfer function from e to v defined by the system of equations[4]

$$\begin{bmatrix}v\\ \bar{y}\end{bmatrix} = \begin{bmatrix}\mathrm{X}_{\mathbb{C}} & -\mathrm{Y}_{\mathbb{C}}\\ \mathrm{Y}_{\mathbb{P}} & \mathrm{X}_{\mathbb{P}}-I\end{bmatrix}\circ\begin{bmatrix}\bar{u}\\ e-\bar{y}\end{bmatrix}, \qquad \bar{u}=\mathrm{K}_{\mathbb{C}}(p)\circ(e-\bar{y}) \tag{3.13}$$

Because of (3.12),

$$\left\{A_E, \begin{bmatrix}B_E & -D_E\end{bmatrix}, \begin{bmatrix}F_E\\ C_E\end{bmatrix}, \begin{bmatrix}I & -G_E\\ 0 & 0\end{bmatrix}\right\}$$

is a realization for $\left[\begin{smallmatrix}\mathrm{X}_{\mathbb{C}} & -\mathrm{Y}_{\mathbb{C}}\\ \mathrm{Y}_{\mathbb{P}} & \mathrm{X}_{\mathbb{P}}-I\end{smallmatrix}\right]$, thus picking any minimal realization $\{\hat{A}(p),\hat{B}(p),\hat{C}(p),\hat{D}(p)\}$ of $\mathrm{K}_{\mathbb{C}}(p)$, the system (3.13) (and therefore $\mathrm{S}(p)$) can be realized by $\{\tilde{A}(p),\tilde{B}(p),\tilde{C}(p),\tilde{D}(p)\}$ with

$$\tilde{A}(p) := \begin{bmatrix}A_E+D_EC_E-B_E\hat{D}(p)C_E & B_E\hat{C}(p)\\ -\hat{B}(p)C_E & \hat{A}(p)\end{bmatrix} \tag{3.14}$$

and $\tilde{B}(p)$, $\tilde{C}(p)$, $\tilde{D}(p)$ appropriately defined. Since $\mathrm{K}_{\mathbb{C}}(p)$ stabilizes $\mathrm{N}_{\mathbb{P}}$ with stability margin λ and $\{A_E+D_EC_E, B_E, C_E\}$ is a stabilizable and detectable realization of $\mathrm{N}_{\mathbb{P}}$ with stability margin λ, the poles of $\tilde{A}(p)$ must have real part smaller then $-\lambda$ (cf. $\tilde{A}(p)$ in (3.14) against (3.2)). Thus, for each $p\in\mathcal{P}$, the poles of $\mathrm{S}(p)$ must also have real part smaller then $-\lambda$. By virtue of Lemma 3.2, one thus concludes that there exists an integer n_S, a symmetric positive definite matrix $Q\in\mathbb{R}^{n_S\times n_S}$, and stabilizable and detectable n_S-dimensional realizations $\{A(p),B(p),C(p),D(p)\}$ for each $\mathrm{S}(p)$ with $p\in\mathcal{P}$, such that

$$QA(p)+A(p)'Q<-2\lambda Q\,, \qquad p\in\mathcal{P} \tag{3.15}$$

[2]Cf. Lemma A.1 in Appendix A, which is a reformulation of results that can be found in [105, 106, 107].

[3]A realization $\{A,B,C,D\}$ is *stabilizable and detectable with stability margin* λ if $\{A+\lambda I, B\}$ and $\{C, A+\lambda I\}$ are stabilizable and detectable pairs, respectively.

[4]Given a transfer matrix $\mathrm{H}:\mathbb{C}\to\mathbb{C}^{m\times n}$ and a piecewise constant signal $u:[0,\infty)\to\mathbb{R}^n$, $\mathrm{H}\circ u$ denotes the signal $y:[0,\infty)\to\mathbb{R}^m$ defined by the convolution of the impulse response of H with u.

Suppose now that for every $p \in \mathcal{P}$ one defines

$$A_{\mathbb{C}}(p) := \begin{bmatrix} A_E - B_E F_E + B_E D(p) C_E & B_E C(p) \\ B(p) C_E & A(p) \end{bmatrix} \qquad B_{\mathbb{C}}(p) := \begin{bmatrix} -D_E + B_E(D(p)+G_E) \\ B(p) \end{bmatrix} \tag{3.16}$$

$$C_{\mathbb{C}}(p) := \begin{bmatrix} -F_E + D(p) C_E & C(p) \end{bmatrix} \qquad D_{\mathbb{C}}(p) := D(p) + G_E \tag{3.17}$$

Because of (3.12), it is straightforward to verify that, for each $p \in \mathcal{P}$, the transfer function of $\{A_{\mathbb{C}}(p), B_{\mathbb{C}}(p), C_{\mathbb{C}}(p), D_{\mathbb{C}}(p)\}$ is equal to the transfer function from $\mathbf{e_T}$ to u defined by the system of equations

$$\begin{bmatrix} \bar{u} \\ e \end{bmatrix} = \begin{bmatrix} \mathrm{X}_{\mathbb{C}} - I & -\mathrm{Y}_{\mathbb{C}} \\ \mathrm{Y}_{\mathbb{P}} & \mathrm{X}_{\mathbb{P}} \end{bmatrix} \circ \begin{bmatrix} u \\ \mathbf{e_T} \end{bmatrix}, \qquad v = \mathrm{S}(p) \circ e, \qquad u = v - \bar{u}$$

and therefore that

$$C_{\mathbb{C}}(p)\big(sI - A_{\mathbb{C}}(p)\big)^{-1} B_{\mathbb{C}}(p) + D_{\mathbb{C}}(p) = \big(\mathrm{X}_{\mathbb{C}} - \mathrm{S}(p)\mathrm{Y}_{\mathbb{P}}\big)^{-1}\big(\mathrm{Y}_{\mathbb{C}} + \mathrm{S}(p)\mathrm{X}_{\mathbb{P}}\big) \tag{3.18}$$

But, solving (3.11) with respect to $\mathrm{K}_{\mathbb{C}}(p)$ yields

$$\mathrm{K}_{\mathbb{C}}(p) = \big(\mathrm{X}_{\mathbb{C}} - \mathrm{S}(p)\mathrm{Y}_{\mathbb{P}}\big)^{-1}\big(\mathrm{Y}_{\mathbb{C}} + \mathrm{S}(p)\mathrm{X}_{\mathbb{P}}\big) \tag{3.19}$$

From this and (3.18) one concludes that $\{A_{\mathbb{C}}(p), B_{\mathbb{C}}(p), C_{\mathbb{C}}(p), D_{\mathbb{C}}(p)\}$ is a realization for $\mathrm{K}_{\mathbb{C}}(p)$. The main result of this chapter can now be stated:

Theorem 3.3. *There exists a symmetric positive definite matrix $\bar{Q}$ such that*

$$\bar{Q}\,\bar{A}(p) + \bar{A}(p)'\,\bar{Q} \leq -2\lambda\bar{Q}\,, \qquad \forall p \in \mathcal{P} \tag{3.20}$$

where

$$\bar{A}(p) := \begin{bmatrix} A_{\mathbb{P}} - B_{\mathbb{P}} D_{\mathbb{C}}(p) C_{\mathbb{P}} & B_{\mathbb{P}} C_{\mathbb{C}}(p) \\ -B_{\mathbb{C}}(p) C_{\mathbb{P}} & A_{\mathbb{C}}(p) \end{bmatrix} \tag{3.21}$$

with the realization $\{A_{\mathbb{C}}(p), B_{\mathbb{C}}(p), C_{\mathbb{C}}(p), D_{\mathbb{C}}(p)\}$ given by (3.16)–(3.17) and

$$A_{\mathbb{P}} := A_E + D_E C_E, \qquad B_{\mathbb{P}} := B_E, \qquad C_{\mathbb{P}} := C_E \tag{3.22}$$

Before proving Theorem 3.3, it should be noted that, in general, the realizations given by (3.16)–(3.17) are not minimal. However, denoting by n_K the McMillan degree of $\mathrm{K}_{\mathbb{C}}$, by n_H the McMillan degree of $\mathrm{H}_{\mathbb{C}}$, and by $n_{\mathcal{K}}$ the McMillan degree of the transfer matrix in $\mathcal{K}_{\mathbb{C}}$ with largest McMillan degree, the size of A_E need not be larger than $n_H + n_K$ (cf. Lemma A.1) and therefore the dimension of the state of the realizations (3.16)–(3.17) need not be larger than $2(n_H + n_K) + n_{\mathcal{K}}$ *no matter what the number of controllers in $\mathcal{K}_{\mathbb{C}}$ is.* When $\mathrm{K}_{\mathbb{C}}$ is chosen to have the structure of an observer with state feedback , i.e., when $\mathrm{N}_{\mathbb{P}}$ and $\mathrm{K}_{\mathbb{C}}$ have realizations $\{A, B, C\}$ and $\{A + HC - BF, H, F\}$, respectively, the size of the matrix A_E need not be larger than n_H (cf. Remark A.2) and therefore the dimension of the state of the realizations (3.16)–(3.17) can be reduced to $2n_H + n_{\mathcal{K}}$.

Proof of Theorem 3.3. Replacing (3.16)–(3.17) and (3.22) in (3.21), for each $p \in \mathcal{P}$, one obtains

$$\bar{A}(p) = \begin{bmatrix} A_E + D_E C_E - B_E(D(p)+G_E)C_E & -B_E F_E + B_E D(p) C_E & B_E C(p) \\ D_E C_E - B_E(D(p)+G_E)C_E & A_E - B_E F_E + B_E D(p) C_E & B_E C(p) \\ -B(p) C_E & B(p) C_E & A(p) \end{bmatrix}$$

and, defining $T := \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & 0 & I \\ -I & I & 0 \end{smallmatrix}\right]$, one further concludes that

$$T\bar{A}(p)T^{-1} = \begin{bmatrix} A_E + D_E C_E - B_E F_E - B_E G_E C_E & B_E C(p) & -B_E F_E + B_E D(p) C_E \\ 0 & A(p) & B(p) C_E \\ 0 & 0 & A_E \end{bmatrix} \tag{3.23}$$

Since $\mathrm{K}_{\mathbb{C}}$ stabilizes $N_{\mathbb{P}}$ with stability margin λ, and $\{A_E + D_E C_E, B_E, C_E\}$ and $\{A_E - B_E F_E, D_E - B_E G_E, F_E, G_E\}$ are stabilizable and detectable realizations of $\mathrm{N}_{\mathbb{P}}$ and $\mathrm{K}_{\mathbb{C}}$, respectively, with stability margin λ, all the poles of

$$\bar{A}_E := \begin{bmatrix} A_E + D_E C_E - B_E G_E C_E & B_E F_E \\ -(D_E - B_E G_E) C_E & A_E - B_E F_E \end{bmatrix} \tag{3.24}$$

have real part smaller then $-\lambda$ (cf. right-hand side of (3.24) against (3.2)). But, defining $T_E := \left[\begin{smallmatrix} I & 0 \\ I & I \end{smallmatrix}\right]$,

$$T_E \bar{A}_E T_E^{-1} = \begin{bmatrix} A_E + D_E C_E - B_E F_E - B_E G_E C_E & B_E F_E \\ 0 & A_E \end{bmatrix}$$

Thus $A_E + D_E C_E - B_E F_E - B_E G_E C_E + \lambda I$ must be asymptotically stable. From this and the asymptotic stability of $A_E + \lambda I$ one concludes that there exist positive definite symmetric matrices Q_1, Q_2 such that

$$\begin{aligned} Q_1(A_E + &D_E C_E - B_E F_E - B_E G_E C_E + \lambda I) \\ &+ (A_E + D_E C_E - B_E F_E - B_E G_E C_E + \lambda I)' Q_1 = -I \end{aligned} \tag{3.25}$$

$$Q_2(A_E + \lambda I) + (A_E + \lambda I)' Q_2 = -I \tag{3.26}$$

Moreover, because of (3.15), each

$$P(p) := -Q(A(p) + \lambda I) - (A(p) + \lambda I)' Q, \qquad p \in \mathcal{P}$$

is positive definite. Therefore there must exist a positive constant ϵ small enough such that

$$P(p) - \epsilon Q B(p) C_E C_E' B(p)' Q > 0, \qquad \forall p \in \mathcal{P}$$

which guarantees that each

$$R(p) := \epsilon \begin{bmatrix} P(p) & -Q B(p) C_E \\ -C_E' B(p)' Q & \epsilon^{-1} I \end{bmatrix} \qquad p \in \mathcal{P} \tag{3.27}$$

is also positive definite (cf. [33, Section 2.1]). Let now

$$\bar{Q} := T' \begin{bmatrix} \epsilon_1 Q_1 & 0 & 0 \\ 0 & \epsilon Q & 0 \\ 0 & 0 & Q_2 \end{bmatrix} T \tag{3.28}$$

with

$$\epsilon_1 := \left(\max_{p \in \mathcal{P}} \| Q_1 S(p) R(p)^{-1} S(p)' Q_1 \| \right)^{-1} \tag{3.29}$$

where, for each $p \in \mathcal{P}$,

$$S(p) := \begin{bmatrix} B_E C(p) & -B_E F_E + B_E D(p) C_E \end{bmatrix} \tag{3.30}$$

From (3.23), (3.25)–(3.26), (3.27), (3.28), and (3.30) one concludes that

$$\bar{Q}(\bar{A}(p) + \lambda I) + (\bar{A}(p) + \lambda I)' \bar{Q} = -\epsilon_1 T' \begin{bmatrix} I & -Q_1 S(p) \\ -S(p)' Q_1 & \epsilon_1^{-1} R(p) \end{bmatrix} T, \qquad p \in \mathcal{P} \tag{3.31}$$

But, because of (3.29), $I - \epsilon_1 Q_1 S(p) R(p)^{-1} Q_1 S(p) > 0$ for each $p \in \mathcal{P}$, thus

$$\begin{bmatrix} I & -Q_1 S(p) \\ -S(p)' Q_1 & \epsilon_1^{-1} R(p) \end{bmatrix} > 0, \qquad p \in \mathcal{P}$$

(cf. [33, Section 2.1]). From this and (3.31) one concludes that (3.20) holds. ■

Remark 3.4. Denoting by $\mathrm{RH}_\infty^\lambda$ the ring of transfer matrices whose entries are proper rational functions with real coefficients and poles with real part smaller then $-\lambda$, the transfer matrices $\mathrm{X}_\mathbb{P}, \mathrm{Y}_\mathbb{P}, \mathrm{Y}_\mathbb{C}, \mathrm{X}_\mathbb{C}$ defined in (3.12) form a simultaneous right-coprime factorization of $\mathrm{N}_\mathbb{P}$ and $\mathrm{K}_\mathbb{C}$ in that $\mathrm{X}_\mathbb{P}$ and $\mathrm{X}_\mathbb{C}$ have causal inverse, $\left[\begin{smallmatrix} \mathrm{X}_\mathbb{C} & -\mathrm{Y}_\mathbb{C} \\ \mathrm{Y}_\mathbb{P} & \mathrm{X}_\mathbb{P} \end{smallmatrix}\right]$ is a unit in $\mathrm{RH}_\infty^\lambda$, and $\mathrm{N}_\mathbb{P} = \mathrm{X}_\mathbb{P}^{-1} \mathrm{Y}_\mathbb{P}$ and $\mathrm{K}_\mathbb{C} = \mathrm{X}_\mathbb{C}^{-1} \mathrm{Y}_\mathbb{C}$. Thus, the existence of the family of transfer matrices $\{\mathrm{S}(p) : p \in \mathcal{P}\} \subset \mathrm{RH}_\infty^\lambda$ such that (3.19) holds is not surprising in light of the Youla parameterization of all controllers that stabilize $\mathrm{N}_\mathbb{P}$ (with stability margin λ) given by [105]. Note also that since $\mathrm{K}_\mathbb{C} = \mathrm{X}_\mathbb{C}^{-1} \mathrm{Y}_\mathbb{C}$, if one chooses $\mathrm{K}_\mathbb{C} = \mathrm{K}_\mathbb{C}(p_0)$ for some $p_0 \in \mathcal{P}$, then the corresponding transfer matrix $\mathrm{S}(p_0)$ given by (3.11) with $p = p_0$ is equal to 0.

Chapter 4

Scale-Independent Hysteresis Switching

"Scale-independence" is a property of certain switching algorithms used in an adaptive context which is key to proving an algorithm's correctness when operating in the face of noise and disturbance inputs [41]. The concept of *dwell-time switching*—exploited in [41] and elsewhere—has the advantage of being scale-independent. However, the existence of a prescribed dwell-time makes it impossible to rule out the possibility of finite escape in applications of dwell-time switching to the adaptive control of nonlinear systems [42]. On the other hand, the popular idea of *hysteresis switching* [43, 9] does not have this shortcoming. Unfortunately, hysteresis switching is not a scale-independent algorithm. The objective of this chapter is to introduce a new form of chatter-free switching that does not employ a prescribed dwell-time and which is scale independent. We call this logic "scale-independent hysteresis switching" and we prove its correctness for applications to adaptive control [42].

Consider the hybrid dynamical system

$$\dot{x} = f_\sigma(x,t), \qquad\qquad x(0) = x_0 \tag{4.1}$$

where $\{f_p : p \in \mathcal{P}\}$ is an indexed family of locally Lipschitz functions taking values on a finite dimensional space $\mathcal{X}$ and defined on $\mathcal{X} \times [0,\infty)$, and σ is a piecewise constant *switching signal* taking values in $\mathcal{P}$. The switching signal σ is chosen so as to cause the *performance signals*

$$\pi_p := \Pi(p,x,t), \qquad\qquad p \in \mathcal{P} \tag{4.2}$$

to have certain desired properties. Here, Π is a *performance function* from $\mathcal{P} \times \mathcal{X} \times [0,\infty)$ to $\mathbb{R}$ that is continuous with respect to the second and third arguments for frozen values of the first.

The algorithm used to generate σ considered in this chapter is called a *scale-independent hysteresis switching logic* and can be regarded as a hybrid dynamical system $\mathbb{S}_\mathbb{H}$ whose input is x and whose state and output are both σ. To specify $\mathbb{S}_\mathbb{H}$ it is necessary to first pick a positive number $h > 0$ called a *hysteresis constant*. $\mathbb{S}_\mathbb{H}$'s internal logic is then defined by the computer diagram shown in Figure 4.1 where the π_p are defined by (4.2) and, at each time t, $q := \arg\min_{p\in\mathcal{P}} \Pi(p,x,t)$. In interpreting this diagram it is to be understood that σ's value at each of its switching times $\bar{t}$ is equal to its limit from the right as $t \downarrow \bar{t}$. Thus if $\bar{t}_i$ and $\bar{t}_{i+1}$ are two consecutive switching times, then σ is constant on $[\bar{t}_i, \bar{t}_{i+1})$. The functioning of $\mathbb{S}_\mathbb{H}$ is roughly as follows. Suppose that at some time t_0, $\mathbb{S}_\mathbb{H}$ has just changed the value of σ to p. σ

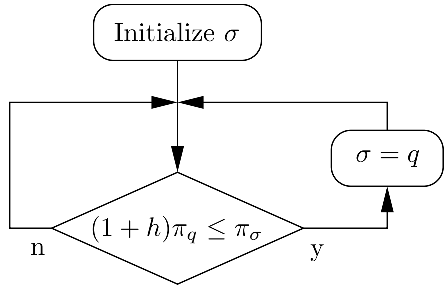

Figure 4.1: Computer Diagram of $\mathbb{S}_{\mathbb{H}}$.

is then held fixed at this value unless and until there is a time $t_1 > t_0$ at which $(1+h)\pi_q \leq \pi_p$ for some $q \in \mathcal{P}$. If this occurs, σ is set equal to q and so on. This type of logic has numerous applications in adaptive and supervisory control [43, 9, 34, 44, 42, 45, 46, 47]. In Chapter 6 the scale-independent hysteresis switching logic is used in the supervisory control of nonlinear systems.

The main result of this chapter is the scale-independent hysteresis switching theorem. This theorem states that under appropriate "open-loop" assumptions, switching will stop at some finite time. Being able to establish that switching stops in finite time is crucial to the stability analysis of adaptive and supervisory control algorithms using hysteresis switching.

The switching logic described above is new. Its main advantage over the hysteresis switching logic considered in [43, 9] is that it is "scale-independent" in that its output σ remains unchanged if its performance function/input signal pair $\{\Pi, x\}$ is replaced by another pair $\{\bar{\Pi}, \bar{x}\}$ satisfying

$$\bar{\Pi}(p, \bar{x}, t) = \vartheta \Pi(p, x, t), \qquad \forall p \in \mathcal{P},\ t \geq 0$$

where ϑ is a positive time function. This is because, for any fixed time t, (i) the value of p that minimizes $\Pi(p, x, t)$ is the same value of p that minimizes $\bar{\Pi}(p, \bar{x}, t)$ and (ii) $(1+h)\Pi(q, x, t) \leq \Pi(p, x, t)$ is exactly equivalent to $(1+h)\bar{\Pi}(q, \bar{x}, t) \leq \bar{\Pi}(p, \bar{x}, t)$ for every $p, q \in \mathcal{P}$. Scale-independence often simplifies considerably the analysis of estimator-based supervisory control algorithms [41, 48, 42, 45, 46, 47].

This chapter is organized as follows. Section 4.1 contains the statement and proof of the scale-independent hysteresis switching theorem. This theorem states that under appropriate "open-loop" assumptions, switching will stop at some finite time. In Section 4.2 it is illustrated how the scale-independence property can be used to apply the scale-independent hysteresis switching theorem to some systems for which the "open-loop" assumptions stated in Section 4.1 might be violated.

4.1 Scale-Independent Hysteresis Switching Theorem

Let $\mathcal{X}_0$ denote a given subset of $\mathcal{X}$, and $\mathcal{S}$ the class of all piecewise-constant functions $s : [0, \infty) \rightarrow \mathcal{P}$. In what follows, for each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$, $T_{\{x_0, s\}}$ is the length of the maximal interval of existence of solution to the equations

$$\dot{x} = f_{s(t)}(x, t), \qquad x(0) = x_0$$

and $x_{\{x_0, s\}}$ is the corresponding solution. The following "open-loop" assumptions are made:

Assumption 4.1 (Open-Loop). For each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$ the following is true:

1. There exists a positive constant ϵ such that for each $p \in \mathcal{P}$, the performance signal $\pi_p := \Pi(p, x_{\{x_0,s\}}, t)$ is bounded below on $[0, T_{\{x_0,s\}})$ by ϵ.

2. For each $p \in \mathcal{P}$, the performance signal $\pi_p := \Pi(p, x_{\{x_0,s\}}, t)$ has a limit (which may be infinite) as $t \to T_{\{x_0,s\}}$.

3. There exists at least one $p^* \in \mathcal{P}$ such that the performance signal $\pi_{p^*} := \Pi(p^*, x_{\{x_0,s\}}, t)$ is bounded on $[0, T_{\{x_0,s\}})$.

First note that, because of the definition of $\mathbb{S}_{\mathbb{H}}$, $\sigma(0)$ must be such that $\pi_{\sigma(0)}(0)$ is strictly smaller than $(1+h)\pi_p(0)$ for every $p \in \mathcal{P}$. Thus, because of the Lipschitz continuity of the f_p and the continuity of Π, there must exist an interval $[0, t_1)$ of maximal length on which $\pi_{\sigma(0)}$ remains strictly smaller than $(1+h)\pi_p$ for every $p \in \mathcal{P}$. This means that σ is constant on $[0, t_1)$. Either this interval is the maximal interval of existence for x or it is not, in which case x is bounded on $[0, t_1)$ (cf. [108]). If the latter is true, a switch must occur at t_1 and therefore $\pi_{\sigma(t_1)}(t_1) \leq \pi_p(t_1)$ for every $p \in \mathcal{P}$. Again, because of the Lipschitz continuity of the f_p, the continuity of Π, and the assumed boundedness below of each $h\pi_p$ by a positive constant, there must exist an interval $[t_1, t_2)$ of maximal length on which $\pi_{\sigma(t_1)}$ remains strictly smaller than $(1+h)\pi_p$ for every $p \in \mathcal{P}$. σ will then be constant on such an interval. Continuing this reasoning one concludes that there must be an interval $[0, T)$ of maximal length on which there is a unique pair $\{x, \sigma\}$ with x continuous and σ piecewise constant, which satisfies (4.1) with σ generated by $\mathbb{S}_{\mathbb{H}}$. Moreover, on each strictly proper subinterval $[0, \tau) \subset [0, T)$, σ can switch at most a finite number of times.

To establish existence of solution to (4.1) with σ generated by $\mathbb{S}_{\mathbb{H}}$, only the first Open-Loop Assumption was used. The remaining assumptions enable us to draw conclusions regarding the limiting behavior of σ as $t \to T$. The following is the main result of this chapter.

Theorem 4.2 (Scale-Independent Hysteresis Switching). *Let $\mathcal{P}$ be a finite set, assume that the Open-Loop Assumptions 4.1 hold and, for fixed initial state $\{x_0, \sigma_0\} \in \mathcal{X}_0 \times \mathcal{P}$, let $\{x, \sigma\}$ denote the unique solution to* (4.1) *with σ generated by $\mathbb{S}_{\mathbb{H}}$ with input x. If $[0, T)$ is the largest interval on which this solution is defined, there is a time $T^* < T$ beyond which σ is constant and $\pi_{\sigma(T^*)} := \Pi(\sigma(T^*), x, t)$ is bounded on $[0, T)$.*

Proof of Theorem 4.2. Let $\{x, \sigma\}$ denote the unique solution to (4.1) with σ generated by $\mathbb{S}_{\mathbb{H}}$ and suppose that $[0, T)$ is the largest interval on which this solution is defined. Also, let ϵ be a positive constant that bounds below each performance signal π_p, $p \in \mathcal{P}$. For each $t \in [0, T)$, the switching logic guarantees that

$$\pi_{\sigma(t)}(t) < (1+h)\pi_q(t) \tag{4.3}$$

with $\pi_q(t) \leq \pi_p(t)$ for every $p \in \mathcal{P}$. This is because if at some time t (4.3) were violated, σ would have to switch (at that precise instant of time) to that value q in $\mathcal{P}$ that minimizes $\pi_q(t)$ and therefore (4.3) would still be valid.

Because of the third Open-Loop Assumption 4.1, there is some $p^* \in \mathcal{P}$ such that $\pi_{p^*}(t)$ is bounded on $[0, T)$ by some positive constant K. From this and (4.3) one must have $\pi_{\sigma(t)}(t) \leq (1+h)K$ on $[0, T)$. Because of this and the monotonicity of the π_p, if a performance signal π_p

becomes larger than $(1+h)K$, σ will not switch to p ever again that. Thus, after some finite time $\bar{T} < T$, $\sigma(t)$ must remain inside the following nonempty set

$$\mathcal{P}^* := \Big\{p \in \mathcal{P} : |\pi_p(\tau)| \le (1+h)K,\ \tau \in [0, T)\Big\} \tag{4.4}$$

To conclude the proof it is then enough to shown that there is a time $T^* \in [\bar{T}, T)$ beyond which σ actually becomes constant. Because of the second Open-Loop Assumption 4.1, for each $p \in \mathcal{P}^*$, $\pi_p(t)$ converges to a finite limit as $t \to \infty$. Thus, for each $p \in \mathcal{P}^*$, there must exist a time $T_p \in [\bar{T}, T)$ such that

$$|\pi_p(t) - \pi_p(\tau)| < \frac{h\epsilon}{2}, \qquad \forall t, \tau \in [T_p, T) \tag{4.5}$$

Because $\mathcal{P}$ is a finite set, $\mathcal{P}^*$ is also a finite set and therefore the set $\{T_p : p \in \mathcal{P}^*\}$ has a maximum element. Let $T_1 \in [\bar{T}, T)$ be such element. If there is no switching in $[T_1, T)$ then one can take $T^* := T_1$ and the proof is finished. Otherwise, let $T_2 \in (T_1, T)$ be the time instant at which the next switching occurs. For a given $t \in [T_2, T)$, let q be that element in $\mathcal{P}$ that minimizes $\pi_q(t)$. Since $\sigma(T_2), q \in \mathcal{P}^*$, $T_1 \ge T_{\sigma(T_2)}$, and $T_1 \ge T_q$, from (4.5) one concludes that

$$\pi_{\sigma(T_2)}(T_2) > \pi_{\sigma(T_2)}(t) - \frac{h\epsilon}{2} \qquad \text{and} \qquad \pi_q(T_2) < \pi_q(t) + \frac{h\epsilon}{2} \tag{4.6}$$

But $\sigma(T_2)$ is the element $p \in \mathcal{P}$ that minimizes $\pi_p(T_2)$, thus $\pi_{\sigma(T_2)}(T_2) \le \pi_q(T_2)$. From this and (4.6) one concludes that

$$\pi_{\sigma(T_2)}(t) - \frac{h\epsilon}{2} < \pi_{\sigma(T_2)}(T_2) \le \pi_q(T_2) < \pi_q(t) + \frac{h\epsilon}{2}$$

and therefore that

$$\pi_{\sigma(T_2)}(t) < \pi_q(t) + h\epsilon \tag{4.7}$$

Moreover, because of the first Open-Loop Assumption 4.1, $\pi_q(t) > \epsilon$. From this and (4.7) one concludes that

$$\pi_{\sigma(T_2)}(t) < (1+h)\pi_q(t)$$

Thus there can be no more switching at any time $t \in [T_2, T)$ and one can take $T^* := T_2$. ∎

4.2 Relaxing the Open-Loop Assumptions

The scale-independence property can be use to somewhat relax the "open-loop" assumptions stated in the previous section. Suppose that the following assumptions hold.

Assumption 4.3 (Relaxed Open-Loop). There exists a positive time-function ϑ such that for each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$ the following is true:

1. There exists a positive constant ϵ such that for each $p \in \mathcal{P}$, the scaled performance signal $\bar{\pi}_p := \vartheta\Pi(p, x_{\{x_0,s\}}, t)$ is bounded below on $[0, T_{\{x_0,s\}})$ by ϵ.

2. For each $p \in \mathcal{P}$, the scaled performance signal $\bar{\pi}_p := \vartheta\Pi(p, x_{\{x_0,s\}}, t)$ has a limit (which may be infinite) as $t \to T_{\{x_0,s\}}$.

3. There exists at least one $p^* \in \mathcal{P}$ such that the scaled performance signal $\bar{\pi}_{p^*} := \vartheta\Pi(p^*, x_{\{x_0,s\}}, t)$ is bounded on $[0, T_{\{x_0,s\}})$.

In light of the scale independence property, $\mathbb{S}_{\mathbb{H}}$'s output σ remains unchanged if its performance function/input signal pair $\{\Pi, x\}$ is replaced by another pair $\{\bar{\Pi}, \bar{x}\}$ satisfying

$$\bar{\Pi}(p, \bar{x}, t) = \vartheta\Pi(p, x, t), \qquad \forall p \in \mathcal{P}$$

Since Assumptions 4.3 guarantee that the Open-Loop Assumptions 4.1 hold for the pair $\{\bar{\Pi}, \bar{x}\}$, we obtain the following corollary of Theorem 4.2.

Corollary 4.4 (Scale-Independent Hysteresis Switching). *Let $\mathcal{P}$ be a finite set, assume that the Relaxed Open-Loop Assumptions 4.3 hold and, for fixed initial state $\{x_0, \sigma_0\} \in \mathcal{X}_0 \times \mathcal{P}$, let $\{x, \sigma\}$ denote the unique solution to* (4.1) *with σ generated by $\mathbb{S}_{\mathbb{H}}$ with input x. If $[0, T)$ is the largest interval on which this solution is defined, there is a time $T^* < T$ beyond which σ is constant and $\bar{\pi}_{\sigma(T^*)} := \vartheta\Pi(\sigma(T^*), x, t)$ is bounded on $[0, T)$.*

It should be emphasized that in supervisory control it is often the case that Assumptions 4.1 might be violated, whereas Assumptions 4.3 can be shown to hold [42, 45, 46, 47]. In fact, Corollary 4.4 is used in Chapter 6 to prove stability of a supervisory control algorithm.

Part II

Supervisory Control of Families of Nonlinear Systems

Chapter 5

Certainty Equivalence Implies Detectability

In an adaptive context "certainty equivalence" is a well known heuristic idea which advocates that, at each instant of time, one should design the feedback control to an imprecisely modeled process on the basis of the current estimate of what the process model is, with the understanding that each such estimate is to be viewed as correct even though it may not be.

On the surface justification for certainty equivalence seems self-evident: if process model estimates converge to the "true" process model, then a certainty equivalence based controller ought to converge to the nonadaptive controller that would have been implemented had there been no process uncertainty. The problem with this justification is that, because of noise and unmodeled dynamics, process model estimates don't typically converge to the true process model—even in those instances where certainty equivalence controls can be shown to perform in a satisfactory manner. A more plausible justification stems from the fact that any (stabilizing) certainty equivalence control causes the familiar interconnection of a controlled process and associated output estimator to be *detectable* through the estimator's output error e_p, for every frozen value of the index or parameter vector p upon which both the estimator and controller dynamics depend. Detectability is key because adaptive controller tuning/switching algorithms are invariably designed to make e_p small—and so with detectability, smallness of e_p ensures smallness of the state of the controlled process and estimator interconnection.

The fact that certainty equivalence implies detectability has been known for some time—this has been shown to be so whenever the process model is linear and the controller and estimator models are also linear for every frozen value of p [64, 109]. In this chapter use is made of recently introduced concepts of input-to-state stability [65] and detectability [34, 66] for nonlinear systems, to explain why the same implication is valid in a more general, nonlinear setting.

Historical Note

Reference [110] attributes the term "certainty equivalence" to H. A. Simon who, in turn, gives the following historical account of the term's origin [111].

> I used the term "certainty equivalence" in my paper "Dynamic Programming under Uncertainty with a Quadratic Criterion Function," *Econometrica* 24:74-81 (1956). There is no indication in the paper as to whether I thought I was coining a new

> term or had borrowed one that I already found in use. Perhaps Theil used it in the reference cited in the bibliography—I don't know.
>
> Precisely what I meant by the phrase is stated in the third paragraph of my paper: "When the criterion function is quadratic, the planning problem for the case of uncertainty can be reduced to the problem for the case of certainty simply by replacing, in the computation of the optimal first-period action, the 'certain' future values of variables by their unconditional expectations. In this sense, the unconditional expected values of these variables may be regarded as a set of sufficient statistics for the entire joint probability distribution, or alternatively, as a set of 'certainty equivalents'." Stated otherwise, under the given assumptions, the higher moments of the distribution function are irrelevant to the optimal decision.
>
> This work was done as a part of a project on decision rules for factory scheduling, reported later in a book co-authored with Holt, Muth and Modigliani, and led to ideas on control theory quite similar to some developed soon thereafter, but I think quite independently, by servomechanism theorists who were trying to create the new optimal control theory in engineering. It soon became clear that the quadratic character of the criterion function was essential to the proof, and that it could not be effectively generalized to other functions (old Gauss knew what he was doing when he introduced least squares!). Dick Bellman and members of his group worked for some time seeking non-trivial generalizations, and found none.
>
> H. A. Simon, February, 1997

This chapter is organized as follows. Section 5.1 introduces the overall control problem addressed in this chapter, namely the stabilization of uncertain process. This section also describes the basic model-based approach to the control of uncertain processes. In Section 5.2 the notion of "input-to-state stability" is reviewed and an appropriate definition of detectability for nonlinear systems is introduced. Section 5.3 describes the basic structure of an estimator-based supervisor. This type of supervisor is based on the idea of certainty equivalence. In Section 5.4 the main result of this paper—Theorem 5.3—is stated and a precise meaning is given to the phrase "certainty equivalence implies detectability." In Section 5.5 an "injected system" is constructed. This injected system does not depend on the process model and is therefore especially useful when one tries to take unmodeled dynamics into account. In Section 5.6 it is explained how to construct state sharing estimators. These estimators are central to estimator-based supervisor. The proof of Theorem 5.3 is finally given in Section 5.7. Section 5.8 contains some concluding remarks. The application of the results in this chapter to the analysis of adaptive systems is deferred to Chapter 6.

Notation: In the sequel, prime denotes transpose and $\|x\|$ denotes the 2-norm of a vector x in a real, finite dimensional space $\mathcal{X}$. The exterior direct sum of two real linear spaces $\mathcal{X}$ and $\mathcal{Y}$, is denoted by $\mathcal{X} \oplus \mathcal{Y}$. $\mathcal{L}^1[t_1, t_2)$ denotes the space of all real, vector-valued functions f on $[t_1, t_2)$ for which $\int_{t_1}^{t_2} \|f(\tau)\| d\tau < \infty$.

5.1 Overall Problem

Let $\mathbb{P}$ denote the model of a process with a control input u, a measured output y, and a piecewise-continuous disturbance/noise input w that cannot be measured. Suppose that u, y, and w take values in real, finite-dimensional vector spaces $\mathcal{U}$, $\mathcal{Y}$, and $\mathcal{W}$, respectively. Assume that $\mathbb{P}$ is an unknown member of some suitably defined family of dynamical systems $\mathcal{F}$ whose elements each have a real, finite-dimensional state space $\mathcal{X}_{\mathbb{P}}$ and a pair of defining equations of the form

$$\dot{x}_{\mathbb{P}} = A_{\mathbb{P}}(x_{\mathbb{P}}, w, u), \qquad y = C_{\mathbb{P}}(x_{\mathbb{P}}, w)$$

where $A_{\mathbb{P}}$ and $C_{\mathbb{P}}$ are at least locally Lipschitz continuous on $\mathcal{X}_{\mathbb{P}} \oplus \mathcal{W} \oplus \mathcal{U}$ and $\mathcal{X}_{\mathbb{P}} \oplus \mathcal{W}$, respectively. Assume, in addition, that $\mathcal{F}$ can be written as

$$\mathcal{F} = \bigcup_{p \in \mathcal{P}} \mathcal{F}_p \tag{5.1}$$

where $\mathcal{P}$ is either a finite set of indices or a compact subset of parameter values within a real, finite-dimensional, normed linear space. Here each $\mathcal{F}_p$ denotes a subfamily consisting of a given *nominal process model* $\mathbb{M}_p$ together with a collection of "perturbed versions" of $\mathbb{M}_p$. Typically one would require the allowable perturbations to be "small" enough so that for each possible process model $\mathbb{P}$ in $\mathcal{F}$, there is a nominal process model within the set $\mathcal{M} := \{\mathbb{M}_p : p \in \mathcal{P}\}$, which is "close" to $\mathbb{P}$ in some suitably defined sense. We shall not explicitly demand that this be so, since the main result of this chapter does not require it.

The overall problem of interest is to devise a feedback control which regulates y about the value 0 and, in addition, causes y to tend to 0 in the event that w tends to zero. The main result of this chapter, Theorem 5.3, applies to this problem and (with minor modification) to more general tracking problems such as set-point control where the references to be tracked are generated by exogenous, input-free, time-invariant dynamical systems. In the event that the class within which $\mathbb{P}$ resides—say $\mathcal{F}_{p^*}$—were known and fixed, the problem might be dealt with using standard nonadaptive techniques. On the other hand, if $\mathbb{P}$ were to change slowly or intermittently from time to time, and the corresponding value of p^* over time were known or could be determined from measured data, then the problem might be approached using gain scheduling with p^* playing the role of a scheduling variable—but if the evolution of p^* could not be determined, then the problem would typically call for an adaptive solution. This chapter is concerned with this situation.

Model-Based Control

Assume that one has chosen a family of off-the-shelf, candidate loop-controllers $\mathcal{C} := \{\mathbb{C}_p : p \in \mathcal{P}\}$, in such a way that for each $p \in \mathcal{P}$, $\mathbb{C}_p$ would "solve" the regulation problem, were $\mathbb{P}$ to be any element of $\mathcal{F}_p$. The idea then is to generate a *switching signal* σ taking values in $\mathcal{P}$, which causes the output y of the process model $\mathbb{P}$ in closed-loop with $\mathbb{C}_\sigma$—as shown in Figure 5.1—to be regulated about zero. We call $\mathbb{C}_\sigma$ a *multi-controller* and we require it to be a dynamical

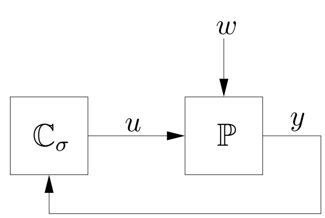

Figure 5.1: Process and Multi-Controller Feedback Loop

system with a real, finite dimensional state space $\mathcal{X}_{\mathbb{C}}$ and defining equations of the form

$$\dot{x}_{\mathbb{C}} = F_\sigma(x_{\mathbb{C}}, y), \qquad u = G_\sigma(x_{\mathbb{C}}, y) \tag{5.2}$$

where for each fixed $p \in \mathcal{P}$, the equations

$$\dot{\bar{x}}_{\mathbb{C}} = F_p(\bar{x}_{\mathbb{C}}, y), \qquad u_p = G_p(\bar{x}_{\mathbb{C}}, y)$$

model $\mathbb{C}_p$ and the F_p and G_p are locally Lipschitz continuous functions on $\mathcal{X}_{\mathbb{C}} \oplus \mathcal{Y}$. Thus we are implicitly assuming that all $\mathbb{C}_p$ have the same finite dimension. This is in fact a very mild assumption, since for $p \in \mathcal{P}$, $\mathbb{C}_p$ is not required to be "observable" through u_p. For example, if one initially had two controllers—say $\mathbb{C}_p$ and $\mathbb{C}_q$—with the dimension of $\mathbb{C}_p$ less than that of $\mathbb{C}_q$, one could increase the dimension of $\mathbb{C}_p$ to match that of $\mathbb{C}_q$ by simply adjoining to $\mathbb{C}_p$ any stable subsystem Σ of dimension $\dim(\mathbb{C}_q) - \dim(\mathbb{C}_p)$, in such a way that Σ has no influence on $\mathbb{C}_p$'s input-output behavior.

Depending on the way in which σ is varied (which might be in either a piecewise-continuous or even a piecewise-constant manner), the algorithm which generates σ might be called either a "tuner" or a "supervisor." In the sequel we use the term supervisor and we restrict σ to be piecewise-constant since this case is somewhat easier to understand. Much of which follows however, is also valid for piecewise-continuously tuned σ.

The types of supervisors that seem to be the most promising are those which utilize "estimators" and base controller selection on certainty equivalence. As mentioned before, certainty equivalence is a heuristic idea which advocates that in an adaptive context, the feedback control applied to an imprecisely modeled process should, at each instant of time, be designed on the basis of a current estimate of what the process is, with the understanding that each such estimate is to be viewed as correct even though it may not be. The heuristic is usually justified by reasoning that if process model estimates converge to the actual process model, then a certainty equivalence based controller ought to converge to the nonadaptive controller which would have been implemented had the correct process model been known in the first place. The problem with this justification is that process model estimates don't typically converge to the actual process model because of noise and unmodeled dynamics. So why then does certainty equivalence prove to be especially useful in an adaptive context? To answer this question we need the notion of "input-to-state stability" and an appropriate definition of detectability for nonlinear systems. These ideas are discussed next.

5.2 Input-to-State Stability and Detectability

Let

$$\dot{x} = A(x, u), \qquad y = C(x, u) \tag{5.3}$$

be a finite dimensional dynamical system whose state, input, and output take values in real, finite dimensional spaces $\mathcal{X}, \mathcal{U}$, and $\mathcal{Y}$, respectively. Suppose that A and C are at least locally Lipschitz continuous on $\mathcal{X} \oplus \mathcal{U}$ and that $\tilde{u}$ is an *equilibrium input* of A; i.e., a fixed vector in $\mathcal{U}$ satisfying $A(\tilde{x}, \tilde{u}) = 0$ for some $\tilde{x} \in \mathcal{X}$. The following definition extends to nonzero equilibrium inputs, the concept of "input-to-state stability" given in [112].

Input-to-State Stability: The system defined by (5.3) (or just the state dynamic $\dot{x} = A(x, u)$, or even just A) is said to be *input-to-state stable about* $\tilde{u}$, if $A(\tilde{x}, \tilde{u}) = 0$ for some state $\tilde{x} \in \mathcal{X}$, and there exist continuous, positive definite, radially unbounded functions $V, X : \mathcal{X} \to [0, \infty)$, $U : \mathcal{U} \to [0, \infty)$ such that V is continuously differentiable, $V(0) = 0$, $X(0) = 0$,

$U(0) = 0$, and

$$\frac{\partial V(x-\tilde{x})}{\partial x} A(x,u) \leq -X(x-\tilde{x}) + U(u-\tilde{u}), \qquad \forall (x,u) \in \mathcal{X} \oplus \mathcal{U} \tag{5.4}$$

Note that for each equilibrium input $\tilde{u} \in \mathcal{U}$ about which (5.3) is input-to-state stable, there can be only one state $\tilde{x} \in \mathcal{X}$ at which $A(\tilde{x},\tilde{u}) = 0$[1]. We call $\tilde{x}$ the *equilibrium state induced by* $\tilde{u}$. In the event that (5.3) is input-to-state stable about each of its equilibrium inputs, we say that (5.3) is input-to-state stable. In the sequel we call any list of functions $\{V,\ X,\ U\}$ with the aforementioned properties, a *stability triple* for (5.3) about $\tilde{u}$. Thus (5.3) is input-to-state stable about $\tilde{u}$ just in case $\tilde{u}$ is an equilibrium input of (5.3) and (5.3) possesses a stability triple about $\tilde{u}$.

As defined, input-to-state stability implies that if u is bounded then so is x and if $u(t) = \tilde{u}$, $t \geq 0$, then $x \to \tilde{x}$ as $t \to \infty$. A more precise statement of these implications, for the case $\tilde{u} = 0$, can be found in [113].

It is possible to define detectability in a number of different ways (See [114] and references therein). An especially useful characterization is in terms of an inequality much like (5.4). A definition along these lines, for systems without inputs, appears in [34]. A generalization of this definition which is particularly well-suited to our application, appears in [66] and for nonzero equilibrium input-output pairs[2] is as follows.

Detectability: The system defined by (5.3) (or just the pair $(C,\ A)$) is said to be *detectable about an input-output pair* $\{\tilde{u},\tilde{y}\} \in \mathcal{U} \oplus \mathcal{Y}$ if $A(\tilde{x},\tilde{u}) = 0$ and $\tilde{y} = C(\tilde{x},\tilde{u})$ for some state $\tilde{x} \in \mathcal{X}$, and there exist continuous, positive definite, radially unbounded functions $V, X : \mathcal{X} \to [0,\infty)$, $U : \mathcal{U} \to [0,\infty)$, $Y : \mathcal{Y} \to [0,\infty)$ such that V is continuously differentiable, $V(0) = 0$, $X(0) = 0$, $U(0) = 0$, $Y(0) = 0$, and

$$\frac{\partial V(x-\tilde{x})}{\partial x} A(x,u) \leq -X(x-\tilde{x}) + Y\big(C(x,u)-\tilde{y}\big) + U(u-\tilde{u}), \quad \forall (x,u) \in \mathcal{X} \oplus \mathcal{U} \tag{5.5}$$

It can be shown that for each equilibrium input-output pair $\{\tilde{u},\ \tilde{y}\}$ about which (5.3) is detectable there is exactly one state $\tilde{x} \in \mathcal{X}$ at which $A(\tilde{x},\tilde{u}) = 0$ and $\tilde{y} = C(\tilde{x},\tilde{u})$. We call $\tilde{x}$ the *equilibrium state* induced by $\{\tilde{u},\ \tilde{y}\}$. In addition, we call any list of functions $\{V,\ X,\ U,\ Y\}$ with the aforementioned properties, a *detectability quadruple* for (5.3) about $\{\tilde{u},\ \tilde{y}\}$. Thus (5.3) is detectable about $\{\tilde{u},\ \tilde{y}\}$ just in case $\{\tilde{u},\ \tilde{y}\}$ is an equilibrium input-output pair and (5.3) possesses a detectability quadruple about $\{\tilde{u},\ \tilde{y}\}$. In case (5.5) holds without the term $U(u-\tilde{u})$, (5.3) is said to be *strongly detectable about the equilibrium output*[3] $\tilde{y}$ and the list of functions $\{V,\ X,\ Y\}$ is called a *strong detectability triple* about $\tilde{y}$. Clearly strong detectability implies detectability. It can be shown that for each equilibrium output $\tilde{y}$ about which (5.3) is strongly detectable there is exactly one state $\tilde{x} \in \mathcal{X}$ at which $A(\tilde{x},\tilde{u}) = 0$ and $\tilde{y} = C(\tilde{x},\tilde{u})$, for some $\tilde{u}$. We call $\tilde{x}$ the *equilibrium state* induced by the equilibrium output $\tilde{y}$.

The preceding definition of detectability reduces to the familiar one in the event that (5.3) is a linear system and $\tilde{u}$ and $\tilde{y}$ are both zero. In addition, the definition proves to capture the

[1] This can be proved by supposing that $\bar{x}$ is another such state and then setting $x := \bar{x}$ and $u := \tilde{u}$ in (5.4). From this it can be seen that $0 \leq -X(\bar{x}-\tilde{x})$ and consequently that $\bar{x} = \tilde{x}$.

[2] A pair $\{\tilde{u},\ \tilde{y}\} \in \mathcal{U} \oplus \mathcal{Y}$ is an *equilibrium input-output pair* of (5.3) if for some state $\tilde{x}$, $A(\tilde{x},\tilde{u}) = 0$ and $\tilde{y} = C(\tilde{x},\tilde{u})$.

[3] A vector $\tilde{y} \in \mathcal{Y}$ is an *equilibrium output* of (5.3) if for some $\tilde{u} \in \mathcal{U}$, $\{\tilde{u},\tilde{y}\}$ is an equilibrium input-output pair of (5.3).

intuitive notion of detectability, namely that if u and y are bounded, then so is x and if u and y converge to $\tilde{u}$ and $\tilde{y}$, respectively, then x converges to $\tilde{x}$. More precisely, we can state the following lemma [114].

Lemma 5.1. *Suppose* (5.3) *is detectable about an equilibrium input-output pair* $\{\tilde{u}, \tilde{y}\}$ *and that* $\tilde{x}$ *is the equilibrium state of* (5.3) *induced by this pair. There exist continuous, positive definite, strictly monotone increasing functions* $\gamma_y : [0,\infty) \to [0,\infty)$, $\gamma_u : [0,\infty) \to [0,\infty)$, *and a continuous, positive definite function* $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ *which is strictly monotone increasing on the first argument for each fixed value of the second, and has* $\lim_{\tau\to\infty} \beta(s,\tau) = 0$ *for each fixed* $s \geq 0$, *such that the following is true. For each initial state* $x(0) \in \mathcal{X}$ *and each piecewise-continuous signal* u,

$$\|x(t) - \tilde{x}\| \leq \beta\big(\|x(t_0) - \tilde{x}\|, t - t_0\big) + \gamma_y\bigg(\sup_{\tau\in[t_0,t]} \|y(\tau) - \tilde{y}\|\bigg) + \gamma_u\bigg(\sup_{\tau\in[t_0,t]} \|u(\tau) - \tilde{u}\|\bigg), \qquad t \geq t_0 \geq 0$$

along the corresponding solution to (5.3).

If $\{V,\ X,\ U\}$ is a stability triple for (5.3) about $\tilde{u}$, and Y is any scalar-valued, locally Lipschitz continuous, positive definite function defined on $\mathcal{Y}$, then $\{V,\ X,\ U,\ Y\}$ is a detectability quadruple for (5.3) about $\{\tilde{u},\ C(\tilde{x}, \tilde{u})\}$ where $\tilde{x}$ is the equilibrium state of (5.3) induced by $\tilde{u}$; thus input-to-state stability about $\tilde{u}$ implies detectability about $\{\tilde{u},\ C(\tilde{x},\ \tilde{u}\}$. Even more is true if the point at which u enters (5.3) is an *injection channel*; i.e., if $\mathcal{U} = \mathcal{Y}$ and there exists a locally Lipschitz continuous function $C^{-1} : \mathcal{X} \oplus \mathcal{Y} \to \mathcal{Y}$ such that $C(x, C^{-1}(x,u)) = u$ for all $u \in \mathcal{U}$ and all $x \in \mathcal{X}$.

Lemma 5.2. *Suppose that the point at which* u *enters* (5.3) *is an injection channel and that* (5.3) *is input-to-state stable about an equilibrium input* $\tilde{u}$. *Then* $\tilde{y} := \tilde{u}$ *is an equilibrium output of the system*

$$\dot{x} = A(x, C(x,u)), \qquad\qquad y = C(x,u) \tag{5.6}$$

and (5.6) *is strongly detectable about* $\tilde{y}$. *In addition, any stability triple for* (5.3) *about* $\tilde{u}$ *is a strong detectability triple for* (5.6) *about* $\tilde{y}$. *Moreover, the equilibrium state of* (5.6) *induced by its equilibrium output* $\tilde{y}$ *is the same as the equilibrium state of* (5.3) *induced by its equilibrium input* $\tilde{u}$.

Proof of Lemma 5.2. Set $\bar{u} = C^{-1}(\tilde{x}, \tilde{u})$ where $\tilde{x}$ is the equilibrium state of (5.3) induced by $\tilde{u}$. Since $\tilde{y} = \tilde{u} = C(\tilde{x}, \bar{u})$ and $A(\tilde{x}, C(\tilde{x}, \bar{u})) = A(\tilde{x}, \tilde{u}) = 0$, $\tilde{y}$ must be an equilibrium output of (5.6). From this and the fact that $A(x, C(x,u)) = A(x,y)$, it follows that any stability triple for (5.3) about $\tilde{u}$ must be a strong detectability triple for (5.6) about $\tilde{y}$. Thus (5.6) must be strongly detectable about $\tilde{y}$ and $\tilde{x}$ must be the equilibrium state of (5.6) induced by $\tilde{y}$. ■

5.3 Estimator-Based Supervisor

An estimator-based supervisor consists of three subsystems, a "multi-estimator" $\mathbb{E}$, a performance signal generator $\mathbb{PS}$, and a switching logic $\mathbb{S}$ (cf. Figure 5.2).

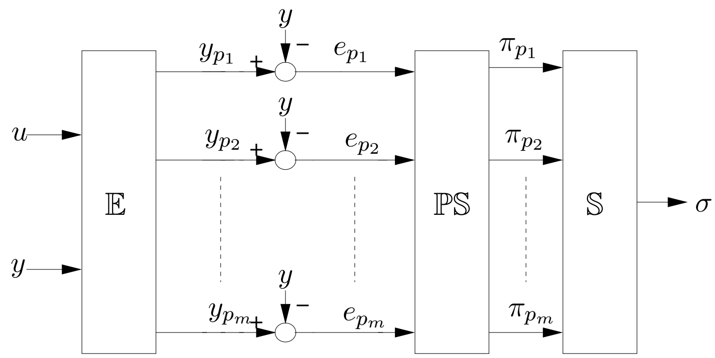

Figure 5.2: Estimator-Based Supervisor

A multi-estimator is a parallel implementation of a family of "estimators," one for each $\mathbb{M}_p \in \mathcal{M}$. By a *estimator* $\mathbb{E}_{\mathbb{M}_p}$ for a given nominal process model $\mathbb{M}_p$, is meant any finite-dimensional, input-to-state stable dynamical system whose input is the pair $\{u,\ y\}$ and whose output is a signal y_p which would be an asymptotically correct estimate of y, if $\mathbb{M}_p$ were the actual process model and there were no measurement noise or disturbances. For $\mathbb{E}_{\mathbb{M}_p}$ to have this property, it would have to exhibit (under the feedback interconnection $y := y_p$ and an appropriate initialization) the same input-output behavior between u and y_p as $\mathbb{M}_p$ does between its input and output[4]. For linear systems such estimators would typically be observers or identifiers [34]. Estimators can also be defined quite easily for certain types of nonlinear systems including those which are linearizable by output injection; in this category is any system whose state and measured output is one and the same.

By a *multi-estimator* $\mathbb{E}$ for a given family of nominal process models $\mathcal{M} = \{\mathbb{M}_p : p \in \mathcal{P}\}$ is meant a finite dimensional, input-to-state stable system of the form

$$\dot{x}_{\mathbb{E}} = A_{\mathbb{E}}(x_{\mathbb{E}}, u, y), \qquad\qquad y_p = C_p(x_{\mathbb{E}}), \quad p \in \mathcal{P} \tag{5.7}$$

where, for each fixed $p \in \mathcal{P}$,

$$\dot{x}_{\mathbb{E}} = A_{\mathbb{E}}(x_{\mathbb{E}}, u, y), \qquad\qquad y_p = C_p(x_{\mathbb{E}}) \tag{5.8}$$

is an estimator for $\mathbb{M}_p$. $\mathbb{E}$ is thus a *parallel implementation* of a family of estimators defined by (5.8) for each $p \in \mathcal{P}$. In the sequel we write $\mathbb{E}_p$ for the pth such estimator and $\mathcal{X}_{\mathbb{E}}$ for the state space of $\mathbb{E}$. We require $A_{\mathbb{E}}$ and each C_p to be locally Lipschitz continuous on $\mathcal{X}_{\mathbb{E}} \oplus \mathcal{U} \oplus \mathcal{Y}$ and $\mathcal{X}_{\mathbb{E}}$, respectively[5].

Note that all of the estimators implemented by $\mathbb{E}$, *share* the same state $x_{\mathbb{E}}$. The problem of constructing a family of state-sharing estimators $\{\mathbb{E}_p : p \in \mathcal{P}\}$ from a given family of non state-sharing estimators $\{\mathbb{E}_{\mathbb{M}_p} : p \in \mathcal{P}\}$ proves to be quite easy if $\mathcal{P}$ is a finite set. If $\mathcal{P}$ is not finite, such a construction can still be carried out under certain conditions. This point is discussed further in Section 5.6.

[4]The main reason for not simply defining $\mathbb{E}_{\mathbb{M}_p} := \mathbb{M}_p$ is the requirement that $\mathbb{E}_{\mathbb{M}_p}$ be input-to-state stable. To require input-to-state stability of $\mathbb{M}_p$ would rule out open-loop unstable nominal process models which is too restrictive.

[5]The results which follow readily generalize to the case when the readout maps in (5.8) depend on y (i.e., when $y_p = C_p(x_{\mathbb{E}}, y)$) provided, for each such C_p, there is a map $\bar{C}_p : \mathcal{X}_{\mathbb{E}} \rightarrow \mathcal{Y}$ such that $\bar{C}_p(x_{\mathbb{E}}) = C(x_{\mathbb{E}},\ \bar{C}_p(x_{\mathbb{E}}))$ for all $x_{\mathbb{E}} \in \mathcal{X}_{\mathbb{E}}$. Readout maps depending on y in this manner are sometimes useful in multi-output representations [115].

A *performance signal generator* $\mathbb{PS}$ is a dynamical system whose inputs are *output estimation errors* $e_p := y_p - y$, $p \in \mathcal{P}$ and whose outputs are *performance signals* π_p, $p \in \mathcal{P}$. For each $p \in \mathcal{P}$, π_p is intended to be a suitably defined measure of the size of the e_p.

The third subsystem of an estimator-based supervisor is a *switching logic* $\mathbb{S}$. The role of $\mathbb{S}$ is to generate σ. Although there are many different ways to define $\mathbb{S}$, in each case the underlying strategy for generating σ is more or less that same: From time to time set σ equal to that value of $p \in \mathcal{P}$ for which π_p is the smallest[6]. The motivation for this idea is obvious: the nominal process model whose associated performance signal is the smallest, "best" approximates what the process is and thus the candidate controller designed on the basis of that model ought to be able to do the best job of controlling the process. The origin of this idea is the concept of certainty equivalence which we've discussed above.

5.4 The Implication of Certainty Equivalence

To understand what certainty equivalence actually implies, let us assume that for each $p \in \mathcal{P}$, $\mathbb{C}_p$ has been chosen so that the system shown in Figure 5.3 has $\tilde{v} := 0$ as an equilibrium input and that about this input the system is at least input-to-state stable[7]. Suppose in addition, that $\bar{y}_p = 0$ at the equilibrium state of this system induced by $\tilde{v}$.

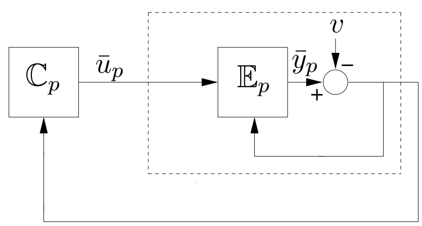

Figure 5.3: Feedback Interconnection

By this we mean that for each $p \in \mathcal{P}$, the interconnected system

$$\left.\begin{array}{ll} \dot{\bar{x}}_{\mathbb{E}} = A_{\mathbb{E}}(\bar{x}_{\mathbb{E}}, \bar{u}_p, \bar{y}_p - v) & \qquad \bar{y}_p = C_p(\bar{x}_{\mathbb{E}}) \\ \dot{\bar{x}}_{\mathbb{C}} = F_p(\bar{x}_{\mathbb{C}}, \bar{y}_p - v) & \qquad \bar{u}_p = G_p(\bar{x}_{\mathbb{C}}, \bar{y}_p - v) \end{array}\right\} \tag{5.9}$$

with input v, is input-to-state stable about the equilibrium input $\tilde{v} = 0$, and moreover that at the equilibrium state induced by this input, $\bar{y}_p = 0$. Justification for placing these requirements[8] on $\mathbb{C}_p$ stems first from the fact that the subsystem enclosed within the dashed box in Figure 5.3 is input-output equivalent to $\mathbb{M}_p$ if $v \equiv 0$ and second from the control objective which is to regulate y about zero. Here v is to be regarded as a fictitious perturbing signal. Later v is identified with e_p.

For each $p \in \mathcal{P}$, let Σ_p then denote the system shown in Figure 5.4 consisting of the interconnection of the process model $\mathbb{P}$, the estimator $\mathbb{E}_p$, and the controller $\mathbb{C}_\sigma$ with σ held fixed at p.

[6]Were we talking about a conventional adaptive control, $\mathcal{P}$ would typically be a continuum, there would be no performance signals, and in place of $\mathbb{S}$ there would be a tuning algorithm $\mathbb{T}$ driven by e_σ, which continuously tunes σ according to some parameter adjustment rule.

[7]In practice, one would demand much more of the $\mathbb{C}_p$. For example, for each $p \in \mathcal{P}$ one would require the feedback connection of $\mathbb{C}_p$ with each model in $\mathcal{F}_p$ to be input-to-state stable with respect to w, about an equilibrium state at which the model's output is zero. The results which follow do not depend on this requirement.

[8]These requirements are implicit in most standard adaptive algorithms.

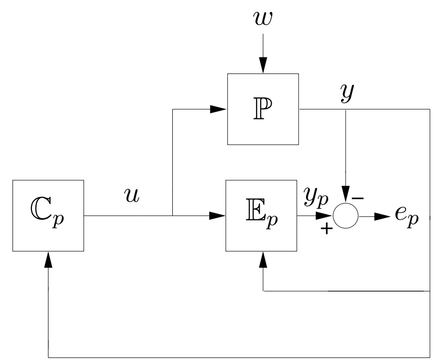

Figure 5.4: Σ_p

Thus Σ_p is a dynamical system with input w and output

$$e_p = y_p - y \tag{5.10}$$

defined by the process model

$$\dot{x}_{\mathbb{P}} = A_{\mathbb{P}}(x_{\mathbb{P}}, w, u), \qquad y = C_{\mathbb{P}}(x_{\mathbb{P}}, w) \tag{5.11}$$

together with the pth estimator-controller equations

$$\left.\begin{array}{ll} \dot{x}_{\mathbb{E}} = A_{\mathbb{E}}(x_{\mathbb{E}}, u, y) & y_p = C_p(x_{\mathbb{E}}) \\ \dot{x}_{\mathbb{C}} = F_p(x_{\mathbb{C}}, y) & u = G_p(x_{\mathbb{C}}, y) \end{array}\right\} \tag{5.12}$$

The main result of this chapter is as follows.

Theorem 5.3 (Certainty Equivalence Stabilization). *Let $\mathbb{P}$ be a process model in $\mathcal{F}$ that is detectable about each of its equilibrium input-output pairs. Let $p \in \mathcal{P}$ be fixed. The system Σ_p shown in Figure 5.4 and defined by* (5.10)–(5.12) *is detectable about any equilibrium input-output pair $\{\tilde{w},\ \tilde{e}_p\}$ it has, for which $\tilde{e}_p = 0$.*

The proof of Theorem 5.3 is deferred to Section 5.7.

The implication of Theorem 5.3 is clear. For each process model $\mathbb{P} \in \mathcal{F}$ which is detectable about all of its equilibrium inputs, and for each $p \in \mathcal{P}$, the input-to-state stabilization of the system in Figure 5.3 by $\mathbb{C}_p$ causes the system Σ_p shown in Figure 5.4 to be detectable about each equilibrium input-output pair $\{\tilde{w},\ \tilde{e}_p\}$ it has (if any), for which $\tilde{e}_p = 0$. This is what is meant by the phrase *certainty equivalence implies detectability.*

With the preceding in mind, recall the underlying decision making strategy of an estimator-based supervisor: From time to time select for σ, that value $q \in \mathcal{P}$ such that the performance signal π_q is the smallest among the π_p, $p \in \mathcal{P}$. Justification for this strategy is now clear: By choosing σ to maintain smallness of π_σ and consequently e_σ, the supervisor is also maintaining smallness of the composite state of the interconnection of $\mathbb{P}$, $\mathbb{C}_\sigma$, and $\mathbb{E}$ because of detectability through e_σ for each fixed value of σ.

5.5 Injected Systems

In this section we digress to define and briefly discuss a useful family of systems which do not depend on $\mathbb{P}$ and which turn out to be detectable because of certainty equivalence. In particular, for each pair $p, q \in \mathcal{P}$, let

$$\dot{\bar{x}} = A_{pq}(\bar{x}, v), \qquad\qquad e_{pq} = C_{pq}(\bar{x}) + v \tag{5.13}$$

abbreviate the system with input v, state $\bar{x} := \begin{bmatrix} \bar{x}'_{\mathbb{E}} & \bar{x}'_{\mathbb{C}} \end{bmatrix}'$ and output $e_{pq} = \bar{y}_p - \bar{y}_q + v$ defined by the equations

$$\left.\begin{aligned} \dot{\bar{x}}_{\mathbb{E}} &= A_{\mathbb{E}}(\bar{x}_{\mathbb{E}}, \bar{u}_p, \bar{y}_q - v) & \qquad \bar{y}_l &= C_l(\bar{x}_{\mathbb{E}}),\ l \in \{p, q\} \\ \dot{\bar{x}}_{\mathbb{C}} &= F_p(\bar{x}_{\mathbb{C}}, \bar{y}_q - v) & \qquad \bar{u}_p &= G_p(\bar{x}_{\mathbb{C}}, \bar{y}_q - v) \end{aligned}\right\} \tag{5.14}$$

Note that if $p = q$, the system defined by the differential equation in (5.14), namely $\dot{x} = A_{pp}(\bar{x}, v)$, is the same as the system defined by (5.9). By assumption, the latter is input-to-state stable about $\tilde{v} = 0$ and, in addition, $\bar{y}_p = 0$ at the equilibrium state induced by this equilibrium input. Moreover, if $C(x, v) := C_{pq}(x) + v$, then $C^{-1}(x, v) = v - C_{pq}(x)$, so the point at which v enters (5.9) is an injection channel. It follows from Lemma 5.2 that the system

$$\dot{\bar{x}} = A_{pp}(\bar{x}, C_{pq}(\bar{x}) + v), \qquad\qquad e_{pq} = C_{pq}(\bar{x}) + v \tag{5.15}$$

is strongly detectable about the equilibrium output $\tilde{e}_{pq} := 0$, that the equilibrium state of (5.15) induced by this equilibrium output is the same as the equilibrium state of (5.9) induced by $\tilde{v}$, and that any stability triple for (5.9) about $\tilde{v}$ is a strong detectability triple for (5.15) about $\tilde{e}_{pq}$. Thus for any p and q in $\mathcal{P}$, (5.15) is strongly detectable about $\tilde{e}_{pq} = 0$, and $\bar{y}_p = 0$ at the equilibrium state of (5.15) induced by this output. Since for any fixed $p, q \in \mathcal{P}$, the definition of A_{pq} implies that $A_{pq}(\bar{x}, v) = A_{pp}(\bar{x}, C_{pq}(\bar{x}) + v)$, we can therefore conclude the following:

Theorem 5.4. *For each $p, q \in \mathcal{P}$, the system described by* (5.13)*, with input v and output e_{pq}, is strongly detectable about the equilibrium output $\tilde{e}_{pq} := 0$. In addition, any stability triple about $\tilde{v} = 0$ for the input-to-state stable system defined by* (5.14) *with $q := p$, is a strong detectability triple about $\tilde{e}_{pq} = 0$ for* (5.13) *for all $q \in \mathcal{P}$. Moreover, $\bar{y}_p = 0$ at the equilibrium state induced by $\tilde{e}_{pq}$.*

In analyzing adaptive and supervisory control systems, it is often convenient to focus attention on subsystems of the form

$$\left.\begin{aligned} \dot{x}_{\mathbb{E}} &= A_{\mathbb{E}}(x_{\mathbb{E}}, u, y) & \qquad e_l &= C_l(x_{\mathbb{E}}) - y,\ l \in \{p, q\} \\ \dot{x}_{\mathbb{C}} &= F_p(x_{\mathbb{C}}, y) & \qquad u &= G_p(x_{\mathbb{C}}, y) \end{aligned}\right\} \tag{5.16}$$

where q is typically the index p^* of the subfamily $\mathcal{F}_{p^*}$ within which $\mathbb{P}$ resides [41]. By the *pqth injected system* is meant the system which results when the equation $y = C_q(x_{\mathbb{E}}) - e_q$ from (5.16) is used to eliminate y from $A_{\mathbb{E}}(\cdot), F_p(\cdot)$ and $G_p(\cdot)$ in (5.16). Once this is done, the pqth injected system can be written as

$$\dot{x} = A_{pq}(x, e_q), \qquad\qquad e_p = C_{pq}(x) + e_q \tag{5.17}$$

where $x = \begin{bmatrix} x'_{\mathbb{E}} & x'_{\mathbb{C}} \end{bmatrix}'$. We have at once the following.

Corollary 5.5. *For each $p, q \in \mathcal{P}$, the pqth injected subsystem* (5.17)*, with input e_q and output e_p, is strongly detectable about the equilibrium output $\tilde{e}_p := 0$. In addition, any stability triple about the equilibrium input $\tilde{v} = 0$ of the input-to-state stable system $\dot{x} = A_{pp}(x, v)$, is a strong detectability triple about $\tilde{e}_p$ for* (5.17)*. Moreover, $y_p = 0$ at the equilibrium state induced by $\tilde{e}_p$.*

The advantage of working with the injected systems $\dot{x} = A_{pp^*}(x, e_{p^*})$, $e_p = C_{pp^*}(x) + e_{p^*}$, $p \in \mathcal{P}$, rather than the overall system comprised of $\mathbb{P}$, $\mathbb{C}_\sigma$ and $\mathbb{E}$, is that the maps defining the former, namely the A_{pp^*} and C_{pp^*}, depend only on the subfamily $\mathcal{F}_{p^*}$ within which $\mathbb{P}$ resides and not on $\mathbb{P}$ itself. This is especially useful when one tries to take unmodeled dynamics into account.

5.6 State Sharing

Let $\{\mathbb{E}_{\mathbb{M}_p} : p \in \mathcal{P}\}$ be a given family of estimators, each of the form

$$\dot{z}_p = D_p(z_p, u, y), \qquad\qquad \bar{y}_p = B_p(z_p) \tag{5.18}$$

where z_p takes values in a real finite-dimensional space $\mathcal{Z}_p$, and D_p and B_p are locally Lipschitz continuous on $\mathcal{Z}_p \oplus \mathcal{U} \oplus \mathcal{Y}$ and $\mathcal{Z}_p$, respectively. Our aim is to explain how to construct a family of input-to-state stable, state-sharing estimators $\{\mathbb{E}_p : p \in \mathcal{P}\}$ in such a way that for each $p \in \mathcal{P}$, $\mathbb{E}_p$ and $\mathbb{E}_{\mathbb{M}_p}$ have the same input-output behavior. Each $\mathbb{E}_p$ is thus to be of the form

$$\dot{x}_{\mathbb{E}} = A_{\mathbb{E}}(x_{\mathbb{E}}, u, y), \qquad\qquad y_p = C_p(x_{\mathbb{E}}) \tag{5.19}$$

with shared state space $\mathcal{X}_{\mathbb{E}}$. We require $A_{\mathbb{E}}$ and the C_p to be locally Lipschitz continuous on $\mathcal{X}_{\mathbb{E}} \oplus \mathcal{U} \oplus \mathcal{Y}$ and $\mathcal{X}_{\mathbb{E}}$, respectively.

Suppose first that $\mathcal{P}$ is a finite set. It is then clearly possible to represent all of the estimators given by (5.18) together as a single dynamical system of the form (5.19) where $x_{\mathbb{E}} :=$ column$\{z_p : p \in \mathcal{P}\}$ and $A_{\mathbb{E}}$ and the C_p are locally Lipschitz continuous functions defined by

$$A_{\mathbb{E}}(x_{\mathbb{E}}, u, y) := \text{column}\{D_p(z_p, y, u) : p \in \mathcal{P}\}, \qquad C_p(x_{\mathbb{E}}) := B_p(z_p), \;\; p \in \mathcal{P} \tag{5.20}$$

In fact, for each fixed $p \in \mathcal{P}$, (5.19) has the same input-output behavior between $\{u,\ y\}$ and y_p as $\mathbb{E}_{\mathbb{M}_p}$ does and is input-to-state stable. Indeed, as a stability triple for (5.19) about any given equilibrium input $\{\tilde{u},\ \tilde{y}\}$, one could choose

$$\left\{ \sum_{p\in\mathcal{P}} V_p(z_p),\ \sum_{p\in\mathcal{P}} X_p(z_p),\ \sum_{p\in\mathcal{P}} U_p(y,u) \right\}$$

where for each $p \in \mathcal{P}$, $\{V_p,\ X_p,\ U_p\}$ is a stability triple for $\mathbb{E}_{\mathbb{M}_p}$ about $\{\tilde{u},\ \tilde{y}\}$. In summary, *any* finite family of (parallel) estimators (sometimes called "multiple models") can be viewed as a single multi-estimator whose state is the composite of the states of the constitute estimators in the family.

If $\mathcal{P}$ is not a finite set and we were to define the $\mathbb{E}_p$ as above, then the $\mathbb{E}_p$'s shared state space would clearly not be finite dimensional. It turns out that it is possible to construct finite-dimensional, state-sharing estimators, for a given family $\{\mathbb{E}_{\mathbb{M}_p} : p \in \mathcal{P}\}$—even when $\mathcal{P}$ is not finite—provided all the $\mathbb{E}_{\mathbb{M}_p}$ are of the same dimension n and the functions D_p appearing in (5.18) are "affinely separable". The D_p are *affinely separable* if there is a constant $n \times n$ stability matrix D, a positive integer m, a locally Lipschitz continuous function $R : \mathcal{U} \oplus \mathcal{Y} \rightarrow \mathbb{R}^{n\times m}$, and a vector-valued function $p \longmapsto k_p$ from $\mathcal{P}$ to $\mathbb{R}^m$ such that

$$D_p(z_p, u, y) = Dz_p + R(u, y)k_p$$

Under these conditions the multi-estimator maps appearing in (5.19) might take the form

$$A_{\mathbb{E}}(x_{\mathbb{E}}, u, y) := \bar{D}x_{\mathbb{E}} + \text{stack}\{R(u, y)\}, \qquad C_p(x_{\mathbb{E}}) := B_p(K_p x_{\mathbb{E}}), \;\; p \in \mathcal{P} \tag{5.21}$$

where $x_{\mathbb{E}} \in \mathbb{R}^{nm}$, $\bar{D} :=$ block diagonal$\{\overbrace{D, \ldots, D}^{m \text{ times}}\}$, $K_p := \begin{bmatrix} k_{p1}I_{n\times n} & k_{p2}I_{n\times n} & \cdots & k_{pm}I_{n\times n}\end{bmatrix}$, k_{pi} is the ith element of k_p, and stack$\{R(u, y)\}$ is the nm-vector which results when the m columns of $R(u, y)$ are stack one atop the next. One can show that, with $A_{\mathbb{E}}$ so defined,

$$z_p(t) - K_p x_{\mathbb{E}}(t) = e^{Dt}(z_p(0) - K_p x_{\mathbb{E}}(0)), \qquad t \geq 0, \quad p \in \mathcal{P},$$

for each input $\{u,\ y\}$. This, the definitions of the C_p, and the Lipschitz continuity of the B_p can then be used to verify that $\bar{y}_p(t) - y_p(t) \to 0$, $p \in \mathcal{P}$, as $t \to \infty$, for each input $\{u,\ y\}$. In this sense, (5.19) has the same input-output behavior between $\{u,\ y\}$ and y_p as $\mathbb{E}_{\mathbb{M}_p}$ does. Moreover, (5.19) is input-to-state stable. Indeed, as a stability triple about $\{\tilde{u},\ \tilde{y}\}$[9], one could use $\{x_{\mathbb{E}}'Px_{\mathbb{E}},\ x_{\mathbb{E}}'x_{\mathbb{E}},\ \mathfrak{U}(\|\{u,\ y\}\|)\}$ where P is the unique positive definite solution to $P\bar{D}' + \bar{D}P + 2I = 0$,

$$\mathfrak{U}(s) = s + \lambda^2 \sup_{\|\{\bar{u},\ \bar{y}\}\| \leq s} \|\, \text{stack}\{R(\bar{u} + \tilde{u}, \bar{y} + \tilde{y}) - R(\tilde{u}, \tilde{y})\}\|^2$$

and λ is the largest eigenvalue of P.

The preceding constructions are given for illustrative purposes only. In many cases one can exploit the detailed structure of the $\mathbb{E}_{\mathbb{M}_p}$ (e.g., linearity) to obtain more appealing (e.g., lower dimensional) $\mathbb{E}_p$.

5.7 Proof of the Certainty Equivalence Stabilization Theorem

Let $\dot{x} = A(x, u)$, $y = C(x, u)$ be a finite dimensional dynamical system whose state, input and output take values in real, finite dimensional spaces $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$, respectively—and suppose that A and C are at least locally Lipschitz continuous on $\mathcal{X} \oplus \mathcal{U}$. To prove Theorem 5.3 we use the following generalizations of concepts and results from [116].

By a *supply pair* for A on a nonempty subset $\mathcal{S} \subset \mathcal{X} \oplus \mathcal{U}$ about an equilibrium input $\tilde{u}$, is meant any ordered pair of scalar-valued, positive definite, continuous, unbounded, strictly increasing functions $\mathfrak{X}$ and $\mathfrak{U}$ on $[0, \infty)$, for which there is a continuously differentiable, positive definite, radially unbounded (storage) function V such that

$$\frac{\partial V(x - \tilde{x})}{\partial x} A(x, u) \leq -\mathfrak{X}(\|x - \tilde{x}\|) + \mathfrak{U}(\|u - \tilde{u}\|), \qquad \forall (x, u) \in \mathcal{S}$$

Here $\tilde{x}$ is any state such that $A(\tilde{x}, \tilde{u}) = 0$. By a *supply triple* for the pair (C, A) about an equilibrium input-output pair $\{\tilde{u}, \tilde{y}\}$ is meant any ordered triple of scalar-valued, positive definite, continuous, unbounded, strictly increasing functions $\mathfrak{X}$, $\mathfrak{U}$, and $\mathfrak{Y}$ on $[0, \infty)$, for which there is a continuously differentiable, positive definite, radially unbounded (storage) function V such that

$$\frac{\partial V(x - \tilde{x})}{\partial x} A(x, u) \leq -\mathfrak{X}(\|x - \tilde{x}\|) + \mathfrak{U}(\|u - \tilde{u}\|) + \mathfrak{Y}(\|C(x, u) - \tilde{y}\|), \quad \forall (x, u) \in \mathcal{X} \oplus \mathcal{U}$$

[9] Because D is nonsingular, every $\{\tilde{u},\ \tilde{y}\} \in \mathcal{U} \oplus \mathcal{Y}$ is an equilibrium input of each $\mathbb{E}_{\mathbb{M}_p}$.

Here $\tilde{x}$ is any vector such that $A(\tilde{x},\tilde{u})=0$ and $C(\tilde{x},\tilde{u})=\tilde{y}$. Like supply pairs, supply triples can also be defined on subsets $\mathcal{S}\subset\mathcal{X}\oplus\mathcal{U}$.

Suppose that $\tilde{x}\in\mathcal{X}$, $\tilde{u}\in\mathcal{U}$ and $\tilde{y}\in\mathcal{Y}$ are vectors such that $A(\tilde{x},\tilde{u})=0$ and $C(\tilde{x},\tilde{u})=\tilde{y}$. Then, if $\{\mathfrak{X},\ \mathfrak{U}\}$ is a supply pair for A on $\mathcal{X}\oplus\mathcal{U}$ about $\tilde{u}$, and V is a corresponding storage function, then $\{V,\ \mathfrak{X}(||\cdot||),\ \mathfrak{U}(||\cdot||)\}$ is a stability triple for A about $\tilde{u}$. Conversely, if $\{V,\ X,\ U\}$ is a stability triple for A about $\tilde{u}$, and we define

$$\mathfrak{X}(r):=\frac{r}{r+1}\inf_{||x||\geq r}X(x)\text{ and }\mathfrak{U}(r):=r+\sup_{||u||\leq r}U(u)$$

for $r\geq 0$, then $\{\mathfrak{X},\ \mathfrak{U}\}$ is a supply pair for A on $\mathcal{X}\oplus\mathcal{U}$ about $\tilde{u}$. In other words, A is input-to-state stable about $\tilde{u}$ just in case it possesses a supply pair on $\mathcal{X}\oplus\mathcal{U}$ about $\tilde{u}$. By similar reasoning, (C,A) is detectable about $\{\tilde{u},\ \tilde{y}\}$ if and only if $(C,\ A)$ possesses a supply triple about $\{\tilde{u},\ \tilde{y}\}$. The following result, proved in [116] for the case $\mathcal{S}=\mathcal{U}\oplus\mathcal{X}$, $\bar{\mathcal{S}}=\bar{\mathcal{U}}\oplus\bar{\mathcal{X}}$ enables one to establish the detectability of the cascade connection of an input-to-state stable system with a detectable system.

Lemma 5.6. *Let $A:\mathcal{X}\oplus\mathcal{U}\rightarrow\mathcal{U}$ and $\bar{A}:\bar{\mathcal{X}}\oplus\bar{\mathcal{U}}\rightarrow\bar{\mathcal{U}}$ be maps respectively possessing supply pairs on $\mathcal{S}\subset\mathcal{X}\oplus\mathcal{U}$ about $\tilde{u}$ and on $\bar{\mathcal{S}}\subset\bar{\mathcal{X}}\oplus\bar{\mathcal{U}}$ about $\tilde{\bar{u}}$. There exist functions $\mathfrak{X},\mathfrak{U}$ and $\bar{\mathfrak{U}}$ such that $\{\mathfrak{X},\ \mathfrak{U}\}$ is a supply pair on $\mathcal{S}$ about $\tilde{u}$ for the former and $\{\bar{\mathfrak{X}},\ \bar{\mathfrak{U}}\}$ is a supply pair on $\bar{\mathcal{S}}$ about $\tilde{\bar{u}}$ for the the latter, with $\bar{\mathfrak{X}}(r):=2\mathfrak{U}(3r)$, $r\geq 0$.*

With Lemma 5.6 in hand, one can prove the following.

Lemma 5.7. *Let Σ_1 and Σ_2 be dynamical systems defined by*

$$\dot{x}_1=A_1(x_1,u_1),\qquad\qquad y_1=C_1(x_1,u_1),\tag{5.22}$$

and

$$\dot{x}_2=A_2(x_2,u_2),\qquad\qquad y_2=C_2(x_2,u_2),\tag{5.23}$$

respectively. Suppose that Σ_1 is detectable about $\{\tilde{u}_1,\tilde{y}_1\}$, that Σ_2 is input-to-state stable about $\tilde{u}_2$, that $\mathcal{U}_1=\mathcal{Y}_2$, and that $\tilde{u}_1=C_2(\tilde{x}_2,\tilde{u}_2)$ where $\tilde{x}_2$ is the equilibrium state of Σ_2 induced by $\tilde{u}_2$. Then the cascade interconnection of Σ_1 and Σ_2 determined by setting $u_1=y_2$, is detectable about $\{\tilde{u}_2,\ \tilde{y}_1\}$.

Proof of Lemma 5.7. Set $v:=\{x_2,u_2\}$, $\tilde{v}:=\{\tilde{x}_2,\ \tilde{u}_2\}$, $\bar{A}_1(x_1,v):=A_1(x_1,C_2(x_2,u_2))$, $\bar{C}_1(x_1,v):=C_1(x_1,C_2(x_2,u_2))$, and $\bar{C}_2(v):=C_2(x_2,u_2)$. By assumption $\tilde{u}_1=C_2(\tilde{x}_2,\tilde{u}_2)$ so $\tilde{u}_1=\bar{C}_2(\tilde{v})$.

Since Σ_1 is detectable about $\{\tilde{u}_1,\ \tilde{y}_1\}$, Σ_1 must have a supply triple $\{\mathfrak{X},\ \mathfrak{U},\mathfrak{Y}\}$ about $\{\tilde{u}_1,\ \tilde{y}_1\}$. It follows that if we define

$$\bar{\mathfrak{U}}(r):=r+\sup_{||u||\leq s}\mathfrak{U}(||\bar{C}_2(u+\tilde{v})-\bar{C}_2(\tilde{v})||),\qquad\qquad r\geq 0$$

then $\{\mathfrak{X},\ \bar{\mathfrak{U}},\mathfrak{Y}\}$ must be a supply triple for $(\bar{C}_1,\bar{A}_1)$ about $\{\tilde{v},\ \tilde{y}_1\}$.

Set $w:=\{v,\ \bar{C}_1(x_1,v)\}$, $\tilde{w}:=\{\tilde{v},\ \tilde{y}_1\}$, $A(x_1,w)=\bar{A}_1(x_1,v)$, $\widehat{\mathfrak{U}}:=\bar{\mathfrak{U}}+\mathfrak{Y}$, and

$$\mathcal{S}:=\big\{\{x_1,w\}:w=\{v,\bar{C}_1(x_1,v)\},\ \{x_1,v\}\in\mathcal{X}_1\oplus\mathcal{X}_2\oplus\mathcal{U}_2\big\}$$

Then $\{\mathfrak{X}_1,\ \widehat{\mathfrak{U}}\}$ must be a supply pair for A on $\mathcal{S}$ about $\tilde{w}$. Since A_2 is input-to-state stable about $\tilde{u}_2$, it must have a supply pair on $\mathcal{X}_2 \oplus \mathcal{U}_2$ about $\tilde{u}_2$. Therefore, by Lemma 5.6, there must be functions $\mathfrak{X}_1$, $\mathfrak{U}_1$ and $\mathfrak{U}_2$ such that $\{\mathfrak{X}_1, \mathfrak{U}_1\}$ is a supply pair for A on $\mathcal{S}$ about $\tilde{w}$, and with $\mathfrak{X}_2(r) := 2\mathfrak{U}_1(3r)$, $r \geq 0$, $\{\mathfrak{X}_2,\ \mathfrak{U}_2\}$ is a supply pair for A_2 on $\mathcal{X}_2 \oplus \mathcal{U}_2$ about $\tilde{u}_2$. Thus there must exist storage functions V_1 and V_2 such that

$$\frac{\partial V_1(x_1 - \tilde{x}_1)}{\partial x_1} A(x_1, w) \leq -\mathfrak{X}_1(\|x_1 - \tilde{x}_1\|) + \mathfrak{U}_1(\|w - \tilde{w}\|), \qquad \{x_1, w\} \in \mathcal{S} \tag{5.24}$$

$$\frac{\partial V_2(x_2 - \tilde{x}_2)}{\partial x_2} A_2(x_2, u_2) \leq -2\mathfrak{U}_1(3\|x_2 - \tilde{x}_2\|) + \bar{\mathfrak{U}}_2(\|u_2 - \tilde{u}_2\|), \quad \{x_2, u_2\} \in \mathcal{X}_2 \oplus \mathcal{U}_2 \tag{5.25}$$

But $w = \{v,\ y_1\} = \{x_2, u_2, y_1\}$ so $\mathfrak{U}_1(||w - \tilde{w}||) = \mathfrak{U}_1(||\{x_2 - \tilde{x}_2,\ u_2 - \tilde{u}_2,\ y_1 - \tilde{y}_1\}||) \leq (\mathfrak{U}_1(3||x_2 - \tilde{x}_2||) + \mathfrak{U}_1(3||u_2 - \tilde{u}_2||) + \mathfrak{U}_1(3||y_1 - \tilde{y}_1||)$. Moreover, $A(x_1, w) = A_1(x_1, C_2(x_2, u_2))$. In view of (5.24) we can therefore write

$$\begin{aligned}\frac{\partial V_1(x_1 - \tilde{x}_1)}{\partial x_1} A_1(x_1, C_2(x_2, u_2)) \leq -\mathfrak{X}_1(\|x_1 - \tilde{x}_1\|) + \mathfrak{U}_1(3||x_2 - \tilde{x}_2||) \\ + \mathfrak{U}_1(3||u_2 - \tilde{u}_2||) + \mathfrak{U}_1(3||C_1(x_1, C_2(x_2, u_2)) - \tilde{y}_1||)\end{aligned}$$

for all $(x_1, x_2, u_2) \in \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{U}_2$. In the light of this and (5.25) it is now clear that if we define $V(x_1, x_2) := V_1(x_1) + V_2(x_2)$, $X(x_1, x_2) := \mathfrak{X}_1(||x_1||) + \mathfrak{U}_1(3||x_2||)$, $U(u_2) = \mathfrak{U}_1(3||u_2||) + \bar{\mathfrak{U}}_2(||u_2||)$, and $Y(y_1) = \mathfrak{U}_1(||y_1||)$, then $\{V, X, U, Y\}$ is a detectability quadruple about $\{\tilde{u}_2, \tilde{y}_1\}$ for the cascade connection defined by (5.22)–(5.23). ■

Proof of Theorem 5.3. With $\tilde{e}_p := 0$, let $\{\tilde{w},\ \tilde{e}_p\}$ be an equilibrium input-output pair of the system Σ_p shown in Figure 5.4 and defined by (5.10)–(5.12). Let $\tilde{x}_{\mathbb{P}}$, $\tilde{x}_{\mathbb{E}}$, $\tilde{x}_{\mathbb{C}}$, $\tilde{u}$, $\tilde{y}$, and $\tilde{y}_p$ be such that $A_{\mathbb{P}}(\tilde{x}_{\mathbb{P}}, \tilde{w}, \tilde{u}) = 0$, $A_{\mathbb{E}}(\tilde{x}_{\mathbb{E}}, \tilde{u}, \tilde{y}) = 0$, $F_p(\tilde{x}_{\mathbb{C}}, \tilde{y}) = 0$, $\tilde{y} = C_{\mathbb{P}}(\tilde{x}_{\mathbb{P}}, \tilde{w})$, $\tilde{u} = G_p(\tilde{x}_{\mathbb{C}}, \tilde{y})$, and $\tilde{y}_p = C_p(\tilde{x}_{\mathbb{E}})$. Then $\{\{\tilde{w}, \tilde{u}\}, \tilde{y}\}$ is an equilibrium input-output pair of $\mathbb{P}$. By assumption, $\mathbb{P}$ is detectable about each such pair. Thus there must be a detectability quadruple $\{V, X, Y, U\}$ for $\mathbb{P}$ about $\{\{\tilde{w}, \tilde{u}\}, \tilde{y}\}$.

Let Σ_1 denote the system consisting of the dynamics of $\mathbb{P}$ given by (5.11) and the output estimation error e_p defined by (5.10); i.e., the system

$$\dot{x}_{\mathbb{P}} = A_{\mathbb{P}}(x_{\mathbb{P}}, w, u), \qquad\qquad e_p = y_p - C_{\mathbb{P}}(x_{\mathbb{P}}, w), \tag{5.26}$$

In the sequel we regard Σ_1 as a dynamical system with input $\{w, u, y_p\}$ and output e_p. Since $y_p = C_{\mathbb{P}}(x_{\mathbb{P}}, w)$ when $e_p = 0$, it must be true that $\{\{\tilde{w}, \tilde{u}, \tilde{y}\},\ 0\}$ is an equilibrium input-output pair of Σ_1. Moreover, defining

$$\mathfrak{Y}(r) := \sup_{||y|| \leq r} Y(||y||), \qquad\qquad r \geq 0,\ y \in \mathcal{Y},$$

$\bar{U}(w, u, y_p) := U(w, u) + \mathfrak{Y}(2||y_p||)$, and $\bar{Y}(e_p) := \mathfrak{Y}(2||e_p||)$, then $\{V, X, \bar{U}, \bar{Y}\}$ is a detectability quadruple for Σ_1 about $\{\{\tilde{w}, \tilde{u}, \tilde{y}\},\ 0\}$. Therefore Σ_1 is detectable about this pair.

Using the definition of A_{pq} in (5.13) together with the definition of e_p in (5.10) it is possible to write the estimator-controller subsystem defined by (5.12) as

$$\dot{x} = A_{pp}(x, v) \tag{5.27}$$

where $v = e_p$ and $x := \begin{bmatrix} x'_{\mathbb{E}} & x'_C \end{bmatrix}'$. In the sequel it is convenient to regard (5.27) as a dynamical system Σ_2 with input $\{w, v\}$ and output $\{w, u, y_p\}$ where $y_p = C_p(x_{\mathbb{E}})$; the readout formula for

u, namely $u = G_p(x_{\mathbb{C}}, C_p(x_{\mathbb{E}}) - v)$, is obtained from (5.12) by substituting $C_p(x_{\mathbb{E}}) - e_p$ for y; and w is just a direct feedthrough term. As noted in the discussion just preceding Theorem 5.4, $A_{pp}(x, v)$ is input-to-state stable about the equilibrium input $\tilde{v} = 0$. Thus Σ_2 is input-to-state stable about $\{\tilde{w}, 0\}$.
Let Σ denote the system with input $\{w, v\}$ and output e_p which is defined by the cascade connection of Σ_1 and Σ_2. Since Σ_1 is detectable about $\{\{\tilde{w}, \tilde{u}, \tilde{y}\},\ 0\}$ and Σ_2 is input-to-state stable about $\{\tilde{w}, 0\}$, we can conclude, by Lemma 5.7, that Σ is detectable about $\{\{\tilde{w}, 0\}, 0\}$. In view of the definition of detectability, it is clear that this remains true even if v is set equal to e_p. ∎

5.8 Concluding Remarks

Throughout this chapter we have made use of "linear" estimation errors of the form $e_p = y_p - y$. Given the nonlinear nature of the processes under consideration, it is natural to ask if the results in this chapter would still hold if one were to utilize more general "nonlinear" errors. Towards this end, let us agree to call a locally Lipschitz continuous binary operation $\vee : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$, a *generalized sum* if, for all $y, \bar{y} \in \mathcal{Y}$, $y \vee \bar{y} = 0 \iff y + \bar{y} = 0$ and there is a locally Lipschitz continuous binary operation $\wedge : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ such that $y = -(-y \wedge \bar{y}) \vee \bar{y}$ for all $y, \bar{y} \in \mathcal{Y}$. For example, if $\vee$ were defined so that $y \vee \bar{y} := y^3 + \bar{y}^3$, then the definition of $\wedge$ would have to be $y \wedge \bar{y} := (y + \bar{y}^3)^{\frac{1}{3}}$. If $\vee$ were an associative operation, then $\wedge$ and $\vee$ would be the same.

Now consider a supervisory control system in which for $p \in \mathcal{P}$, e_p is of the form $e_p := -y \vee y_p$ where $\vee$ is a suitably defined generalized sum, possibly depending on p. Suppose that $\wedge$ is a corresponding binary operation such that $y = -(-y \wedge \bar{y}) \vee \bar{y}$ for all $y, \bar{y} \in \mathcal{Y}$. It is a simple matter to prove that if the e_p are so defined, and if the signals coming out of the summing junctions in Figures 5.3 and 5.4 are interpreted as $-v \wedge \bar{y}_p$ and $-y \vee y_p$, respectively, then Theorems 5.3 and 5.4 as well as Corollary 5.5 is still true. Nonlinear estimation errors are used in Section 6.2.3 in the next chapter.

It should be emphasized that the assumptions of Theorem 5.3 are very mild. This Theorem applies to off-the-shelf, nonadaptive controllers such as back-stepping designed controllers [117, 118], as well as to those specifically crafted to adaptive applications. The theorem's validity does *not* require the process model to be exactly matched by one of the nominal models in $\mathcal{M}$. In fact, all Theorem 5.3 requires of the process model is that it be detectable about each of its equilibrium input-output pairs.

Recently several control algorithms motivated by the model validation paradigm [67, 68] have been proposed in the literature. The basic idea consists in starting by hypothesizing an a priori family of admissible process models $\mathcal{F}$—for example as in (5.1)—and then using data collected in real-time to "falsify" those elements of $\mathcal{F}$ that are incompatible with the measured data. In this way, the family $\mathcal{F}(t)$ of processes in which the actual process $\mathbb{P}$ is known to lie at time t, shrinks as time goes by, i.e.,

$$t_2 > t_1 \quad \Rightarrow \quad \mathcal{F}(t_2) \subset \mathcal{F}(t_1), \qquad\qquad \forall t_1, t_2$$

At each instant of time t, the closed-loop controller is then selected on the basis of the current family $\mathcal{F}(t)$ of admissible processes [69, 70, 71]. In [72, 73, 74] a different approach is proposed. In this work, one starts with a family of closed-loop controllers $\mathcal{C}$, at least one of which is guaranteed to achieve some desired level of performance when in feedback with the process $\mathbb{P}$. Data collected in real-time is then used to "falsify" those controllers in $\mathcal{C}$ that are unable to achieve the desired level of performance and, at each instant of time, a controller is selected from those that have not yet been falsified. In the implementations of many of these algorithms, it is possible to recognize structures like the one in Figure 5.4, where e_p typically corresponds to a "model validation error" or to a "performance specification error". This same structure can be found in multi-model $\mathcal{H}_\infty$ optimal control [75, 76]. The detectability property proved in this thesis can be used in the analysis of several of these control algorithms.

Chapter 6

Supervision Under Exact Matching

The main objective of Chapter 5 was to make precise and justify a statement applicable to many adaptive algorithms, namely that certainty equivalence implies detectability. The intent of this chapter is to demonstrate how this implication can be used in analyzing an adaptive system.

In Section 6.1 the analysis of a supervisory control system is carried out in a fairly general setting. The main assumptions are that the disturbance/noise input w is identically zero and that one of the y_p is an asymptotically correct estimate of y. It is also assumed that $\mathcal{P}$ is a finite set. The switching logic used is the scale-independent hysteresis switching logic introduced in Chapter 4. Under the above assumptions, global boundedness and asymptotic convergence are deduced. For illustrative purposes, in Section 6.2, these results are applied to the control of three nonlinear systems.

6.1 Analysis of a Supervisory Control Algorithm

Suppose that $\mathcal{P}$ is a finite set—say $\mathcal{P} := \{1, 2, \dots, m\}$—and that the disturbance/noise input w is identically zero. The equations for $\mathbb{P}$, $\mathbb{E}$, and $\mathbb{C}_\sigma$ are

$$\dot{x}_{\mathbb{P}} = A_{\mathbb{P}}(x_{\mathbb{P}}, 0, u) \qquad y = C_{\mathbb{P}}(x_{\mathbb{P}}), \tag{6.1}$$

$$\dot{x}_{\mathbb{E}} = A_{\mathbb{E}}(x_{\mathbb{E}}, u, y) \qquad e_p = C_p(x_{\mathbb{E}}) - y, \quad p \in \mathcal{P} \tag{6.2}$$

and

$$\dot{x}_{\mathbb{C}} = F_\sigma(x_{\mathbb{C}}, y) \qquad u = G_\sigma(x_{\mathbb{C}}, y), \tag{6.3}$$

respectively, where $A_{\mathbb{P}}$, $A_{\mathbb{E}}$, and $C_{\mathbb{P}}$, as well as the C_p, F_p, and G_p, $p \in \mathcal{P}$, are at least locally Lipschitz continuous functions.

Corollary 5.5 guarantees that for any $p, q \in \mathcal{P}$, the pqth injected system

$$\dot{x} = A_{pq}(x, e_q), \qquad e_p = C_{pq}(x) + e_q \tag{6.4}$$

determined by (6.1)–(6.3) with σ frozen at p and

$$x := \begin{bmatrix} x_{\mathbb{E}} \\ x_{\mathbb{C}} \end{bmatrix}, \tag{6.5}$$

is strongly detectable about the equilibrium output $\tilde{e}_p := 0$ and, in addition, $y_q = 0$ at the equilibrium state induced by this output. For each such p, let $\{V_p,\ X_p,\ Y_p\}$ be a stability triple for $\dot{x} = A_{pp}(x, v)$ about the equilibrium input $\tilde{v} = 0$. Corollary 5.5 also guarantees that $\{V_p,\ X_p,\ Y_p\}$ is a strong detectability triple for (6.4) about its equilibrium output 0.

For the performance signal generator $\mathbb{PS}$ we consider the dynamical system

$$\dot{\pi}_p = -\lambda\pi_p + Y_p(e_p), \qquad p \in \mathcal{P}, \tag{6.6}$$

whose state and outputs are the performance signals $\{\pi_1,\ \pi_2, \dots, \pi_m\}$ and λ is a prespecified positive number. It is assumed that (6.6) is initialized so that

$$\pi_p(0) > 0, \qquad p \in \mathcal{P} \tag{6.7}$$

The interconnected system defined by (6.1)–(6.3) and (6.6) is a dynamical system of the form

$$\dot{z} = f_\sigma(z), \qquad p \in \mathcal{P} \tag{6.8}$$

where $z := \{x_{\mathbb{P}},\ x_{\mathbb{E}},\ x_{\mathbb{C}},\ \pi_1,\ \pi_2, \dots, \pi_m\}$ and each f_p, $p \in \mathcal{P}$, is locally Lipschitz. For $\mathbb{S}$ we consider the scale-independent hysteresis switching logic $\mathbb{S}_{\mathbb{H}}$ described in Chapter 4, with the performance function Π defined by

$$\Pi(p, z) = \pi_p, \qquad p \in \mathcal{P},\ \ z \in \mathcal{X}_{\mathbb{P}} \oplus \mathcal{X}_{\mathbb{E}} \oplus \mathcal{X}_{\mathbb{C}} \oplus \mathbb{R}^m \tag{6.9}$$

Three assumptions are made.

Assumption 6.1. Each process model in $\mathcal{F}$ is detectable about at least one equilibrium input-output pair.

With Assumption 6.1, Lemma 5.1 allows one to conclude that if u and y are bounded then so is the state $x_{\mathbb{P}}$ of the process.

Assumption 6.2. Each Y_p is continuously differentiable.

Assumption 6.3. There exists an index $p^* \in \mathcal{P}$ such that, for each initial state $\{x_{\mathbb{P}}(0),\ x_{\mathbb{E}}(0)\} \in \mathcal{X}_{\mathbb{P}} \oplus \mathcal{X}_{\mathbb{E}}$ and each piecewise-continuous, open-loop control signal u, $\|e_{p^*}\|$ and

$$\int_0^t e^{\lambda\tau}\|e_{p^*}(\tau)\|d\tau$$

are bounded on the interval of maximal length on which a solution to (6.1)–(6.2) exists.

$\mathbb{E}$ can typically be constructed so that Assumption 6.3 is satisfied in the noise/disturbance free case, provided $\mathbb{P}$ is input-output equivalent to a nominal model (say $\mathbb{M}_{p^*}$) which is linearizable by output injection.

The implication of Assumptions 6.2 and 6.3 is that for any piecewise-constant signal $s : [0, \infty) \rightarrow \mathcal{P}$, and any initialization of (6.1)–(6.3) and (6.6),

$$\int_0^{T_s} e^{\lambda t} Y(e_{p^*})dt < \infty \tag{6.10}$$

where $[0, T_s)$ is the interval of maximal length on which there exists a solution to (6.1)–(6.3) and (6.6) when $\sigma := s$. Suppose now that one defines the scaled performance signals

$$\bar{\pi}_p := e^{\lambda t}\pi_p, \qquad p \in \mathcal{P}$$

In view of the definitions of the π_p in (6.6), the $\bar{\pi}_p$ satisfy

$$\bar{\pi}_p = \pi_p(0) + \int_0^t e^{\lambda\tau} Y(e_p) d\tau, \qquad p \in \mathcal{P} \tag{6.11}$$

From this and (6.10) it follows that for any piecewise-constant signal $s : [0, \infty) \to \mathcal{P}$ and any initialization of (6.1)–(6.3) and (6.6) for which (6.7) holds, (i) each $\bar{\pi}_p$ is positive on $[0, T_s)$; (ii) each $\bar{\pi}_p$ has a limit as $t \to T_s$ because it is monotone on $[0, T_s)$; and (iii) $\bar{\pi}_{p^*}$ is bounded on $[0, T_s)$. The fact that the $\bar{\pi}_p$ possess properties (i)-(iii) enable us to exploit Corollary 4.4 of the Scale-Independent Hysteresis Switching Lemma, and consequently to draw the following conclusion.

Lemma 6.4. *For fixed initial states $x_{\mathbb{P}}(0) \in \mathcal{X}_{\mathbb{P}}$, $x_{\mathbb{E}}(0) \in \mathcal{X}_{\mathbb{E}}$, $x_{\mathbb{C}}(0) \in \mathcal{X}_{\mathbb{C}}$, $\pi_p(0) > 0$, $p \in \mathcal{P}$, $\sigma(0) \in \mathcal{P}$, let $\{x_{\mathbb{P}},\ x_{\mathbb{E}},\ x_{\mathbb{C}},\ \pi_1,\ \pi_2, \ldots, \pi_m,\ \sigma\}$ denote the unique solution to (6.1)–(6.3) and (6.6) with σ the output of $\mathbb{S}_{\mathbb{H}}$—and suppose $[0, T)$ is the largest interval on which this solution is defined. There is a time $T^* < T$ beyond which σ is constant and no more switching occurs. In addition, the scaled performance signal $\bar{\pi}_{\sigma(T^*)} := e^{\lambda t}\pi_{\sigma(T^*)}$ is bounded on $[0, T)$.*

Let $x_{\mathbb{P}}$, $x_{\mathbb{E}}$, $x_{\mathbb{C}}$, $\pi_1, \pi_2, \ldots, \pi_m$, σ, T and T^* be as in Lemma 6.4 and set $q^* := \sigma(T^*)$. In view of (6.10), (6.11) and the observation that $e^{\lambda t}\pi_{q^*}$ must be bounded on $[0, T)$,

$$\int_0^T Y_{q^*}(e_{q^*}(\tau)) d\tau < \infty \tag{6.12}$$

Since σ is frozen at q^* for $t \in [T^*, T)$ and $\{V_{q^*},\ X_{q^*},\ Y_{q^*}\}$ is a strong detectability triple for (6.4) (with $p := q^*$) about the equilibrium output $\tilde{e}_{q^*} = 0$, we can write

$$\dot{V}_{q^*}(x - \tilde{x}) \leq -X_{q^*}(x - \tilde{x}) + Y_{q^*}(e_{q^*}), \qquad t \in [T^*, T) \tag{6.13}$$

where $\tilde{x}$ is the equilibrium state of (6.4) (with $p := q^*$) induced by $\tilde{e}_{q^*}$. Therefore, by integrating the preceding, we obtain

$$V_{q^*}(x(t) - \tilde{x}) \leq V_{q^*}(x(t_0) - \tilde{x}) - \int_{t_0}^t X_{q^*}(x(\tau) - \tilde{x}) d\tau + \int_{t_0}^t Y_{q^*}(e_{q^*}(\tau)) d\tau, \quad T^* \leq t_0 \leq t \leq T \tag{6.14}$$

From this and (6.12) it follows that $V_{q^*}(x(t) - \tilde{x}) < \infty$ for $T^* \leq t_0 \leq t < T$. Thus $V_{q^*}(x(t) - \tilde{x})$ is bounded on $[T^*, T)$ and consequently on $[0, T)$. But $V_{q^*}(\cdot)$ is radially unbounded, so x and consequently $x_{\mathbb{E}}$ and $x_{\mathbb{C}}$ (cf., (6.5)) must be bounded on $[0, T)$ as well.

In view of (6.2), $y = e_{p^*} + C_{p^*}(x_{\mathbb{E}})$. By Assumption 6.3, e_{p^*} is bounded on $[0, T)$, so y must also be. Boundedness of u on $[0, T)$ then follows from the formula $u = G_\sigma(x_{\mathbb{C}}, y)$. Therefore $x_{\mathbb{P}}$ is bounded on $[0, T)$ because of Assumption 6.1 and Lemma 5.1. So is each e_p, $p \in \mathcal{P}$, because of the defining formula $e_p = C_p(x_{\mathbb{E}}) - y$. Therefore each π_p will be bounded on $[0, T)$ because the differential equations (6.6) defining the π_p are input-to-state stable systems and the $Y_p(e_p)$ are bounded on $[0, T)$. In other words $x_{\mathbb{P}}$, $x_{\mathbb{E}}$, $x_{\mathbb{C}}$, and the π_p are all bounded on $[0, T)$.

Now if T were finite, the solution to (6.1)–(6.3) and (6.6) could be continued onto at least an open half interval of the form $[T, T_1)$ thereby contradicting the hypothesis that $[0, T)$ is the system's interval of maximal existence. By contradiction one can therefore conclude that $T=\infty$ and that $x_{\mathbb{P}}$, $x_{\mathbb{E}}$, $x_{\mathbb{C}}$, and the π_p are bounded on $[0,\infty)$.

It is shown next that x converges to $\tilde{x}$ as $t\to\infty$. Define

$$\bar{V}(t):=V_{q^*}(x(t)-\tilde{x})+\int_t^\infty Y_{q^*}(e_{q^*})d\tau \qquad\qquad t\geq T^*$$

Then $\bar{V}$ is a nonnegative bounded function. Moreover from (6.13), $\dot{\bar{V}}\leq -X_{q^*}(x-\tilde{x})$, $t\geq T^*$ so $\bar{V}$ is nonincreasing on $[T^*,\infty)$. We claim that $\bar{V}\to 0$ a $t\to\infty$. Suppose this was not so. Then there would be a positive number μ such that $\bar{V}(t)\geq 2\mu$ for $t\geq T^*$. Pick $t_1\geq T^*$ so large that

$$\int_t^\infty Y_{q^*}(e_{q^*})d\tau\leq\mu$$

Then $V_{q^*}(x(t)-\tilde{x})\geq\mu$ for $t\geq t_1$. But $V_{q^*}(\cdot)$ is positive definite so there would have to be a positive number $\bar{\mu}$ such that $\|x(t)-\tilde{x}\|\geq\bar{\mu}$, $t\geq t_1$. As $X_{q^*}(\cdot)$ is also positive definite, there would have to be another positive number δ such that $X_{q^*}(x(t)-\tilde{x})\geq\delta$, $t\geq t_1$. But then one would have that $\dot{\bar{V}}\leq-\delta$, $t\geq t_1$ and thus that $\bar{V}(t)-\bar{V}(t_1)\leq-\delta(t-t_1)$, $t\geq t_1$. Hence for t sufficiently large $\bar{V}$ would become negative which is impossible. Therefore $\bar{V}\to 0$ a $t\to\infty$. From this it follows that $\lim_{t\to\infty}V_{q^*}(x(t)-\tilde{x})=0$. This proves that $x\to\tilde{x}$ as $t\to\infty$ since V_{q^*} is positive definite and radially unbounded. With $\tilde{x}$ partitioned as $\begin{bmatrix}\tilde{x}'_{\mathbb{E}} & \tilde{x}'_{\mathbb{C}}\end{bmatrix}'$ it can therefore be concluded that $x_{\mathbb{E}}$ converges to $\tilde{x}_{\mathbb{E}}$ and $x_{\mathbb{C}}$ converges to $\tilde{x}_{\mathbb{C}}$ as $t\to\infty$ because of (6.5).

Corollary 5.5 guarantees that $y_{q^*}=0$ at $\tilde{x}$ or equivalently that $C_{q^*}(\tilde{x}_{\mathbb{E}})=0$. In view of (6.2), $y=e_{q^*}+C_{q^*}(x_{\mathbb{E}})$. Therefore $\lim_{t\to\infty}y=\lim_{t\to\infty}e_{q^*}$. Now $\lim_{t\to\infty}Y_{q^*}(e_{q^*})=0$ because $\frac{\mathrm{d}}{\mathrm{dt}}Y_{q^*}(e_{q^*})$ is bounded on $[0,\infty)$ and $Y_{q^*}(e_{q^*})\in\mathcal{L}^1[0,\infty)$ (cf. [119, Lemma 1, p. 58]). From this it follows that $\lim_{t\to\infty}y=\lim_{t\to\infty}e_{q^*}=0$ since Y_{q^*} is positive definite and radially unbounded. The following was proved.

Theorem 6.5. *Let Assumptions 6.1 to 6.3 hold. For each initial state $x_{\mathbb{P}}(0)\in\mathcal{X}_{\mathbb{P}}$, $x_{\mathbb{E}}(0)\in\mathcal{X}_{\mathbb{E}}$, $x_{\mathbb{C}}(0)\in\mathcal{X}_{\mathbb{C}}$, $\pi_p(0)>0$, $p\in\mathcal{P}$, $\sigma(0)\in\mathcal{P}$, the solution $\{x_{\mathbb{P}},\ x_{\mathbb{E}},\ x_{\mathbb{C}},\ \pi_1,\ \pi_2,\dots,\pi_m\}$ to* (6.1)–(6.3) *and* (6.6) *(with σ the output of $\mathbb{S}_{\mathbb{H}}$) exists and is bounded on* $[0,\infty)$*. Moreover, y converges to zero as $t\to\infty$.*

6.2 Examples

The remainder of this chapter is devoted to the presentation of examples that illustrate the application of the above results to the control of nonlinear systems.

6.2.1 Example #1

Suppose that $\mathbb{P}$ is the one-dimensional system

$$\dot{x}=-x+p^*x^2\sin(p^*x)+u, \qquad\qquad y=x \tag{6.15}$$

where y denotes the measured output, u the control input, and p^* an uncertain parameter taking its value in $\mathcal{P}:=\{1,2,3\}$. The nonlinear way in which p^* appears in (6.15) makes most nonlinear adaptive control techniques inapplicable to the control of (6.15).

From (6.15) one concludes that $\mathbb{P}$ is an element of the family of process models[1] $\mathcal{F} := \{\mathbb{M}_p : p \in \mathcal{P}\}$ where, for each fixed $p \in \mathcal{P}$, $\mathbb{M}_p$ denotes the one-dimensional system

$$\dot{x} = -x + px^2 \sin(px) + u, \qquad y = x \tag{6.16}$$

Since the state and output of $\mathbb{P}$ are the same, it is straightforward to verify that Assumption 6.1 is satisfied. A multi-estimator $\mathbb{E}$ for $\mathcal{F}$ is the three-dimensional system

$$\dot{x}_p = -x_p + py^2 \sin(py) + u \qquad y_p = x_p, \qquad p \in \mathcal{P} \tag{6.17}$$

and the corresponding estimation errors are

$$e_p := y_p - y, \qquad p \in \mathcal{P} \tag{6.18}$$

Since $\mathbb{P}$ is equal to $\mathbb{M}_{p^*}$, from (6.16), (6.17), and (6.18), one concludes that

$$e_{p^*}(t) = e^{-t}\big(x_{p^*}(0) - x(0)\big), \qquad t \geq 0$$

and therefore that

$$e^{\lambda t}\|e_{p^*}(t)\| \leq e^{-(1-\lambda)t}\|x_{p^*}(0) - x(0)\|, \qquad t \geq 0$$

Thus Assumption 6.3 is satisfied for any $\lambda \in (0, 1)$.

For the pth controller $\mathbb{C}_p$ one can take

$$u_p = -py^2 \sin(py)$$

The feedback interconnected system shown in Figure 5.3 is then defined by the equations

$$\dot{\bar{x}}_q = -\bar{x}_q + (\bar{x}_p - v)^2\Big(q \sin\big(q(\bar{x}_p - v)\big) - p \sin\big(p(\bar{x}_p - v)\big)\Big), \qquad q \in \mathcal{P} \setminus \{p\} \tag{6.19}$$

$$\dot{\bar{x}}_p = -\bar{x}_p \tag{6.20}$$

$$\bar{y}_p = \bar{x}_p \tag{6.21}$$

This system is input-to-state stable about the equilibrium input $\tilde{v} := 0$ since (6.19) can be regarded as a system with input $\{\bar{x}_p,\ v\}$ which is input-to-state stable about the zero equilibrium input and, because of (6.20), $\bar{x}_p$ converges to zero exponentially fast. Moreover, the origin is the equilibrium state induced by the equilibrium input $\tilde{v}$ and, at this state, $\bar{y}_p = 0$. Because of Corollary 5.5, one thus concludes that the pp^*th injected system, which is defined by

$$\begin{aligned}
\dot{x}_q &= -x_q + (x_{p^*} - e_{p^*})^2\Big(q \sin\big(q(x_{p^*} - e_{p^*})\big) - p \sin\big(p(x_{p^*} - e_{p^*})\big)\Big), \qquad q \in \mathcal{P} \setminus \{p\} \\
\dot{x}_p &= -x_p \\
e_p &= x_p - x_{p^*} + e_{p^*}
\end{aligned}$$

is strongly detectable about the equilibrium output $\tilde{e}_p := 0$. For example, for $p = 2$ the specific equations are

$$\begin{aligned}
\dot{x}_1 &= -x_1 + (x_{p^*} - e_{p^*})^2\Big(\sin(x_{p^*} - e_{p^*}) - 2 \sin\big(2(x_{p^*} - e_{p^*})\big)\Big) \\
\dot{x}_2 &= -x_2 \\
\dot{x}_3 &= -x_3 + (x_{p^*} - e_{p^*})^2\Big(3 \sin\big(3(x_{p^*} - e_{p^*})\big) - 2 \sin\big(2(x_{p^*} - e_{p^*})\big)\Big) \\
e_2 &= x_2 - x_{p^*} + e_{p^*}
\end{aligned}$$

[1] Since there is no unmodeled dynamics, the family of process models $\mathcal{F}$ is the same as the family of nominal process models $\mathcal{M}$.

For this system it can be verified that $\{x_1^2 + 70x_2^4 + x_3^2,\ x_1^2 + 8x_2^4 + x_3^2,\ 300e_2^4\}$ is a strong detectability triple about $\tilde{e}_2 = 0$ [2]. By similar reasoning, it is also possible to construct for $p = 1$ and $p = 3$, strong detectability triples $\{V_p, X_p, Y_p\}$ for the pp^*th injected system, in such a way that Y_p is a positive, quartic function of e_p, V_1 and X_1 are positive definite quadratic function of x_1^2, x_2, x_3, and V_3 and X_3 are positive definite quadratic function of x_1, x_2, x_3^2. Assumption 6.2 is thus satisfied.

Were the nonlinearity in the original process model separable (say of the form p^*x^2 rather than $p^*x^2\sin(p^*x)$), this example could be dealt with more efficiently using a state-shared multi-estimator of the form

$$\dot{z}_1 = -z_1 + u, \qquad \dot{z}_2 = -z_2 + y^2, \qquad y_p = z_1 + pz_2, \qquad p \in \mathcal{P}$$

and a multi-controller of the form $u_\sigma = -\sigma y^2$. In this case the Y_p could still be taken as quartic functions of the e_p.

6.2.2 Example #2

Suppose that $\mathbb{P}$ is the two-dimensional system

$$\dot{x} = -p^*y^2 + u, \qquad \dot{y} = x + p^*y^2 - u \tag{6.22}$$

where y denotes the measured output, u the control input, and p^* an uncertain parameter taking values in a finite subset $\mathcal{P}$ of $[-1, 1]$ with m elements. Along any solution to (6.22) for which y is identically zero, $\dot{x} = x$, which means that the system (6.22) has unstable zero dynamics. This makes many nonlinear adaptive control techniques inapplicable to the control of (6.22).

From (6.22) one concludes that $\mathbb{P}$ is an element of the family of process models[3] $\mathcal{F} := \{\mathbb{M}_p : p \in \mathcal{P}\}$ where, for each fixed $p \in \mathcal{P}$, $\mathbb{M}_p$ denotes the two-dimensional system

$$\dot{z} = Az + b(py^2 - u), \qquad y = cz \tag{6.23}$$

with $z := [x \ \ y]'$ and

$$A := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad b := \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad c := \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Because of the observability of the pair (c, A), it is straightforward to verify that Assumption 6.1 is satisfied. A multi-estimator $\mathbb{E}$ for $\mathcal{F}$ is the $2m$-dimensional system

$$\dot{z}_p = (A + kc)z_p + b(py^2 - u) - ky, \qquad y_p = cz_p, \qquad p \in \mathcal{P} \tag{6.24}$$

where k is any vector that makes $A + kc$ asymptotically stable. The corresponding estimation errors are

$$e_p := y_p - y, \qquad p \in \mathcal{P} \tag{6.25}$$

[2] One can also check that $\tilde{x} := 0$ is the equilibrium state induced by the equilibrium output $\tilde{e}_2 = 0$, and that at the equilibrium state, $y_2 = 0$.

[3] Since there is no unmodeled dynamics, the family of process models $\mathcal{F}$ is the same as the family of nominal process models $\mathcal{M}$.

Since $\mathbb{P}$ is equal to $\mathbb{M}_{p^*}$, from (6.23), (6.24), and (6.25), one concludes that

$$e_{p^*}(t) = ce^{(A+kc)t}\big(z_{p^*}(0) - z(0)\big), \qquad t \geq 0$$

with $z := [x \;\; y]'$, and therefore

$$e^{\lambda t}\|e_{p^*}(t)\| \leq \|c\| \, \|e^{(A+kc+\lambda I)t}\| \, \|z_{p^*}(0) - z(0)\|, \qquad t \geq 0$$

Thus Assumption 6.3 is satisfied for any $\lambda > 0$ such that $A + kc + \lambda I$ is asymptotically stable.

For the pth controller $\mathbb{C}_p$ one can take the following two-dimensional, feedback linearizing, control law

$$\dot{x}_{\mathbb{C}} = (A + kc + bf)x_{\mathbb{C}} - ky, \qquad u_p = py^2 - fx_{\mathbb{C}} \tag{6.26}$$

where f is any vector that makes $A + bf$ asymptotically stable. The feedback interconnected system shown in Figure 5.3 is then defined by the equations

$$\dot{\bar{z}}_q = (A + kc)\bar{z}_q + b(q - p)(c\bar{z}_p - v)^2 - k(c\bar{z}_p - v) + bf\bar{x}_{\mathbb{C}}, \qquad q \in \mathcal{P} \setminus \{p\} \tag{6.27}$$
$$\dot{\bar{z}}_p = A\bar{z}_p + bf\bar{x}_{\mathbb{C}} - kv \tag{6.28}$$
$$\dot{\bar{x}}_{\mathbb{C}} = -kc\bar{z}_p + (A + kc + bf)\bar{x}_{\mathbb{C}} + kv \tag{6.29}$$
$$\bar{y}_p = c\bar{z}_p \tag{6.30}$$

This system can be regarded as a cascade connection of a system Σ_1 defined by (6.28)–(6.29) with input v and output[4] $\{\bar{z}_p, \; \bar{x}_{\mathbb{C}}, \; v\}$ and a system Σ_2 defined by (6.27) with input $\{\bar{z}_p, \; \bar{x}_{\mathbb{C}}, \; v\}$. Both Σ_1 and Σ_2 are input-to-state stable about zero equilibrium inputs since $A+kc$ and $A+bf$ are asymptotically stable. The system (6.27)–(6.29) is then the cascade of two input-to-state stable system and is therefore input-to-state stable about the equilibrium input $\tilde{v} := 0$ [116]. Moreover, the origin is the equilibrium state induced by the equilibrium input $\tilde{v}$ and, at this state, $\bar{y}_p = 0$. Because of Corollary 5.5, one thus concludes that the pp^*th injected system, which is defined by

$$\begin{aligned}
\dot{z}_q &= (A + kc)z_q + bfx_{\mathbb{C}} + b(q - p)(cz_{p^*} - e_{p^*})^2 - k(cz_{p^*} - e_{p^*}), \qquad q \in \mathcal{P} \setminus \{p\} \\
\dot{z}_p &= Az_p + kcz_p + bfx_{\mathbb{C}} - k(cz_{p^*} - e_{p^*}) \\
\dot{x}_{\mathbb{C}} &= (A + kc + bf)x_{\mathbb{C}} - k(cz_{p^*} - e_{p^*}) \\
e_p &= c(z_p - z_{p^*}) + e_{p^*}
\end{aligned}$$

is strongly detectable about the equilibrium output $\tilde{e}_p := 0$. Straightforward, but lengthy computations, allow one to conclude that each pp^*th injected system admits a strong detectability triple about $\tilde{e}_p := 0$ with $Y_p(e_p) = \|e_p\|^2 + \|e_p\|^4$. Assumption 6.2 is thus satisfied.

Since the right-hand sides of the differential equations in (6.24) are affinely separable (cf. Section 5.6), one can also use a state-shared multi-estimator of the form

$$\dot{x}_{\mathbb{E}} = \begin{bmatrix} A + kc & 0 \\ 0 & A + kc \end{bmatrix} x_{\mathbb{E}} + \begin{bmatrix} by^2 \\ -bu - ky \end{bmatrix}, \qquad \bar{y}_p = \begin{bmatrix} pc & c \end{bmatrix} x_{\mathbb{E}}, \qquad p \in \mathcal{P} \tag{6.31}$$

Note that for every piecewise continuous signals y and u, the output estimates generated by (6.24) and by (6.31) differ only by a term that is due to initial conditions and that converges to zero exponentially fast as $t \to \infty$. The dimension of the state-shared estimator (6.31) is

[4]Here v is being regarded as a direct feedthrough term.

always 4 *no matter how many elements* $\mathcal{P}$ *has*. In fact, with this multi-estimator, $\mathcal{P}$ can have infinitely many elements, e.g. $\mathcal{P} := [-1, 1]$.

Figure 6.1 shows a simulation of the supervised system defined by equations (6.22), (6.31), (6.25), (6.26), (6.10), with $\mathcal{P} := [-1, 1]$, and σ the output of $\mathbb{S}_{\mathbb{H}}$. In this simulation the actual parameter p^* is slowly varied to demonstrate the robustness of the closed loop system.

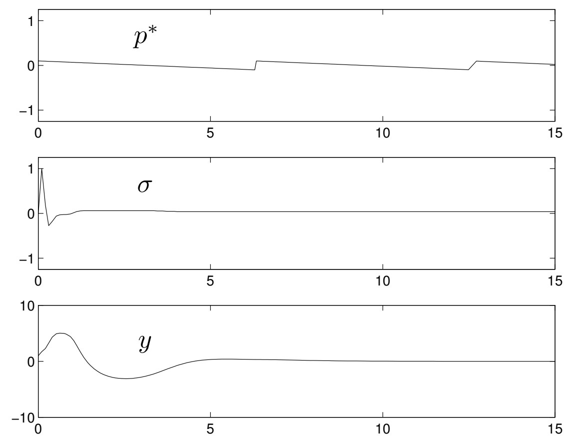

Figure 6.1: Example #2

6.2.3 Example #3

Suppose that $\mathbb{P}$ is the two-dimensional system

$$\dot{x}_1 = p_1^* x_1^3 + p_2^* x_2, \qquad \dot{x}_2 = u \tag{6.32}$$

where u denotes the control input and $p^* := \{p_1^*,\ p_2^*\}$ is an uncertain parameter taking values in a finite subset $\mathcal{P}$ of $[-1, 1] \times \big([-1, 0) \cup (0, 1]\big)$ with m elements. The state $\{x_1,\ x_2\}$ of (6.32) is assumed available for measurement and the control objective is to drive x_1 to a known constant set point r. In this example we take the measured output to be

$$y := \begin{bmatrix} x_1 - r \\ x_2 \end{bmatrix} \tag{6.33}$$

This example differs from the formulation in Chapter 5 in that the control objective is not to drive the whole vector y to zero but just its first component. However, the results in Chapter 5 and the analysis presented in Section 6.1 are still applicable to this problem with a few simple modifications described below.

The system (6.32) is feedback linearizable for every value of $p \in \mathcal{P}$, but for $p_1^* < 0$ the nonlinear term $p_1^* x_1^3$ should not be cancelled since it provides desirable damping. This will be taken into account in the design of the feedback control system.

For each $p := \{p_1,\ p_2\} \in \mathcal{P}$, let C_p denote the invertible function from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by the rule

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \longmapsto \begin{bmatrix} y_1 \\ p_1 (y_1 + r)^3 + p_2 y_2 \end{bmatrix}$$

and let C_p^{-1} denote the inverse of C_p. From (6.32)–(6.33) one concludes that $\mathbb{P}$ is an element of the family of process models[5] $\mathcal{F} := \{\mathbb{M}_p : p \in \mathcal{P}\}$ where, for each fixed $p := \{p_1,\ p_2\} \in \mathcal{P}$, $\mathbb{M}_p$ denotes the two-dimensional system

$$\dot{z} = Az + b(p_1 g(z) + p_2 u), \qquad\qquad y = C_p^{-1}(z) \tag{6.34}$$

with $z := C_p([x_1 - r \ \ x_2]')$, and

$$A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad b := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad g\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right) := 3(z_1 + r)^2 z_2, \qquad z_1, z_2 \in \mathbb{R}$$

It is straightforward to verify that Assumption 6.1 is satisfied. A multi-estimator[6] $\mathbb{E}$ for $\mathcal{F}$ is the $2m$-dimensional system

$$\dot{z}_p = AC_p(y) + b\big(p_1 g\big(C_p(y)\big) + p_2 u\big) - \big(z_p - C_p(y)\big), \qquad y_p = C_p^{-1}(z_p), \qquad p \in \mathcal{P} \tag{6.35}$$

In this example it is convenient to consider the following "nonlinear" estimation errors (cf. comments in Section 5.8)

$$e_p := C_p(y_p) - C_p(y) = z_p - C_p(y), \qquad\qquad p \in \mathcal{P} \tag{6.36}$$

Note that because C_p is an injective function, $e_p = 0$ is equivalent to $y_p = y$. Since $\mathbb{P}$ is equal to $\mathbb{M}_{p^*}$, from (6.34), (6.35), and (6.36), one concludes that

$$e_{p^*}(t) = e^{-t}\big(z_{p^*}(0) - z(0)\big), \qquad\qquad t \geq 0$$

with $z := C_{p^*}(y)$ and therefore

$$e^{\lambda t}\|e_{p^*}(t)\| \leq e^{-(1-\lambda)t}\|z_{p^*}(0) - z(0)\|, \qquad\qquad t \geq 0$$

Thus Assumption 6.3 is satisfied for any $\lambda \in (0,1)$.

Feedback Linearizing Control Law. For the pth controller $\mathbb{C}_p$ we start by considering the feedback linearizing control law

$$u_p = F_p(y) \tag{6.37}$$

with

$$F_p(y) := \frac{1}{p_2}\Big(fC_p(y) - p_1 g\big(C_p(y)\big)\Big), \qquad\qquad p \in \mathcal{P},\ y \in \mathbb{R}^2$$

where f is any vector that makes $A + bf$ asymptotically stable. Since we are using the nonlinear estimation error (6.36), and this equation implies that $y = C_p^{-1}(C_p(y_p) - e_p)$, the signal coming out of the summing junction in Figure 5.3 must be interpreted as

$$\bar{y} = C_p^{-1}(C_p(\bar{y}_p) - v)$$

[5]Since there is no unmodeled dynamics, the family of process models $\mathcal{F}$ is the same as the family of nominal process models $\mathcal{M}$.

[6]Since the nonlinearities in (6.35) are affinely separable, one could also use a state-shared multi-estimator for this example.

Therefore, the feedback interconnected system shown in Figure 5.3 is defined by the equations

$$\dot{\bar{z}}_q = -\bar{z}_q + AC_{pq}(\bar{z}_p - v) + b\Big(q_1 g\big(C_{pq}(\bar{z}_p - v)\big) + \frac{q_2}{p_2}\big(f(\bar{z}_p - v) - p_1 g(\bar{z}_p - v)\big)\Big) + C_{pq}(\bar{z}_p - v), \qquad q \in \mathcal{P} \setminus \{p\} \tag{6.38}$$

$$\dot{\bar{z}}_p = (A + bf)\bar{z}_p - (A + bf + I)v \tag{6.39}$$

$$\bar{y}_p = C_p^{-1}(\bar{z}_p) \tag{6.40}$$

where $C_{pq} := C_q \circ C_p^{-1}$. This system can be regarded as a cascade connection of a system Σ_1 defined by (6.39) with input v and output[7] $\{\bar{z}_p,\ v\}$ and a system Σ_2 defined by (6.38) with input $\{\bar{z}_p,\ v\}$. Since $A + bf$ is a stability matrix, Σ_1 is input-to-state stable about the equilibrium input $\tilde{v} := 0$ and the corresponding induced equilibrium state is $\tilde{z}_p := 0$. Σ_2 is also input-to-state stable about the zero equilibrium input because the homogeneous part of (6.38) is linear and asymptotically stable. The system (6.38)–(6.39) is then the cascade of two input-to-state stable systems and is therefore input-to-state stable about the equilibrium input $\tilde{v}$ [116]. One can verify that, at the equilibrium state of the system (6.38)–(6.39) induced by the equilibrium input $\tilde{v}$, one has $\bar{y}_p = [0 \quad -\frac{p_1}{p_2} r^3]'$. A straightforward adaptation of Corollary 5.5 allows one to conclude that the pp^*th injected system, which is defined by

$$\begin{aligned}
\dot{z}_q &= -z_q + C_{p^*q}(z_{p^*} - e_{p^*}) + AC_{p^*q}(z_{p^*} - e_{p^*}) \\
&\quad + b\bigg(q_1 g\big(C_{p^*q}(z_{p^*} - e_{p^*})\big) + \frac{q_2}{p_2}\Big(fC_{p^*p}(z_{p^*} - e_{p^*}) - p_1 g\big(C_{p^*p}(z_{p^*} - e_{p^*})\big)\Big)\bigg), \quad q \in \mathcal{P} \setminus \{p\} \\
\dot{z}_p &= (A + bf)C_{p^*p}(z_{p^*} - e_{p^*}) - z_p + C_{p^*p}(z_{p^*} - e_{p^*}) \\
e_p &= z_p - C_{p^*p}(z_{p^*} - e_{p^*})
\end{aligned}$$

is strongly detectable about the equilibrium output $\tilde{e}_p := 0$. Moreover, $y_p = [0 \quad -\frac{p_1}{p_2} r^3]'$ at the equilibrium state induced by the equilibrium output $\tilde{e}_p$. Straightforward, but lengthy computations, allow one to conclude that each pp^*th injected system admits a strong detectability triple about $\tilde{e}_p := 0$ with $Y_p(e_p) = \|e_p\|^2 + \|e_p\|^4$. Assumption 6.2 is thus satisfied.

For this system, the analysis in Section 6.1 leads to the conclusion that the output y of the process (6.32)–(6.33) converges to $[0 \quad -\frac{q_1^*}{q_2^*} r^3]'$ as $t \to \infty$, where $q^* = \{q_1^*,\ q_2^*\}$ denotes that element of $\mathcal{P}$ at which switching stops after some finite time (cf. Lemma 6.4). Thus, the first component of y converges to zero as $t \to \infty$ and therefore, from (6.33), one concludes that x_1 converges to r.

Figure 6.2 shows a simulation of the supervised system defined by equations (6.32), (6.33), (6.35), (6.36), (6.37), (6.10) and with σ the output of $\mathbb{S}_{\mathbb{H}}$. For the design of the controllers and estimators the parameter set $\mathcal{P}$ was taken to be

$$\mathcal{P} := \{-1, -.9, -.8, \ldots, -.1, 0, .1, .2, \ldots, .9, 1\} \times \{-1, 1\}$$

To demonstrate the robustness of the closed loop system, in the simulations, the values of the actual parameters p_1^* and p_2^* are not exactly in $\mathcal{P}$. In the simulation shown in Figure 6.2, deviations of up to 1% were allowed.

[7] Here v is being regarded as a direct feedthrough term.

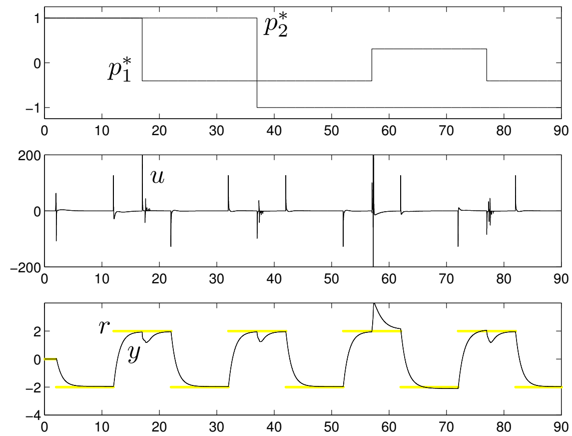

Figure 6.2: Example #3 with feedback linearizing control law

Pointwise Min-Norm Control Law. Let $p = \{p_1,\ p_2\}$ be an arbitrary element of $\mathcal{P}$. As noted above, the system Σ_1 defined by (6.39) with input v is input-to-state stable about the equilibrium input $\tilde{v} := 0$. In fact, it is straightforward to verify that a stability triple for this system is $\{V,\ 2\mu V,\ \|v\|^2\}\}$ where μ is any positive constant such that $A + bf + \mu I$ is a stability matrix and $V(\bar{z}) := \bar{z}'P\bar{z}$ with P symmetric, positive definite, defined by

$$(A + bf + \mu I)'P + P(A + bf + \mu I) + P(A + bf + I)(A + bf + I)'P + I = 0$$

Since (6.39) can also be written as

$$\dot{\bar{z}}_p = A(\bar{z}_p - v) + b\big(p_1 g(\bar{z}_p - v) + p_2 F_p(C_p^{-1}(\bar{z}_p - v))\big) - v$$

one concludes that

$$\frac{\partial V}{\partial \bar{z}}(\bar{z}_p)\Big(A(\bar{z}_p - v) + b\big(p_1 g(\bar{z}_p - v) + p_2 F_p(C_p^{-1}(\bar{z}_p - v))\big) - v\Big) \\ \leq -2\mu V(\bar{z}_p) + \|v\|^2, \qquad \forall v,\ \bar{z}_p \in \mathbb{R}^2$$

Making the change of coordinates $\bar{y} := C_p^{-1}(\bar{z}_p - v)$, one further concludes that

$$\frac{\partial V}{\partial \bar{z}}(C_p(\bar{y}) + v)\Big(A(C_p(\bar{y})) + b\big(p_1 g(C_p(\bar{y})) + p_2 F_p(\bar{y})\big) - v\Big) \\ \leq -2\mu V(C_p(\bar{y}) + v) + \|v\|^2, \qquad \forall v, \bar{y} \in \mathbb{R}^2$$

Thus, for each $\bar{y} \in \mathbb{R}^2$, the set

$$\mathcal{U}_p(\bar{y}) := \Big\{u : \frac{\partial V}{\partial \bar{z}}(C_p(\bar{y}) + v)\Big(A(C_p(\bar{y})) + b\big(p_1 g(C_p(\bar{y})) + p_2 u\big) - v\Big) \\ \leq -2\mu V(C_p(\bar{y}) + v) + \|v\|^2, \quad v \in \mathbb{R}^2\Big\}$$

contains at least $F_p(\bar{y})$, and is therefore nonempty. In the spirit of [120], one can then take for the pth controller $\mathbb{C}_p$ the control law

$$u_p = \bar{F}_p(y) \tag{6.41}$$

with

$$\bar{F}_p(y) := \begin{cases} F_p(y) & p_1 \geq 0 \\ \arg\min\{\|u\| : u \in \mathcal{U}_p(y)\} & p_1 < 0 \end{cases} \qquad p = \{p_1,\ p_2\} \in \mathcal{P},\ y \in \mathbb{R}^2$$

By construction, $\{V,\ 2\mu V,\ \|v\|^2\}$ is then a stability triple for the system

$$\dot{\bar{z}}_p = A(\bar{z}_p - v) + b\big(p_1 g(\bar{z}_p - v) + p_2 \bar{F}_p(C_p^{-1}(\bar{z}_p - v))\big) - v$$

about the equilibrium input $\tilde{v} := 0$ and thus, as before, one concludes that also with the control law (6.41), the pp^*th injected system is strongly detectable about the equilibrium output $\tilde{e}_p := 0$. Also in this case, each pp^*th injected system admits a strong detectability triple about $\tilde{e}_p := 0$ with $Y_p(e_p) = \|e_p\|^2 + \|e_p\|^4$. Assumption 6.2 is thus satisfied.

For $p_1 \geq 0$, (6.41) corresponds to the usual feedback linearizing control law, but for $p_1 < 0$, (6.41) corresponds to a *Pointwise Min-Norm Control Law* [120]. In [120] it is shown that pointwise min-norm control laws are optimal for meaningful games and are therefore solutions to inverse optimal robust stabilization problems for nonlinear systems with disturbances. It is also noted in [120] that, in general, these control laws do not cancel nonlinear terms that cause damping.

Figure 6.3 shows a simulation of the supervised system defined by equations (6.32), (6.33), (6.35), (6.36), (6.41), (6.10) and with σ the output of $\mathbb{S}_{\mathbb{H}}$. Comparing Figures 6.2 and 6.3 one observes that when $p_1^* < 0$, the control law (6.41) results in control signals about 10 times smaller than those produced by (6.37).

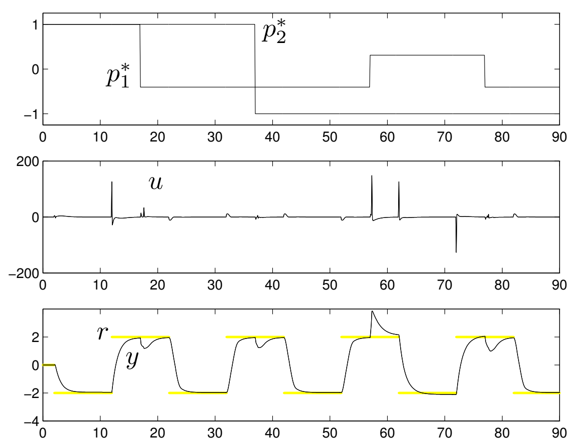

Figure 6.3: Example #3 with pointwise min-norm control law

6.3 Concluding Remarks

In this chapter it was shown how the results in Chapters 5 and 4 can be applied to the supervisory control of uncertain nonlinear system. In Section 6.1 an estimator-based supervisor is proposed and analyzed in a fairly general setting. This supervisor has the key property that it can be built using off-the-shelf controllers and estimators, as illustrated with the examples in Section 6.2.

The main shortcoming of the present results is that the stability analysis in Section 6.1 is only valid in the absence of unmodeled dynamics and disturbance/noise. However, simulation results indicate that the overall closed loop system might, in fact, be robust with respect to unmodeled dynamics, disturbance, and noise.

Part III

Decidability of Robot Positioning Tasks Using Vision Systems

Chapter 7

Task Decidability

Implicit in the formulation of the control problem considered in Part II is the assumption that the signal to be regulated can be directly measured. This does not happen when one wants to control the position of a robot employing a pair of video cameras acting as a position measuring device. Indeed, in this case the control objective—moving the robot to a specified position—is expressed in terms of the Cartesian position and orientation of the robot, but the measured signals are the image coordinates of observed robot "features". If the camera models are accurately known, one can compute a one-to-one correspondence between the robot's Cartesian position/orientation and the image coordinates of the observed robot features. Thus, the signal to be regulated can be accurately "reconstructed" from the measurements, and one falls into the problem addressed in Part II. However, if the cameras are imprecisely modeled, this can not be done. In fact, in this case, it is not clear whether or not it is possible to determine if the control objective has been accomplished just by making use of the cameras' measurements. This observation is the starting point for the main question addressed in the remaining Chapters of this thesis:

> *When is it possible to decide if a prescribed robot positioning task has been accomplished using images acquired by an imprecisely modeled stereo vision system?*

Feedback control systems employing video cameras as sensors have been studied in the robotics community for many years (cf. tutorial on visual servoing [77] and review [78]). Demonstrated applications of vision within a feedback loop—often referred to as visual servoing or, more generally, *vision-based control*—include automated driving [79], flexible manufacturing [80, 81], and tele-operation with large time delays [82] to name a few.

An especially interesting feature of vision-based control systems is that often *both* the process output (e.g., the position and orientation of the robot in its workspace) and the reference set-point (e.g., a set of desired positions and orientations) can be simultaneously observed through the *same* sensors (i.e., cameras). In prior work [83, 84, 85, 86, 87], it has been observed that because of this unusual architectural feature, it is sometimes possible to achieve *precise* positioning (in the absence of measurement noise), despite sensor/actuator and process model imprecision, just as in the case of a conventional set-point control system with a loop-integrator and fixed exogenous reference. But in contrast to a set-point control system where what to choose for an error is usually clear, in vision-based systems there are many choices, each with different attributes. Some of the observations just made are implicit in work extending back more than 15 years [88]. Some of these issues are touched upon in [89] and in the monograph [90].

The aim of this chapter is to give conditions that enable one to decide on the basis of images of point features observed by an imprecisely modeled two-camera vision system, whether or not a prescribed positioning task has been accomplished. By a positioning task is meant, roughly speaking, the objective of bringing the pose of a robot—i.e., its position and orientation—to a target in the robot's workspace. Both the pose of the robot under consideration and the target to which it is to be brought, are determined by a list of simultaneously observed point features $f_1, f_2, \ldots, f_n$ in the two cameras' joint field of view $\mathcal{V}$. A task is then formally defined to be an equation of the form $T(f) = 0$ where T is a function mapping lists of point features of the form $f = \{f_1, f_2, \ldots, f_n\}$ into the integer set $\{0, 1\}$[1]. Such a task is said to be accomplished if the equation $T(f) = 0$ is satisfied by the observed feature list of interest. The images of observed point features appear in the two cameras' joint image space $\mathcal{Y}$ and are available for processing. The two-camera model that maps point features in $\mathcal{V}$ into $\mathcal{Y}$ is not presumed to be known with certainty. Rather the model is assumed to be an unspecified member of some known class of two-camera models $\mathcal{C}$. The only information available for deciding whether or not a given task has been accomplished is thus the task function T, the images of the observed point features in $\mathcal{Y}$, and the class $\mathcal{C}$. A given task is said to be decidable on $\mathcal{C}$ if the available information, namely T, $\mathcal{C}$ and the images of the point features, is sufficient to determine whether or not the task has been accomplished.

One way to formalize the decidability question is by means of task encodings. The concept of a task encoding is discussed briefly in the Doctoral Thesis [91], which contains further references, and is also studied in the Doctoral Thesis [92]. In the present thesis, an encoded task is simply an equation of the form $E_T(y) = 0$ where y is a list of the images of observed point features as they appear in $\mathcal{Y}$, and E_T is a function that maps such lists into the reals. The construction of E_T must be based only on the knowledge of T and $\mathcal{C}$, and *not* on the actual two-camera model in $\mathcal{C}$, which is not assumed known. The encoded task is accomplished if the equation $E_T(y) = 0$ is satisfied by the list of images of observed point features of interest. The task $T(f) = 0$ is said to be verifiable on $\mathcal{C}$ with the encoding $E_T(y) = 0$ if accomplishing the encoded task is equivalent to accomplishing the original task, no matter which model in $\mathcal{C}$ correctly describes the actual two-camera system. The original task $T(f) = 0$ is then decidable on $\mathcal{C}$ if there is an encoded task that verifies $T(f) = 0$ on $\mathcal{C}$.

The remainder of this chapter is structured as follows. Section 7.1 establishes basic nomenclature and definitions needed to formally address the task decidability problem. In Section 7.2, the notion of an encoding of a task is introduced and conditions for a task to be decidable are derived. Two well-known methods of encoding are presented in Section 7.2.3, and a recently proposed one is considered in Section 7.2.4. Finally, Section 7.3 briefly discusses how the results in this chapter can be used to design feedback control systems capable of precise robot positioning using visual feedback, in spite of camera miscalibration.

Notation: In the sequel, prime denotes matrix transpose, $\mathbb{R}^m$ is the real linear space of m-vectors, and $\mathbb{P}^m$ is the real projective space of one-dimensional subspaces of $\mathbb{R}^{m+1}$. Recall that the elements of $\mathbb{P}^m$ are called *points*, *lines* in $\mathbb{P}^m$ are two-dimensional subspaces of $\mathbb{R}^{m+1}$, and for $m > 2$ planes are three-dimensional subspaces of $\mathbb{R}^{m+1}$. A point $p \in \mathbb{P}^m$ is said to be *on* a line ℓ (respectively plane ψ) in $\mathbb{P}^m$ if p is a linear subspace of ℓ (respectively ψ) in $\mathbb{R}^{m+1}$. For each nonzero vector $x \in \mathbb{R}^m$, $\mathbb{R}x$ denotes the one-dimensional linear span of x, and also the point in $\mathbb{P}^m$ that x *represents*. The line in $\mathbb{P}^m$ on which both two distinct points $p_1, p_2 \in \mathbb{P}^m$

[1]The reason for requiring the codomain of such a task function to be $\{0, 1\}$—rather than the reals which is more common—is that this requirement leads to somewhat simpler notation. There is in fact no real loss of generality in so constraining such T's.

lie, is denoted by $p_1 \oplus p_2$. With slight abuse of terminology, the *kernel* of a linear or nonlinear function T with codomain $\mathbb{R}$, written $\operatorname{Ker} T$, is defined to be the set of points x in T's domain such that $T(x) = 0$. As usual, the *image* of a function $H : \mathcal{Z} \to \mathcal{W}$, written $\operatorname{Im} H$, is the set of points $\{H(z) : z \in \mathcal{Z}\}$. The special Euclidean group of rigid body transformations is denoted by SE(3).

7.1 Formulation

This chapter is concerned with the problem of achieving precise control of the pose of a robot that moves in a prescribed *workspace* $\mathcal{W} \subset \mathrm{SE}(3)$, using two cameras functioning as a position measuring system. The data available for this purpose consists of projections onto the cameras' image planes of robot point features[2] as well as point features in the environment. All such features lie within the two cameras' *joint field of view* $\mathcal{V}$. Typically $\mathcal{V}$ will be taken to be either a nonempty subset of $\mathbb{R}^3$ or of $\mathbb{P}^3$.

Point features are mapped into the two cameras' *joint image space* $\mathcal{Y}$ through a fixed but imprecisely known *two-camera model* $C_{\text{actual}} : \mathcal{V} \to \mathcal{Y}$ where, depending on the problem, $\mathcal{Y}$ may be either $\mathbb{R}^2 \times \mathbb{R}^2$ or $\mathbb{P}^2 \times \mathbb{P}^2$. Typically several point features are observed all at once. If f_i is the ith such point feature in $\mathcal{V}$, then f_i's observed position in $\mathcal{Y}$ is given by the measured output vector $y_i = C_{\text{actual}}(f_i)$. The two-camera model C_{actual} is a fixed but unknown element of a prescribed set of injective functions $\mathcal{C}$ that map $\mathcal{V}$ into $\mathcal{Y}$. In the sequel $\mathcal{C}$ is called the *set of admissible two-camera models.* For the present, no constraints are placed on the elements of $\mathcal{C}$ other than they be injective functions mapping $\mathcal{V}$ into $\mathcal{Y}$.

Tasks

By a positioning task or simply a "task" is meant, roughly speaking, the objective of bringing the pose of a robot to a "target" (a set of desired poses) in $\mathcal{W}$. Both the pose of the robot under consideration and the target set are determined by a list of simultaneously observed point features in $\mathcal{V}$. As in [77, 89, 90], tasks are represented mathematically as equations to be satisfied. In this thesis, the term "task function" refers to a function that maps ordered sets (i.e., lists) of n simultaneously observed point features $\{f_1, f_2, \ldots, f_n\}$ in $\mathcal{V}$ into the integer[3] set $\{0, 1\}$. We use an un-subscripted symbol such as f to denote each such list and we henceforth refer to f as a *feature*. In some cases only certain features or lists of point features are of interest (e.g., for $n = 3$, one might want to consider lists whose three point features are collinear). The set of all such lists of interest is denoted by $\mathcal{F}$ and is a nonempty subset of the set $\mathcal{V}^n := \underbrace{\mathcal{V} \times \mathcal{V} \times \cdots \times \mathcal{V}}_{n \text{ times}}$. In the sequel we call $\mathcal{F}$ the *admissible feature space.* A *task function* is then a given function T from $\mathcal{F}$ to $\{0, 1\}$. The *task* specified by T is the equation

$$T(f) = 0. \tag{7.1}$$

[2]Throughout this chapter, the term "feature" always refers to sets (points, lines, etc.) that are observed by cameras. The observations themselves are referred to as "measured data." A "point feature" may be represented by either a point in $\mathbb{R}^3$ or a point in $\mathbb{P}^3$ (i.e., a one-dimensional subspace of $\mathbb{R}^4$). In the examples that follow, points in $\mathbb{R}^m$ are related to points in $\mathbb{P}^m$ by the injective function $x \mapsto \mathbb{R}\bar{x}$ where $\bar{x}$ is the vector $\begin{bmatrix} x' & 1 \end{bmatrix}'$ in $\mathbb{R}^{m+1}$. With this correspondence, geometrically significant points in $\mathbb{R}^3$ such as a camera's optical center can be unambiguously represented as points in $\mathbb{P}^3$.

[3]In the literature a "task function" is usually defined to be mapping to the real line or real vector space with additional properties (e.g. continuity, differentiability) needed to implement a control system. As shown later, a task function in the form we described here can, in most cases of interest be "converted" into a task function with such properties.

In case (7.1) holds we say that the task is *accomplished at f*. Examples of tasks defined in this manner, can be found in [90, 121, 122, 80, 77, 89, 123].

In order to complete the problem formulation, it is helpful to introduce the following notation. For each C in $\mathcal{C}$, let $\bar{C}$ denote the function from $\mathcal{F} \subset \mathcal{V}^n$ to the set $\mathcal{Y}^n := \underbrace{\mathcal{Y} \times \mathcal{Y} \times \cdots \times \mathcal{Y}}_{n \text{ times}}$, which is defined by the rule

$$\{f_1, f_2, \ldots, f_n\} \longmapsto \{C(f_1), C(f_2), \ldots, C(f_n)\}$$

We call $\bar{C}$ the *extension* of C to $\mathcal{F}$. The aim of this chapter is then to give conditions that enable one to decide on the basis of the a priori information, namely $\mathcal{C}$, T, $\mathcal{V}$ and the measured data

$$y := \bar{C}_{\text{actual}}(f) \tag{7.2}$$

whether or not the task (7.1) has been accomplished.

Some examples of task functions that are used later in this thesis include the following: T_{pp} designates the *point-to-point task function*, which is defined on $\mathcal{F}_{\text{pp}} := \mathbb{P}^3 \times \mathbb{P}^3$ by

$$\{f_1, f_2\} \longmapsto \begin{cases} 0 & f_1 = f_2 \\ 1 & f_1 \neq f_2 \end{cases} \tag{7.3}$$

T_{coll} designates the *collinearity task function*, which is defined on $\mathcal{F}_{\text{coll}} := \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$ by

$$\{f_1, f_2, f_3\} \longmapsto \begin{cases} 0 & \text{all the } f_i \text{ are on a line in } \mathbb{P}^3 \\ 1 & \text{otherwise} \end{cases} \tag{7.4}$$

and T_{copl} designates the *coplanarity task function*, which is defined on $\mathcal{F}_{\text{copl}} := \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$ by

$$\{f_1, f_2, f_3, f_4\} \longmapsto \begin{cases} 0 & \text{all the } f_i \text{ are on a plane in } \mathbb{P}^3 \\ 1 & \text{otherwise} \end{cases} \tag{7.5}$$

7.2 Task Encodings

In case C_{actual} were precisely known, a strategy for determining whether or not a task $T(f) = 0$ had been accomplished would be to first compute the feature list f from the measurement y using the formula

$$y = \bar{C}^{-1}_{\text{actual}}(f)$$

where $\bar{C}^{-1}_{\text{actual}}$ is a left inverses of $\bar{C}_{\text{actual}}$, and then to evaluate $T(f)$. The left inverse always exists because $\bar{C}_{\text{actual}}$ is injective, but it cannot be computed since C_{actual} is not presumed known. Because of this, in general, the feature list f cannot be reconstructed from the observed data y. It is thus not clear if, on the basis of this data, it is possible to decide whether or not a given task has been accomplished.

To make precise what the issue is, we formalize the notion of making decisions as to whether a task has been accomplished, using only the measured data y. Toward this end, let us call a function $E_T : \mathcal{Y}^n \to \mathbb{R}$ an *encoded task function* if it can be constructed based only on knowledge of the available a priori information, namely the set of admissible two-camera models $\mathcal{C}$, the joint field of view $\mathcal{V}$, and the task function T. In particular, it must be possible to construct E_T without knowledge of the actual camera model C_{actual}. With E_T so constructed, the equation

$$E_T(y) = 0 \tag{7.6}$$

is said to be an *encoding* of task (7.1) or simply the *encoded task*. In case (7.6) holds we say that the encoded task is *accomplished at y*. In a positioning feedback control system, $E_T(y)$ would thus be a logical choice for a positioning error (cf. Section 7.3).

7.2.1 Verifiability

Thus far, we have not made any formal connection between the accomplishment of a task and the accomplishment of an encoding of that task. To this end, we say that a task $T(f) = 0$ is *verifiable on $\mathcal{C}$* with an encoding $E_T(y) = 0$ if,

$$T(f) = 0 \quad \Longleftrightarrow \quad E_T(y)|_{y=\bar{C}(f)} = 0, \qquad \forall f \in \mathcal{F}, \;\; C \in \mathcal{C} \tag{7.7}$$

In other words, $T(f) = 0$ is verifiable on $\mathcal{C}$ with a given encoding $E_T(y) = 0$, if for each feature $f \in \mathcal{F}$ and each admissible two-camera model C in $\mathcal{C}$, the task $T(f) = 0$ is accomplished at f just in case the encoded task $E_T(y) = 0$ is accomplished at $y = \bar{C}(f)$. Note that

$$E_T(y)|_{y=\bar{C}(f)} = (E_T \circ \bar{C})(f)$$

where $E_T \circ \bar{C}$ is the composition of E_T with $\bar{C}$. From this and the definition of verifiability, it follows that $T(f) = 0$ is verifiable on $\mathcal{C}$ with $E_T(y) = 0$ just in case, for each $C \in \mathcal{C}$, the set of features that T maps into zero is the same as the set of features that $E_T \circ \bar{C}$ maps into zero. We can thus state the following.

Lemma 7.1 (Verifiability). *A task $T(f) = 0$ is verifiable on $\mathcal{C}$ with the encoded task $E_T(y) = 0$ if and only if*

$$\operatorname{Ker} T = \operatorname{Ker} E_T \circ \bar{C}, \qquad \forall C \in \mathcal{C}$$

For example, the point-to-point task $T_{\text{pp}}(f) = 0$ is verifiable on any set of admissible two-camera models $\mathcal{C}$ by an encoding specified by the encoded task function E_{pp} defined on $\mathcal{Y} \times \mathcal{Y}$, $\mathcal{Y} := \mathbb{P}^2 \times \mathbb{P}^2$, by the rule

$$\{y_1, y_2\} \longmapsto \begin{cases} 0 & y_1 = y_2 \\ 1 & y_1 \neq y_2 \end{cases}$$

7.2.2 Decidability

With the preceding definitions in place, we can now pose the central question of this chapter: "When, in the absence of knowledge of the actual camera model, can a task be verified?" Formally, let us call a given task *decidable on $\mathcal{C}$*, if it is verifiable on $\mathcal{C}$ with some encoding. In other words, $T(f) = 0$ is decidable on $\mathcal{C}$ if there exists an encoding $E_T(y) = 0$ for which (7.7) holds. The notion of decidability singles out those tasks that are verifiable without regard to a particular encoding they might be verified with.

In order to further characterize decidable tasks, let $\mathcal{E}\{A_1,\ A_2\}$ denote the set

$$\mathcal{E}\{A_1,\ A_2\} := \{\{z_1,\ z_2\} : A_1(z_1) = A_2(z_2)\}$$

where A_1 and A_2 are given functions with the same codomain. With this definition we can state the following:

Lemma 7.2 (Decidability). *A task $T(f) = 0$ is decidable on $\mathcal{C}$ if and only if for each pair C_1, C_2 in $\mathcal{C}$,*

$$\mathcal{E}\{\bar{C}_1,\ \bar{C}_2\} \subset \mathcal{E}\{T,\ T\} \tag{7.8}$$

This lemma states that $T(f) = 0$ is decidable on $\mathcal{C}$ just in case for each pair $C_1, C_2 \in \mathcal{C}$, the set of pairs $f, g \in \mathcal{F}$ at which $\bar{C}_1(f)$ and $\bar{C}_2(g)$ are equal, consists of pairs at which $T(f)$ and $T(g)$ are equal.

Proof of Lemma 7.2. Suppose $T(f) = 0$ is decidable. Then by Lemma 7.1, there must be an encoded task function E_T for which

$$\operatorname{Ker} T = \operatorname{Ker} E_T \circ \bar{C}, \qquad C \in \mathcal{C} \tag{7.9}$$

Fix $C_1, C_2 \in \mathcal{C}$ and $\{f, g\} \in \mathcal{E}\{\bar{C}_1, \bar{C}_2\}$. Then $\bar{C}_1(f) = \bar{C}_2(g)$, so $E_T \circ \bar{C}_1(f) = E_T \circ \bar{C}_2(g)$. From this and (7.9) it follows that $T(f) = 0$ if and only if $T(g) = 0$. Therefore $\{f, g\} \in \mathcal{E}\{T,\ T\}$. Hence (7.8) holds.

Now suppose that (7.8) is true. Then $T(f) = T(g)$ whenever $C_1, C_2 \in \mathcal{C}$ and $f, g \in \mathcal{F}$ are such that $\bar{C}_1(f) = \bar{C}_2(g)$. Hence the assignment

$$y \longmapsto \begin{cases} T(f) & y = \bar{C}(f) \text{ for some } f \in \mathcal{F} \text{ and } C \in \mathcal{C} \\ 1 & \text{otherwise} \end{cases} \tag{7.10}$$

specifies a well-defined function $E_T : \mathcal{Y}^n \to \mathbb{R}$ that, for every $C \in \mathcal{C}$, satisfies

$$E_T(y)|_{y=\bar{C}(f)} = T(f), \qquad \forall f \in \mathcal{F}$$

Therefore, $E_T \circ \bar{C} = T$, $C \in \mathcal{C}$. It follows that (7.9) is true. Thus by Lemma 7.1, $T(f) = 0$ is verifiable on $\mathcal{C}$ with $E_T(y) = 0$. Therefore $T(f) = 0$ is decidable. ∎

Lemma 7.2 ties the notion of decidability to the properties of the task and the class of two-camera models upon which an encoding of the task is constructed. While guaranteeing the existence of an encoding, it does not, however, say anything about the form of the encoding. It is often desirable for an encoded task function to be continuous so that it may be used, for example, as an error function within a feedback control algorithm. It turns out that under mild technical conditions on T and $\mathcal{C}$, and the assumption that $T(f) = 0$ is decidable, it is always possible to verify $T(f) = 0$ with an encoded task whose task function is continuous. To illustrate this, suppose that $\mathcal{V}$ is a compact subset of $\mathbb{R}^3$, that $\mathcal{Y}$ is $\mathbb{R}^2 \times \mathbb{R}^2$, that $\|\cdot\|$ is a norm on $\mathcal{Y}$, that $\mathcal{C}$ is of the form $\mathcal{C} = \{C_p : p \in \mathcal{P}\}$ where $\mathcal{P}$ a compact subset of a finite dimensional linear space, that $\{p, f\} \mapsto C_p(f)$ is a continuous function on $\mathcal{P} \times \mathcal{V}$, that the kernel of T is compact, and that $T(f) = 0$ is decidable on $\mathcal{C}$. Set

$$\mathcal{I} := C^*(\mathcal{P} \times \operatorname{Ker} T)$$

where $C^* : \mathcal{P} \times \mathcal{F} \to \mathcal{Y}^n$ is the function defined by the rule $\{p, f\} \mapsto \bar{C}_p(f)$, and consider the encoded task function $E_T : \mathcal{Y}^n \to \mathbb{R}$ defined by

$$y \longmapsto \inf_{z \in \mathcal{I}} \|y - z\| \tag{7.11}$$

The following lemma states that E_T has the desired properties.

Lemma 7.3. *Under the above conditions, the task $T(f) = 0$ is verifiable on $\mathcal{C}$ with the encoding $E_T(y) = 0$. Moreover, E_T is globally Lipschitz continuous on $\mathcal{Y}$.*

Proof of Lemma 7.3. We claim that $\mathcal{I}$ is compact. To prove that this is so, first note that $\mathcal{P} \times \operatorname{Ker} T$ is compact because both $\mathcal{P}$ and $\operatorname{Ker} T$ are compact sets. Now the definition of C^* together with the assumed continuity of $\{p, f\} \mapsto C_p(f)$, implies that C^* is continuous. Therefore, $\mathcal{I}$ must be compact since it is the image of a compact set under a continuous function.

To establish the Lipschitz continuity of E_T, let y_1 and y_2 be any two vectors in $\mathcal{Y}^n$. Assume without loss of generality that $E_T(y_1) \le E_T(y_2)$. Since $\mathcal{I}$ is compact, $\inf_{z \in \mathcal{I}} \|y_1 - z\|$ must be attained at some point $y^* \in \mathcal{I}$. In view of (7.11), $E_{T_{\text{new}}}(y_1) = \|y_1 - y^*\|$ and $E_T(y_2) \le \|y_2 - y^*\|$. It follows that

$$|E_T(y_2) - E_T(y_1)| \le \|y_2 - y^*\| - \|y_1 - y^*\| \le \|y_2 - y_1\|$$

Thus E_T is globally Lipschitz continuous as claimed.

In view of Lemma 7.1, to prove that $T(f) = 0$ is verifiable on $\mathcal{C}$ with $E_T(y) = 0$, it is sufficient to show that for each $p \in \mathcal{P}$,

$$\operatorname{Ker}(E_T \circ \bar{C}_p) = \operatorname{Ker} T$$

Toward this end first note that, because $\mathcal{I}$ is a compact set, the kernel of E_T is exactly the set $\mathcal{I}$. Next note that because of the definitions of $\mathcal{I}$ and C^*, $\mathcal{I}$ can also be written as

$$\mathcal{I} = \bigcup_{p \in \mathcal{P}} \bar{C}_p(\operatorname{Ker} T) \tag{7.12}$$

Thus for any $f \in \operatorname{Ker} T$, it must be true that $\bar{C}_p(f) \in \mathcal{I}$, $p \in \mathcal{P}$. But $\mathcal{I} = \operatorname{Ker} E_T$, so $\bar{C}_p(f) \in \operatorname{Ker} E_T$, $p \in \mathcal{P}$. This implies that $f \in \operatorname{Ker}(E_T \circ \bar{C}_p)$, $p \in \mathcal{P}$ and thus that $\operatorname{Ker} T \subset \operatorname{Ker}(E_T \circ \bar{C}_p)$, $p \in \mathcal{P}$.

For the reverse inclusion, fix $p \in \mathcal{P}$ and $f \in \operatorname{Ker}(E_T \circ \bar{C}_p)$. Then $\bar{C}_p(f) \in \operatorname{Ker} E_T$. Therefore $\bar{C}_p(f) \in \mathcal{I}$. In view of (7.12) there must be some $q \in \mathcal{P}$ and $g \in \operatorname{Ker} T$ such that $\bar{C}_p(f) = \bar{C}_q(g)$. Because of Lemma 7.2 this means that $T(f) = T(g)$. But $T(g) = 0$, so $T(f) = 0$ or equivalently $f \in \operatorname{Ker} T$. Hence $\operatorname{Ker}(E_T \circ \bar{C}_p) \subset \operatorname{Ker} T$, $p \in \mathcal{P}$. ∎

7.2.3 Types of Encodings

In the proof of Lemma 7.2 it is shown how to construct an encoding $E_T(y) = 0$ that verifies a given task $T(f) = 0$ that is decidable on $\mathcal{C}$ (cf. equation (7.10)). However, computation of the encoded task function given by (7.10) is often an intractable problem. In the sequel we discuss two well known types of encoding that are more attractive from the tractability point of view.

Cartesian-Based Encoding

The concept of "Cartesian-based" or "position-based" encoding can be motivated by the heuristic idea of "certainty equivalence." In the present context, certainty equivalence advocates that one should use an estimate f_{est} of f to accomplish task (7.1), with the understanding that f_{est} is to be viewed as correct even though it may not be. The construction of such an estimate starts with the selection (by some means) of a two-camera model C_{est} in $\mathcal{C}$ that is considered to be an estimate of C_{actual}. With (7.2) providing motivation, f_{est} is then defined by an equation of the form

$$f_{\text{est}} := \bar{C}_{\text{est}}^{-1}(y) \tag{7.13}$$

where $\bar{C}_{\text{est}}^{-1}$ is one of the left inverses of $\bar{C}_{\text{est}}$. Note that such a left inverse can always be found because $\bar{C}_{\text{est}}$ is injective. In accordance with certainty equivalence, a *Cartesian-based encoding* of $T(f) = 0$ is then

$$T\big(\bar{C}_{\text{est}}^{-1}(y)\big) = 0$$

The encoded task function is thus $E_T := T \circ \bar{C}_{\text{est}}^{-1}$.

In light of Lemma 7.1, it is clear that a given task $T(f) = 0$ will be verifiable on $\mathcal{C}$ by a Cartesian-based encoding of the form

$$(T \circ \bar{C}_{\text{est}}^{-1})(y) = 0$$

only if

$$\operatorname{Ker} T = \operatorname{Ker} T \circ \bar{C}_{\text{est}}^{-1} \circ \bar{C}, \qquad \forall C \in \mathcal{C}$$

It is worth noting that, to encode a given task in this way, it is necessary to pick *both* a model C_{est} from $\mathcal{C}$ and a left inverse of its extension. Because such left inverses are generally not unique, there are many ways to encode $T(f) = 0$ in this manner, even after C_{est} has been chosen.

Image-Based Encoding

As defined above, a Cartesian-based encoding takes the special form $T \circ \bar{C}_{\text{est}}^{-1}$. In practice, it has become common to call only those encodings $E_T(y) = 0$ for which E_T does not so factor, *image-based*. For theoretical purposes this distinction proves to be somewhat awkward, because those encoded task functions that do not so factor are difficult to characterize. This is why, for the purposes of this thesis, the set of image-based encodings of $T(f) = 0$ on $\mathcal{C}$ and the set of *all* encodings of $T(f) = 0$ on $\mathcal{C}$ are taken to be one and the same.

As defined, an image-based encoding may or may not make use of an estimate of C_{actual}. It should be noted however, that even when such an estimate is not used in the encoding, a reasonably good estimate of C_{actual} is often necessary in practice. For example, such an estimate may be necessary to construct a feedback controller that provides (at least) loop-stability and consequently a guarantee that (in the absence of measurement noise) the encoded task can be accomplished in the first place. However, for any (image-based) encoding that verifies $T(f) = 0$ on $\mathcal{C}$, the quality of the approximate two-camera model chosen from $\mathcal{C}$ may affect the feedback controller's ability to accomplish the *encoded* task, but, once the encoded task is accomplished, the initially defined task is guaranteed to be precisely accomplished as well [89].

7.2.4 Equivalent Tasks

It is sometimes desirable to reformulate a positioning task $T(f) = 0$, initially defined on $\mathcal{F} \subset \mathcal{V}^n$, on a new feature space $\mathcal{F}_{\text{new}} \subset \mathcal{V}^m$. In this section we explain how one might go about constructing a new task $T_{\text{new}}(f_{\text{new}}) = 0$ defined on $\mathcal{F}_{\text{new}}$ in such a way that accomplishing the new task is equivalent to accomplishing the original task $T(f) = 0$. In what follows, let $\mathcal{F}_{\text{new}}$ be fixed and write $\bar{C}_{\text{new}}$ for the extension of $C \in \mathcal{C}$ to $\mathcal{F}_{\text{new}}$. We say that a function $H : \mathcal{F} \to \mathcal{F}_{\text{new}}$ *factors through* $\mathcal{C}$ if there is a function K such that

$$\bar{C}_{\text{new}} \circ H = K \circ \bar{C}, \qquad \forall C \in \mathcal{C} \tag{7.14}$$

Intuitively, the functions H and K play the roles of feature and measured data transformations, respectively.

Suppose that T is any given task function that can be factored as

$$T = T_{\text{new}} \circ H$$

where T_{new} is a new task function and H is a surjective function that factors through $\mathcal{C}$ for some K. Let f_{new} and y_{new} denote new feature and output vectors respectively, defined by the equations

$$f_{\text{new}} := H(f) \qquad\qquad y_{\text{new}} := K(y)$$

Then the tasks $T(f) = 0$ and $T_{\text{new}}(f_{\text{new}}) = 0$ are *equivalent* in the sense that

$$T(f) = 0 \iff T_{\text{new}}(f_{\text{new}})|_{f_{\text{new}}=H(f)} = 0, \qquad \forall f \in \mathcal{F}$$

Because of (7.14) one can also verify that

$$y_{\text{new}} = \bar{C}_{\text{new}}(f_{\text{new}})$$

where $\bar{C}_{\text{new}}$ is the extension of C_{actual} to $\mathcal{F}_{\text{new}}$.

Suppose that $E_{T_{\text{new}}}(y_{\text{new}}) = 0$ is an encoding that verifies $T_{\text{new}}(f_{\text{new}}) = 0$ on $\mathcal{C}$. Then, because of the surjectivity of H and the equivalence of $T_{\text{new}}(f_{\text{new}}) = 0$ and $T(f) = 0$ noted above, it must be true that verifiability of $T_{\text{new}}(f_{\text{new}}) = 0$ on $\mathcal{C}$ with $E_{T_{\text{new}}}(y_{\text{new}}) = 0$ implies verifiability of $T(f) = 0$ on $\mathcal{C}$ with $E_{T_{\text{new}}}(y_{\text{new}}) = 0$. It is not difficult to prove that the converse must also be true; that is, if $T(f) = 0$ is verifiable on $\mathcal{C}$ with the encoded task $(E_{T_{\text{new}}} \circ K)(y) = 0$, then $T_{\text{new}}(f_{\text{new}}) = 0$ is necessarily verifiable on $\mathcal{C}$ with the encoded task $E_{T_{\text{new}}}(y_{\text{new}}) = 0$. We have thus found a procedure to construct encodings for "complex" tasks like $(T_{\text{new}} \circ H)(f) = 0$ based on encodings of "simpler" tasks like $T_{\text{new}}(f) = 0$, preserving verifiability.

Modified Cartesian-Based Encoding

One of the motivations for the use of Cartesian-based encodings is that they utilize estimates of features (e.g., f_{est} in equation (7.13)) taking values in Cartesian space, which is the natural space for specifying robot positioning tasks. However, it is also clear that Cartesian-based encodings are more restrictive than image-based encodings. The "modified" Cartesian-based encoding, introduced in [93], is one way of generalizing the idea of Cartesian-based encodings to achieve verifiability for a richer set of tasks.

Formally, a *modified Cartesian-based encoding* of a given task $T(f) = 0$ is any encoding of the form

$$(T_{\text{new}} \circ \bar{C}^{-1}_{\text{est}_{\text{new}}} \circ K)(y) = 0 \tag{7.15}$$

where T_{new} is a factor of T in a formula of the form $T = T_{\text{new}} \circ H$, H is a surjective function that factors through $\mathcal{C}$ with K, and $\bar{C}^{-1}_{\text{est}_{\text{new}}}$ is a left inverse of the extension of an estimate $C_{\text{est}} \in \mathcal{C}$ to $\mathcal{F}_{\text{new}}$. The form of a modified Cartesian-based task function is thus $E_T = T_{\text{new}} \circ \bar{C}^{-1}_{\text{est}_{\text{new}}} \circ K$. Note that

$$(T_{\text{new}} \circ \bar{C}^{-1}_{\text{est}_{\text{new}}})(y_{\text{new}}) = 0, \tag{7.16}$$

is a Cartesian-based encoding of $T_{\text{new}}(f_{\text{new}}) = 0$. Thus a modified Cartesian-based encoding of $T(f) = 0$ can be thought of as a transformed version of a Cartesian-based encoding of $T_{\text{new}}(f_{\text{new}}) = 0$. Moreover, if (7.16) verifies $T_{\text{new}}(f_{\text{new}}) = 0$ on $\mathcal{C}$, then (7.15) verifies $T(f) = 0$ on $\mathcal{C}$.

A Cartesian-based encoding is a special case of a *modified* Cartesian-based encoding in which $\mathcal{F}_{\text{new}} = \mathcal{F}$, H is the identity on $\mathcal{F}$ and $T_{\text{new}} = T$. Thus every Cartesian-based encoding of a given task $T(f) = 0$ on $\mathcal{C}$ is a modified Cartesian-based encoding, whereas every modified Cartesian-based encoding is an image-based encoding. In other words, for any given task $T(f) = 0$, we have the ordering

$$\mathcal{E}_{\text{Cart}}[T] \subset \mathcal{E}_{\text{modCart}}[T] \subset \mathcal{E}_{\text{image}}[T]$$

where $\mathcal{E}_{\text{Cart}}[T]$, $\mathcal{E}_{\text{modCart}}[T]$, and $\mathcal{E}_{\text{image}}[T]$ are respectively, the classes of all Cartesian-based, modified Cartesian-based, and image-based encodings that verify the task $T(f) = 0$ on $\mathcal{C}$.

7.3 Concluding Remarks

In this chapter we addressed the problem of deciding if a prescribed robot positioning task $T(f) = 0$ has been accomplished using images acquired by a two-camera system whose model C_{actual} is an unspecified element of a known class of two-camera models $\mathcal{C}$. The task is said to be decidable on $\mathcal{C}$ if the available information, namely T, $\mathcal{C}$, $\mathcal{V}$, and the images of the point features, is sufficient to determine whether or not the task has been accomplished.

Suppose now that one is given a task $T(f) = 0$ that is decidable on $\mathcal{C}$ and that one is able to find an encoding $E_T(y) = 0$ that verifies $T(f) = 0$ on $\mathcal{C}$. A feedback control architecture that can be used to control the position of a robot so as to asymptotically accomplish the task $T(f) = 0$ is shown in Figure 7.1. In this figure the feature list f is decomposed into robot features and target defining features. Since the error signal $e := E_T(y)$ is fed into an integrator, this architecture has the property that $e = 0$ at any equilibrium point of the closed-loop system. Because the encoding $E_T(y) = 0$ verifies the task $T(f) = 0$ on $\mathcal{C}$ and $C_{\text{actual}} \in \mathcal{C}$, the task is then accomplished at any equilibrium point of the closed loop system, *no matter what the process, control law, and actual two-camera models are.* In general, when the imprecision in any of these models is large, the methodology described in Part II of this thesis may have to be used to design a control law that asymptotically stabilizes the closed loop system around some equilibrium point.

The architecture in Figure 7.1 is given just to illustrate how the results in this chapter can be used to design feedback controllers capable of asymptotically accomplishing a given

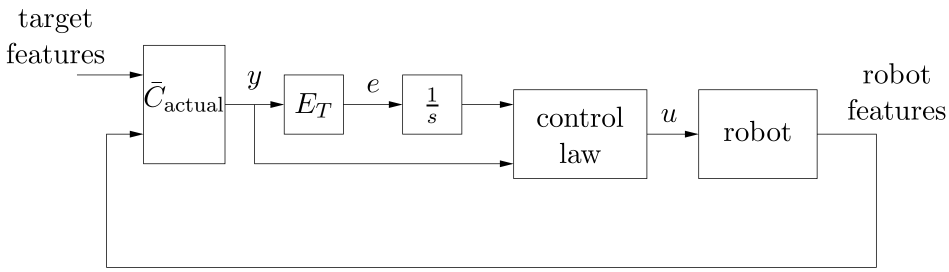

Figure 7.1: Feedback control architecture.

robot positioning task. In fact, it is often difficult to design a control law that asymptotically stabilizes the closed loop system in this figure. More on the design of feedback controllers capable of asymptotically accomplishing robot positioning tasks, can be found in the Doctoral Thesis [92].

Chapter 8

Decidability on Sets of Projective Camera Models

In the previous chapter, the two-camera models are assumed to be injective functions, but nothing more. However, it is well known that two-camera systems have a rich geometric structure. This has led to recent work in the vision literature [94] considering the following question:

> "What can be seen in three dimensions with an uncalibrated stereo rig?"

Roughly speaking, [94] shows that, in the absence of measurement noise, using a two-camera vision system that has been calibrated just using images of point features, it is possible to exactly reconstruct the positions of other point features "up to a projective transformation" on three-dimensional projective space. A two-camera system calibrated using only measured point correspondences is said to be weakly calibrated [95].

These findings suggest that for a weakly calibrated two-camera model class, there ought to be a close relationship between the decidability of a given task $T(f) = 0$ and the invariant properties of the task function T under projective transformations. The main result of this chapter, Theorem 8.3, states that a given task is decidable on a weakly calibrated two-camera class *if and only if* the task is a projective invariant. This result thus serves to underscore the observation made in [89, 96, 84, 97] that accurate metric information is not needed for the accomplishment of many types of positioning tasks with a stereo vision system.

The two-camera models considered in this chapter are pairs of projective camera models which map subsets of $\mathbb{P}^3$ containing $\mathcal{V}$, into $\mathbb{P}^2 \times \mathbb{P}^2$. Projective models of this type have been widely used in computer vision [98, 99] in part because they include as special cases, perspective, affine and orthographic camera models. By restricting our attention to projective models, we are able to provide a complete and concise characterization of decidable tasks in terms of projective invariance.

The remainder of this chapter is structured as follows. In Section 8.1 two-camera systems modeled using projective geometry are discussed. The central result of this chapter—a necessary and sufficient for decidability on a weakly calibrated two-camera class—is presented in Section 8.2. In Section 8.3 a necessary condition for decidability on an uncalibrated two-camera class is also given. Finally, Section 8.4 contains some concluding remarks and directions for future research.

8.1 Camera Models

In the sequel we are concerned with camera models whose fields of view are all the same subset $\mathcal{V} \subset \mathbb{P}^3$. We construct this field of view by first defining a nonempty subset $\mathcal{B}$ of $\mathbb{R}^3$, and then defining $\mathcal{V}$ to be

$$\mathcal{V} := \left\{ \mathbb{R} \begin{bmatrix} w \\ 1 \end{bmatrix} : w \in \mathcal{B} \right\}.$$

Note that, by this construction, we exclude the so-called "points at infinity" in $\mathbb{P}^3$, namely points in $\mathbb{P}^3$ which have no corresponding point in $\mathbb{R}^3$.

For any real 3×4 full-rank matrix M, we write $\mathbb{P}_M$ to denote the set of all points in $\mathbb{P}^3$ except for $\operatorname{Ker} M$, and write $\mathbf{M}$ for the function from $\mathbb{P}^3_M$ to $\mathbb{P}^2$ defined by the rule $\mathbb{R}x \longmapsto \mathbb{R}Mx$. We call $\mathbf{M}$ the *global camera model induced by* M and we call $\operatorname{Ker} M$ the *optical center* of $\mathbf{M}$. In the event that $\operatorname{Ker} M \notin \mathcal{V}$, the restricted function $\mathcal{V} \to \mathbb{P}^2$, $v \longmapsto \mathbf{M}(v)$ is well-defined. We denote this function by $\mathbf{M}|\mathcal{V}$ and refer to it as the *camera model determined by* $\mathbf{M}$ *on* $\mathcal{V}$.

We note that when it is possible to write M in the form

$$M = \begin{bmatrix} H & -Hc \end{bmatrix}$$

where H is a nonsingular 3×3 matrix, and c is a vector in $\mathbb{R}^3$, then $\mathbf{M}$ models a projective camera with center of projection at c [99]. In this case the kernel of M is $\mathbb{R}\left[\begin{smallmatrix} c \\ 1 \end{smallmatrix}\right]$ which justifies calling it the optical center of $\mathbf{M}$. One special case occurs when $H = R$, where R is a 3×3 rotation matrix. In this case, $\mathbf{M}$ models a perspective camera with unit focal length, optical center at $c \in \mathbb{R}^3$, and orientation defined by R. On the other hand, it is also possible for M to be of the form

$$M = \begin{bmatrix} \bar{R} & -\bar{R}c \\ 0_{1\times 3} & 1 \end{bmatrix}$$

where $\bar{R} := \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right] R$ and R is a rotation matrix. In this case, $\mathbf{M}$ models an orthographic camera whose coordinate frame is defined by the rotation matrix R. The kernel of M is $\mathbb{R}\left[\begin{smallmatrix} r_3 \\ 0 \end{smallmatrix}\right]$ where r_3 is the third column of R'. This model also can be regarded as a perspective camera model with optical center at infinity and "orientation" defined by the z-axis of the coordinate frame.

8.1.1 Two-Camera Models

Having established a basic camera model, we now extend these ideas to define what is meant by an uncalibrated, projective two-camera model. We take the joint image-space and joint field of view of the two-camera model[1] to be $\mathcal{Y} := \mathbb{P}^2 \times \mathbb{P}^2$ and $\mathcal{V}$ respectively, where $\mathcal{V}$ is a subset of the form described in the previous subsection. For each pair of real 3×4 full-rank matrices $\{M, N\}$ with distinct kernels, let $\mathbb{P}^3_{\{M,N\}}$ denote the set of all points in $\mathbb{P}^3$ except for $\operatorname{Ker} M$ and $\operatorname{Ker} N$. Let $\{\mathbf{M}, \mathbf{N}\}$ denote the function from $\mathbb{P}^3_{\{M,N\}}$ to $\mathcal{Y}$ defined by the rule $\mathbb{R}x \longmapsto \{\mathbb{R}Mx, \mathbb{R}Nx\}$. We call $\{\mathbf{M}, \mathbf{N}\}$ the *global two-camera model induced by* $\{M, N\}$. For each model $G = \{\mathbf{M}, \mathbf{N}\}$ for which neither $\operatorname{Ker} M$ nor $\operatorname{Ker} N$ are in $\mathcal{V}$, it is possible to define the restricted function $\mathcal{V} \to \mathcal{Y}$, $v \longmapsto G(v)$. We denote this function by $G|\mathcal{V}$ and refer to it as the *two-camera model determined by* G *on* $\mathcal{V}$.

[1]Henceforth, the term "two-camera model" is taken to mean a projective two-camera model.

Given a two-camera model $G = \{\mathbf{M}, \mathbf{N}\}$, the line in $\mathbb{P}^3$ on which the optical centers of $\mathbf{M}$ and $\mathbf{N}$ lie, namely $\operatorname{Ker} M \oplus \operatorname{Ker} N$, is called the *baseline of* G. It is perhaps not surprising that the mapping by G into $\mathcal{Y}$, of points in $\mathbb{P}^3_{\{M,N\}}$ which do not lie on this baseline, is one to one. In the sequel, for each line ℓ in $\mathbb{P}^3$, we write $\mathbb{P}^3[\ell]$ for the set of points in $\mathbb{P}^3$ which are not on ℓ.

Lemma 8.1. *Let ℓ be the baseline of G. The restricted function $\mathbb{P}^3[\ell] \to \mathcal{Y}$, $g \longmapsto G(g)$ is injective.*

Proof of Lemma 8.1. Let $\bar{f}, \bar{g} \in \mathbb{R}^4$ represent two points $f, g \in \mathbb{P}^3[\ell]$ at which $G(f) = G(g)$. Suppose $G = \{\mathbf{M}, \mathbf{N}\}$. It follows that $\bar{f}$, and $\bar{g}$ are representations for which there exist numbers λ and μ such that $\bar{f} - \lambda\bar{g} \in \operatorname{Ker} M$ and $\bar{f} - \mu\bar{g} \in \operatorname{Ker} N$. Then $(\lambda - \mu)\bar{g}$ must be a vector in the baseline $\ell := \operatorname{Ker} M \oplus \operatorname{Ker} N$. Since $g \in \mathbb{P}^3[\ell]$, $\bar{g}$ cannot be in ℓ. Therefore $\lambda = \mu$. Thus $\bar{f} - \lambda\bar{g} \in \operatorname{Ker} M \cap \operatorname{Ker} N$; but $\operatorname{Ker} M \cap \operatorname{Ker} N = 0$, so $\bar{f} = \lambda\bar{g}$. This implies that $f = g$ and thus that the restriction of $\{\mathbf{M}, \mathbf{N}\}$ to $\mathbb{P}^3[\ell]$ is injective. ■

In the sequel we say that the baseline of G lies outside of $\mathcal{V}$ if there is no point on the baseline of G which is also in $\mathcal{V}$. Note that $\mathcal{V}$ is automatically contained in the domain of any global two-camera model whose baseline lies outside of $\mathcal{V}$. Each such global model G thus determines a two-camera model $C = G|\mathcal{V}$. By the set of all *uncalibrated two-camera models on* $\mathcal{V}$, written $\mathcal{C}_{\text{uncal}}[\mathcal{V}]$, is meant the set of all two-camera models which are determined by global two-camera models whose baselines lie outside of $\mathcal{V}$. Any stereo vision system whose two cameras admit a model which is known to be in this class but is otherwise unknown, is said to be *uncalibrated*.

8.1.2 Weakly Calibrated Stereo Vision Systems

It is well-known that, given measurements by a stereo camera system of a sufficient number of points in "general position", it is possible to compute a one-dimensional constraint on the (stereo) projection of any point features in the two-camera field of view [124, 99]. A stereo camera system for which this information, the so-called "epipolar constraint," is known is often referred to as *weakly calibrated* [95]. It has been shown [94] that, with a "weakly calibrated" stereo system, it is possible to reconstruct the position of point features in the two cameras' field of view from image measurements. However, this reconstruction is only unique up to a projective transformation.

These findings suggest that there should be a connection between the decidability of a task $T(f) = 0$ on a weakly calibrated camera class and the properties of $T(f) = 0$ which are invariant under projective transformations. In this section, we demonstrate that this is in fact the case. As a result we are able to concisely characterize the set of all those tasks which are decidable using a weakly calibrated stereo camera system.

The first step is to make precise in the present context what we mean by a weakly calibrated stereo vision system. Toward this end, let us write GL(4) for the general linear group of real, nonsingular, 4×4 matrices. For each such matrix A, $\mathbf{A}$ denotes the corresponding projective transformation $\mathbb{P}^3 \to \mathbb{P}^3$, $\mathbb{R}x \longmapsto \mathbb{R}Ax$.

For each fixed global two-camera model $G_0 := \{\mathbf{M_0}, \mathbf{N_0}\}$ whose baseline ℓ lies outside of $\mathcal{V}$, let $\mathcal{C}[G_0]$ denote the set of two-camera models

$$\mathcal{C}[G_0] := \{(G_0 A)|\mathcal{V} : A \in \mathrm{GL}(4),\ \text{and}\ \mathbf{A}(\mathcal{V}) \subset \mathbb{P}^3[\ell]\}$$

where G_0A is the global two-camera model induced by $\{M_0A, N_0A\}$. Note that $G_0|\mathcal{V}$ is in $\mathcal{C}[G_0]$ because the baseline of G_0 is assumed to lie outside of $\mathcal{V}$. Note also that for any $A \in \mathrm{GL}(4)$ such that $\mathbf{A}(\mathcal{V}) \subset \mathbb{P}^3_{\{M_0,N_0\}}$, $(G_0A)|\mathcal{V}$ is well defined and is equal to $G_0 \circ (\mathbf{A}|\mathcal{V})$ where $\mathbf{A}|\mathcal{V}$ is the restricted function $\mathcal{V} \to \mathbb{P}^3_{\{M_0,N_0\}}, \quad v \longmapsto \mathbf{A}(v)$. Thus

$$\mathcal{C}[G_0] = \{G_0 \circ (\mathbf{A}|\mathcal{V}) : A \in \mathrm{GL}(4), \text{ and } \mathbf{A}(\mathcal{V}) \subset \mathbb{P}^3[\ell]\}$$

since $\mathbb{P}^3[\ell] \subset \mathbb{P}^3_{\{M_0,N_0\}}$.

Let us note that if G_0A is such that $G_0A|\mathcal{V}$ is in $\mathcal{C}[G_0]$, then $G_0A|\mathcal{V}$ must be an uncalibrated model in $\mathcal{C}_{\mathrm{uncal}}[\mathcal{V}]$. If this were not so, then there would have to be a point p on the baseline of G_0A which is also in $\mathcal{V}$. Under these conditions $\mathbf{A}^{-1}(p)$ would have to be on the baseline of G_0; $\mathbf{A}^{-1}(p)$ would also have to be in $\mathbb{P}^3[\ell]$ because of the definition of $\mathcal{C}[G_0]$. But this is impossible since, by definition, no point on the baseline of G_0 is in $\mathbb{P}^3[\ell]$. We can thus state the following.

Lemma 8.2. $G_0|\mathcal{V} \in \mathcal{C}[G_0]$ *and* $\mathcal{C}[G_0] \subset \mathcal{C}_{\mathrm{uncal}}[\mathcal{V}]$.

We say that a pair of two-camera models have the same *epipolar geometry* if they have the same image when viewed as a function from $\mathcal{V}$ to $\mathcal{Y}$. An equivalent definition for the meaning of "same epipolar geometry" can be given in terms of the fundamental matrix of a global two-camera model [99]. One can show that $\mathcal{C}_{\mathrm{wkcal}}(G_0)$ contains all those two-camera models in $\mathcal{C}_{\mathrm{uncal}}[\mathcal{V}]$ determined by global two-camera models with the same epipolar geometry as G_0. It is common to refer to a stereo vision system whose two cameras admit a model which is known be in $\mathcal{C}[G_0]$ but is otherwise unknown as a *weakly calibrated* system [95].

8.2 Main Result

We now define what is meant by a "projectively invariant task". For each $A \in \mathrm{GL}(4)$, let $\bar{\mathbf{A}}$ denote the extended function from $(\mathbb{P}^3)^n$ to $(\mathbb{P}^3)^n$ defined by the rule

$$\{p_1, p_2, \ldots, p_n\} \longmapsto \{\mathbf{A}(p_1), \mathbf{A}(p_2), \ldots \mathbf{A}(p_n)\}$$

where $(\mathbb{P}^3)^n$ denotes the Cartesian product of $\mathbb{P}^3$ with itself n times. Call two points p and q in $(\mathbb{P}^3)^n$ *projectively equivalent*[2] if there exists an A in $\mathrm{GL}(4)$ such that $p = \bar{\mathbf{A}}(q)$. A task $T(f) = 0$ is said to be *projectively invariant on* $\mathcal{F}$ if for each pair of projectively equivalent points $f, g \in \mathcal{F}$,

$$T(f) = T(g)$$

In other words, $T(f) = 0$ is projectively invariant on $\mathcal{F}$, just in case T is constant on each equivalence class of projectively equivalent features within $\mathcal{F}$. It is shown in Chapter 9 that under mild assumptions, for $n \leq 3$ the number of such equivalence classes $k(n)$ is finite and equals 1, 2, and 6 respectively for n equal to 1, 2, and 3. Under these conditions, the number of distinct tasks that are projectively invariant on $\mathcal{F}$ is $2^{k(n)}$ for $n \in \{1, 2, 3\}$. This is because the codomain of each task function contains only two elements.

The main result of this chapter is as follows.

[2] Projective equivalence is an equivalence relation on $(\mathbb{P}^3)^n$ because the set of extensions to $(\mathbb{P}^3)^n$ of all projective transformations on $\mathbb{P}^3$, is a group with composition rule $\{\bar{\mathbf{A}}_1, \bar{\mathbf{A}}_2\} \longmapsto \bar{\mathbf{A}}_1 \circ \bar{\mathbf{A}}_2$. This is further explored in Chapter 9.

Theorem 8.3 (Weak Calibration). *Let G_0 be a fixed global two-camera model whose baseline lies outside of $\mathcal{V}$. A task $T(f) = 0$ is decidable on $\mathcal{C}[G_0]$ if and only if it is projectively invariant.*

In short, with a weakly calibrated camera system, any projectively invariant task is verifiable with at least one encoding. Moreover, any task which is not projectively invariant is *not* verifiable with any type of encoding.

In the proof which follows, for each global two-camera $G : \mathbb{P}^3_{\{M,N\}} \to \mathcal{Y}$, $\bar{G}$ denotes the extended function from

$$(\mathbb{P}_{\{M,N\}})^n := \underbrace{\mathbb{P}^3_{\{M,N\}} \times \mathbb{P}^3_{\{M,N\}} \times \cdots \times \mathbb{P}^3_{\{M,N\}}}_{n \text{ times}}$$

to $\mathcal{Y}^n$ which is defined by the rule $\{p_1, p_2, \ldots, p_n\} \longmapsto \{G(p_1), G(p_2), \ldots, G(p_n)\}$.

Proof of Theorem 8.3. Suppose that $T(f) = 0$ is projectively invariant. To prove that $T(f) = 0$ is decidable on $\mathcal{C}[G_0]$ let $f, g \in \mathcal{F}$ be any pair of features such that $\bar{C}_1(f) = \bar{C}_2(g)$, where $\bar{C}_1$ and $\bar{C}_2$ are extensions of some two-camera models $C_1, C_2 \in \mathcal{C}[G_0]$. It is enough to show that $T(f) = T(g)$.

Since $C_1, C_2 \in \mathcal{C}[G_0]$, there must be matrices $A_i \in \mathrm{GL}(4)$ such that $C_i = \{G_0 \circ \mathbf{A}_i\}|\mathcal{V}$, $i \in \{1, 2\}$ and

$$\mathbf{A_i}(\mathcal{V}) \subset \mathbb{P}^3[\ell], \qquad\qquad i \in \{1, 2\} \tag{8.1}$$

where ℓ is the baseline of G_0. Thus $\bar{C}_1(f) = \bar{G}_0(\bar{\mathbf{A}}_\mathbf{1}(f))$ and $\bar{C}_2(g) = \bar{G}_0(\bar{\mathbf{A}}_\mathbf{2}(g))$. Hence $\bar{G}_0(\bar{\mathbf{A}}_\mathbf{1}(f)) = \bar{G}_0(\bar{\mathbf{A}}_\mathbf{2}(g))$. Now each point feature in the list $\bar{\mathbf{A}}_\mathbf{1}(f)$ and in the list $\bar{\mathbf{A}}_\mathbf{2}(g)$ must be in $\mathbb{P}^3[\ell]$ because of (8.1). Thus $\bar{\mathbf{A}}_\mathbf{1}(f)$ and $\bar{\mathbf{A}}_\mathbf{2}(g)$ must be points in $(\mathbb{P}^3[\ell])^n$. It follows that $\bar{\mathbf{A}}_\mathbf{1}(f) = \bar{\mathbf{A}}_\mathbf{2}(g)$ since by Lemma 8.1, $G_0|\mathbb{P}^3[\ell]$ is injective, and since injectivity of $G_0|\mathbb{P}^3[\ell]$ implies injectivity of $\bar{G}_0|(\mathbb{P}^3[\ell])^n$. This means that $f = \{\bar{\mathbf{A}}_\mathbf{1}^{-1} \circ \bar{\mathbf{A}}_\mathbf{2}\}(g)$ and thus that f and g are projectively equivalent. Hence $T(f) = T(g)$ because of projective invariance.

To prove the converse, now suppose that $T(f) = 0$ is decidable on $\mathcal{C}[G_0]$. Pick two projectively equivalent features $f, g \in \mathcal{F}$ and a matrix $A \in \mathrm{GL}(4)$ such that $f = \bar{\mathbf{A}}(g)$. Suppose that there are two camera models in $\mathcal{C}[G_0]$ that can be written as $C_1 = G_0 \circ (\mathbf{B}|\mathcal{V})$ and $C_2 = G_0 \circ (\mathbf{B} \circ \mathbf{A}|\mathcal{V})$, with $B \in \mathrm{GL}(4)$. Then, it follows that $\bar{C}_1(f) = \bar{C}_2(g)$ and therefore that $T(f) = T(g)$ because of Lemma 7.2 and the hypothesis of decidability on $\mathcal{C}[G_0]$.

It remains to show that such C_1 and C_2 do exist in $\mathcal{C}[G_0]$. To this end, let ℓ_1 be a projective line contained in $\mathcal{S} \cap A\mathcal{S}$, where $\mathcal{S}$ denotes the subspace of $\mathbb{R}^4$ spanned by the first three columns of the 4×4 identity matrix. Such an ℓ_1 exists because $\mathcal{S}$ and $A\mathcal{S}$ are two 3-dimensional linear subspaces of $\mathbb{R}^4$ and so their intersection is a linear subspace of $\mathbb{R}^4$ with dimension no smaller then 2. Note that no point in $\mathcal{V}$ is a linear subspace of $\mathcal{S}$ in $\mathbb{R}^4$.

Let ℓ be the baseline of G_0 and B a matrix in $\mathrm{GL}(4)$ such that $B\ell_1 = \ell$. We show next that $\mathbf{B}(\mathcal{V}) \subset \mathbb{P}^3[\ell]$ and $\{\mathbf{B} \circ \mathbf{A}\}(\mathcal{V}) \subset \mathbb{P}^3[\ell]$ and therefore that $C_1 := G_0 \circ (\mathbf{B}|\mathcal{V})$ and $C_2 := G_0 \circ (\{\mathbf{B} \circ \mathbf{A}\}|\mathcal{V})$ define camera models in $\mathcal{C}[G_0]$. Suppose that $\mathbf{B}(\mathcal{V})$ is not contained in $\mathbb{P}^3[\ell]$ and therefore that there is a point $v \in \mathcal{V}$ such that $\mathbf{B}(v)$ is on ℓ. From this and the definition of B, v must be on ℓ_1 and therefore it must be a linear subspace of $\mathcal{S}$ in $\mathbb{R}^4$. This contradicts the fact that no point in $\mathcal{V}$ is a linear subspace of $\mathcal{S}$ in $\mathbb{R}^4$. To demonstrate that $\{\mathbf{B} \circ \mathbf{A}\}(\mathcal{V}) \subset \mathbb{P}^3[\ell]$ we proceed similarly. By contradiction, suppose that $v \in \mathcal{V}$ and that $\mathbf{B}\big(\mathbf{A}(v)\big)$ is on ℓ. From this and the definition of B, $\mathbf{A}(v)$ must be on ℓ_1 and therefore it must be a linear subspace of $A\mathcal{S}$ in $\mathbb{R}^4$. Thus v must be a linear subspace of $\mathcal{S}$ in $\mathbb{R}^4$ and we arrive at a similar contradiction as before. ∎

8.3 Uncalibrated Stereo Vision Systems

As it stands, Theorem 8.3 applies only to stereo vision systems which are weakly calibrated. However, $\mathcal{C}[G_0]$ is a subset of $\mathcal{C}_{\text{uncal}}[\mathcal{V}]$. Thus any task that is decidable on $\mathcal{C}_{\text{uncal}}[\mathcal{V}]$ must also be decidable on $\mathcal{C}[G_0]$. Therefore since being projectively invariant is a necessary condition for the task $T(f) = 0$ to be decidable on $\mathcal{C}[G_0]$ it must also be a necessary condition for the task $T(f) = 0$ to be decidable on $\mathcal{C}_{\text{uncal}}[\mathcal{V}]$. We can thus state the following.

Proposition 8.4. *If $T(f) = 0$ is decidable on $\mathcal{C}_{\text{uncal}}[\mathcal{V}]$, then $T(f) = 0$ is projectively invariant.*

The reverse implication, namely that task invariance implies decidability on $\mathcal{C}_{\text{uncal}}[\mathcal{V}]$, is false. For example, suppose that the initial positioning objective is to make 3 point features collinear. This corresponds to the previously defined coplanarity task function, T_{coll}. The task $T_{\text{coll}}(f) = 0$ is projectively invariant because projective transformations preserve collinearity. On the other hand this task is not decidable on $\mathcal{C}_{\text{uncal}}[\mathcal{V}]$. Indeed, there are camera models $C_1, C_2 \in \mathcal{C}_{\text{uncal}}[\mathcal{V}]$ and pairs of features $f, g \in \mathcal{F}$ at which $\bar{C}_1(f)$ and $\bar{C}_2(g)$ are equal and yet the task is accomplished at f but not at g. This happens when all point features in the list f and the optical centers of the global camera models that determine C_1 are coplanar, and also all point features in the list g and the optical centers of the global camera models that determine C_2 are coplanar. In geometric terms, the task fails to be decidable when the feature points and the camera centers all lie in the same epipolar plane. Similarly, there are conditions under which the projectively invariant task function T_{copl} is not decidable on $\mathcal{C}_{\text{uncal}}[\mathcal{V}]$.

It should be emphasized that the requirement of task decidability on $\mathcal{C}_{\text{uncal}}[\mathcal{V}]$ does not rule out all nondegenerate tasks[3]. For example, the point-to-point task $T_{\text{pp}}(f) = 0$, is decidable on $\mathcal{C}_{\text{uncal}}[\mathcal{V}]$, because of the injectivity of the cameras in that set.

8.4 Concluding Remarks

The main results of this chapter can be summarized as follows:

- It is possible to verify that a robot positioning task has been accomplished with absolute accuracy using a weakly calibrated, noise-free, stereo vision system, if and only if the task is invariant under projective transformations on $\mathbb{P}^3$.

- If it is possible to verify that such a robot positioning task has been precisely accomplished using an uncalibrated stereo vision system, then the task must be invariant under projective transformations on $\mathbb{P}^3$.

Several questions still remain unanswered. For example: Is there a simple characterization for the set of tasks that are decidable on the set of uncalibrated camera models? Another questions that deserves attention is the choice of encodings for tasks that are decidable on any of the sets of camera models considered. Finally, it should be noted that there are several other sets of admissible camera models that deserve study. For example, the set of uncalibrated (or weakly calibrated) perspective two-camera models.

[3]By a degenerate task is meant any task specified by a task function that is constant on $\mathcal{F}$. Degenerate tasks are always decidable on any set of admissible camera models, but they are not interesting since their accomplishment is independent of the position of the point features.

Chapter 9

Projective Invariance

It was seen in Chapter 8 that projectively invariant tasks are of special importance when dealing with sets of projective camera models. In fact, projective invariance is a necessary condition for decidability on the set of uncalibrated two-camera models, and a necessary and sufficient condition for decidability on any set of weakly calibrated two-camera models. The objective of this chapter is to study the properties of projectively invariant tasks.

This chapter is organized as follows. In Section 9.1 it is shown that the set of all projectively invariant tasks can be completely characterized by the set of equivalence classes determined by projective equivalence. Section 9.2 introduces a few invariants of projective equivalence. These invariants characterize the geometric structure of the features in each equivalence class determined by projective equivalence. In Section 9.3 it is shown that an appropriately defined set of features in "normalized upper triangular form" is a set of canonical forms for projective equivalence. Thus, features in normalized upper triangular form characterize the set of equivalence classes determined by projective equivalence and can therefore be used to construct any projectively invariant task. This leads to the main result of the chapter, Theorem 9.4, which states that there is a one-to-one correspondence between tasks that are projectively invariant and sets of features in normalized upper triangular form. Finally, Section 9.4 contains some concluding remarks and directions for future research.

9.1 Projective Invariance on $\mathcal{F}$

We start by recalling some definitions introduced in Chapter 8. For each matrix $A \in \mathrm{GL}(4)$, let $\bar{\mathbf{A}}$ denote the extended function from $(\mathbb{P}^3)^n$ to $(\mathbb{P}^3)^n$ defined by the rule

$$\{p_1, p_2, \ldots, p_n\} \longmapsto \{\mathbf{A}(p_1), \mathbf{A}(p_2), \ldots \mathbf{A}(p_n)\}$$

where $(\mathbb{P}^3)^n := \overbrace{\mathbb{P}^3 \times \mathbb{P}^3 \times \cdots \times \mathbb{P}^3}^{n \text{ times}}$. The set of extensions to $(\mathbb{P}^3)^n$ of all projective transformations on $\mathbb{P}^3$ is a group $\mathrm{GP}(3;n)$ with composition rule $\{\bar{\mathbf{A}}_1, \bar{\mathbf{A}}_2\} \longmapsto \bar{\mathbf{A}}_1 \circ \bar{\mathbf{A}}_2$. We call two lists $p, q \in (\mathbb{P}^3)^n$ *projectively equivalent* and write

$$p = q \mod \mathrm{GP}(3;n)$$

if there exists an $\bar{\mathbf{A}} \in \mathrm{GP}(3;n)$ such that $p = \bar{\mathbf{A}}(q)$. Projective equivalence is an equivalence relation on $(\mathbb{P}^3)^n$ because $\mathrm{GP}(3;n)$ is a group. For each $p \in (\mathbb{P}^3)^n$ we denote by $[p]_{\mathrm{GP}(3;n)}$ the equivalence class of features in $(\mathbb{P}^3)^n$ that are projectively equivalent to p, i.e.,

$$[p]_{\mathrm{GP}(3;n)} := \{q \in (\mathbb{P}^3)^n : p = q \mod \mathrm{GP}(3;n)\}$$

Given a subset $\mathcal{S}$ of $(\mathbb{P}^3)^n$, the set of all equivalence classes of features projectively equivalent to elements in $\mathcal{S}$ is denoted by $\mathcal{S}|\mathrm{GP}(3;n)$, i.e.,

$$\mathcal{S}|\mathrm{GP}(3;n) := \{[s]_{\mathrm{GP}(3;n)} : s \in \mathcal{S}\}$$

A task $T(f) = 0$ is said to be *projectively invariant on* $\mathcal{F}$ if for each pair of features $f, g \in \mathcal{F}$,

$$f = g \mod \mathrm{GP}(3;n) \quad \Rightarrow \quad T(f) = T(g) \tag{9.1}$$

In other words, $T(f) = 0$ is projectively invariant on $\mathcal{F}$, just in case T is constant on each equivalence class in $\mathcal{F}|\mathrm{GP}(3;n)$. Since task functions can only take two distinct values (0 or 1), each task $T(f) = 0$ that is projectively invariant on $\mathcal{F}$ can be uniquely characterized by the subset of $\mathcal{F}|\mathrm{GP}(3;n)$ consisting of those elements of $\mathcal{F}|\mathrm{GP}(3;n)$ in which T is equal to 1. The following lemma summarizes the above observation.

Lemma 9.1. *A task $T(f) = 0$ is projectively invariant on $\mathcal{F}$ if and only if there exists a subset $\mathcal{I}$ of $\mathcal{F}|\mathrm{GP}(3;n)$ such that $T = \chi_{\mathcal{I}} \circ P_{\mathcal{F}}$ where $\chi_{\mathcal{I}}$ is the characteristic function*[1] *of $\mathcal{I}$ on $\mathcal{F}|\mathrm{GP}(3;n)$ and $P_{\mathcal{F}}$ is the canonical projection from $\mathcal{F}$ to $\mathcal{F}|\mathrm{GP}(3;n)$ defined by the rule $p \longmapsto [p]_{\mathrm{GP}(3;n)}$.*

This Lemma shows that if one is able to enumerate all the elements in $\mathcal{F}|\mathrm{GP}(3;n)$ then one can enumerate all the subsets of $\mathcal{F}|\mathrm{GP}(3;n)$ and thus all the tasks that are projectively invariant on $\mathcal{F}$. For example, if $\mathcal{F}|\mathrm{GP}(3;n)$ has $k < \infty$ elements then it has 2^k distinct subsets and therefore the number of distinct projectively invariant tasks is exactly equal to 2^k. It turns out that for most feature spaces of interest,

$$\mathcal{F}|\mathrm{GP}(3;n) = (\mathbb{P}^3)^n|\mathrm{GP}(3;n) \tag{9.2}$$

In fact, we can state the following:

Lemma 9.2. *Equation* (9.2) *holds for any feature space $\mathcal{F}$ that can be written as*

$$\mathcal{F} := \underbrace{\mathcal{V} \times \mathcal{V} \times \cdots \times \mathcal{V}}_{n \text{ times}}, \qquad \mathcal{V} := \{\mathbb{R}\left[\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right] : w \in \mathcal{B}\} \tag{9.3}$$

for some $\mathcal{B} \subset \mathbb{R}^3$ with nonempty interior.

Before proving Lemma 9.2, we give an example of a feature space for which (9.2) does not hold:

$$\mathcal{F} := \{f \in (\mathbb{P}^3)^n : \text{all points in } f \text{ are collinear}\}$$

Since no projective transformation is able to map a list of collinear points into a list of noncollinear points, any list $p \in (\mathbb{P}^3)^n$ consisting of noncollinear points is not projectively equivalent to any element of $\mathcal{F}$. Therefore, for such a p, $[p]_{\mathrm{GP}(3;n)} \notin \mathcal{F}|\mathrm{GP}(3;n)$ which means that $\mathcal{F}|\mathrm{GP}(3;n)$ is a strict subset of $(\mathbb{P}^3)^n|\mathrm{GP}(3;n)$.

[1] Given a subset $\mathcal{S}$ of a set $\mathcal{X}$, the *characteristic function of $\mathcal{S}$ on $\mathcal{X}$* is the function from $\mathcal{X}$ to $\{0,1\}$ defined by $\chi_{\mathcal{S}}(x) = 1$ if $x \in \mathcal{S}$ and $\chi_{\mathcal{S}}(x) = 0$ if $x \notin \mathcal{S}$.

Proof of Lemma 9.2. Since $\mathcal{F} \subset (\mathbb{P}^3)^n$, it is true that $\mathcal{F}|\mathrm{GP}(3;n) \subset (\mathbb{P}^3)^n|\mathrm{GP}(3;n)$. In order to prove the lemma it is then enough to show that the reverse set inclusion also holds. To this effect take some arbitrary $p \in (\mathbb{P}^3)^n$ and, for each $i \in \{1, 2, \dots, n\}$, let $x_i \in \mathbb{R}^4$ represent the ith point p_i of p, i.e., $p_i = \mathbb{R}x_i$.

We start by showing that there exists a matrix $A \in \mathrm{GL}(4)$ such that for each $i \in \{1, 2, \dots, n\}$

$$Ax_i \notin \mathcal{S} \tag{9.4}$$

where $\mathcal{S}$ denote the subspace of $\mathbb{R}^4$ spanned by the first three columns of the 4×4 identity matrix. Let $\mathcal{I}$ be the set of indices i for which $x_i \notin \mathcal{S}$. Since $\mathbb{R}^4 \setminus \mathcal{S}$ is an open set, for each x_i with $i \in \mathcal{I}$ there exists a ball with radius $\epsilon_i > 0$ around x_i which is completely outside $\mathcal{S}$. Let

$$A := I + \delta B, \qquad B := \begin{bmatrix} 0_{3\times 4} \\ b' \end{bmatrix}$$

where $b \in \mathbb{R}^4$ is some vector which is not orthogonal to any vector in $\{x_i : i \notin \mathcal{I}\}$, and δ is a positive constant small enough so that A is nonsingular and $\delta\|Bx_i\| < \epsilon_i$, $i \in \mathcal{I}$. Picking some $i \in \mathcal{I}$, $Ax_i = x_i + \delta Bx_i$ and therefore $\|Ax_i - x_i\| \le \delta\|Bx_i\| < \epsilon_i$. This means that Ax_i is in the ball with radius $\epsilon_i > 0$ around x_i which is completely outside $\mathcal{S}$, and therefore that (9.4) holds for any $i \in \mathcal{I}$. Picking now some $i \notin \mathcal{I}$,

$$Ax_i = x_i + \delta \begin{bmatrix} 0_{3\times 1} \\ b'x_i \end{bmatrix} \tag{9.5}$$

Since $b'x_i \neq 0$ we conclude that $\left[\begin{smallmatrix} 0_{3\times 1} \\ b'x_i \end{smallmatrix}\right] \notin \mathcal{S}$. From this, the fact that $x_i \in \mathcal{S}$, and (9.5), we conclude that (9.4) also holds for any $i \notin \mathcal{I}$.

Since $\mathcal{B}$ has nonempty interior, it must contain a ball of radius $\bar{\epsilon} > 0$ around some point $x_0 \in \mathbb{R}^3$. Let $\mathbf{A}$ denote the projective transformation determined by the nonsingular matrix

$$B := \begin{bmatrix} \bar{\delta} I_{3\times 3} & x_0 \\ 0 & 1 \end{bmatrix} A$$

where $\bar{\delta}$ is any positive constant such that

$$\bar{\delta}\|E'Ax_i\| < \bar{\epsilon}|e_4'Ax_i|, \qquad \forall i \in \{1, 2, \dots, n\} \tag{9.6}$$

where E is a matrix consisting of the first 3 columns of the 4×4 identity matrix and e_4 is its forth column. Such $\bar{\delta}$ always exists because (9.4) guarantees that each $e_4'Ax_i$ is nonzero. For every $i \in \{1, 2, \dots, n\}$,

$$\mathbf{A}(p_i) = \mathbb{R}(Ax_i) = \mathbb{R} \begin{bmatrix} \bar{\delta} EAx_i + e_4'Ax_i x_0 \\ e_4'Ax_i \end{bmatrix}$$

and since $e_4'Ax_i \neq 0$, one can also write

$$\mathbf{A}(p_i) = \mathbb{R} \begin{bmatrix} \bar{x}_i \\ 1 \end{bmatrix}, \qquad \bar{x}_i := \frac{\bar{\delta}}{e_4'Ax_i} EAx_i + x_0$$

Now, because of (9.6), each $\bar{x}_i$ is contained in the ball of radius $\bar{\epsilon}$ around x_0 and therefore in $\mathcal{B}$. From this and (9.3) one concludes that $\mathbf{A}(p_i) \in \mathcal{V}$ for each $i \in \{1, 2, \dots, n\}$. Therefore, because of (9.3), $f := \bar{\mathbf{A}}(p)$ belongs to $\mathcal{F}$. This shows that, for every vector $p \in (\mathbb{P}^3)^n$, there is a vector $f \in \mathcal{F}$ that is projectively equivalent to p and therefore that $(\mathbb{P}^3)^n|\mathrm{GP}(3;n) \subset \mathcal{F}|\mathrm{GP}(3;n)$. ∎

The remaining of this Chapter is devoted to the characterization of the set of equivalence classes $(\mathbb{P}^3)^n|\mathrm{GP}(3;n)$.

9.2 Invariants

Let p be a list in $(\mathbb{P}^3)^n$. The *subspace spanned by* p—denoted by $\operatorname{span} p$—is the linear subspace of $\mathbb{R}^4$ that results from subspace addition of the points in the list p. The dimension of this subspace is called the *dimension of* p and is denoted by $\dim p$. Since any extension $\bar{\mathbf{A}}$ in $\mathrm{GP}(3; n)$ preserves linear independence of the points in the lists it acts upon, the dimension is an invariant[2] of projective equivalence. Geometrically, if the dimension of p is equal to 1 then all points in p are equal, and if the dimension of p is equal to 2 or 3 then all the points in p are collinear or coplanar, respectively.

The *principal coordinate system of* p—denoted by $\mathrm{PCS}[p]$—is the set of $m := \dim p$ smallest indices $\{i_1, i_2, \ldots, i_m\}$ such that

$$\operatorname{span} p = p_{i_1} \oplus p_{i_2} \oplus \cdots \oplus p_{i_m} \tag{9.7}$$

where each p_i denotes the ith point in the list p. The points $p_{i_1}, p_{i_2}, \ldots, p_{i_m}$ in (9.7) are called the *principal basis of* p. The principal coordinate system is also an invariant of projective equivalence.

Because of (9.7), for each point p_i in the list p, there is a unique subset $\{j_1, j_2, \ldots, j_k\}$ of $\mathrm{PCS}[p]$ with the smallest number of elements, such that

$$p_i \subset p_{j_1} \oplus p_{j_2} \oplus \cdots \oplus p_{j_k}$$

The integers $j_1, j_2, \ldots, j_k$ are called the *coordinates of* p_i *on* $\mathrm{PCS}[p]$. The *principal partition of* p is the partition[3] $\mathcal{S} := \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_s\}$ of $\mathrm{PCS}[p]$ with the largest number of elements $s \in \{1, 2, \ldots, m\}$ such that for each point p_i in p there is a set $\mathcal{S}_j$ in $\mathcal{S}$ in which all the coordinates of p_i lie. The principal partition is also an invariant of projective equivalence. The principal partition of a list p indicates how the points in p are geometrically organized. The following cases are possible:

1. The principal partition contains m sets, each with one element. In this case, all the n points in p are at m fixed distinct positions, no three of these positions are collinear and no four of them are coplanar.

2. The principal partition contains $m - 2$ sets with a single element and one set with 2 elements ($m \geq 2$, $n \geq 3$). In this case, at least 3 distinct points in p are on a given line and the remaining are at $m - 2$ fixed distinct positions. In case $m = 4$, the line and the two positions are not on the same plane.

3. The principal partition contains one set with 3 elements and eventually another set with a single element ($m \geq 3$, $n \geq 4$). In this case, at least 4 points in p are on a given plane and the remaining are at a single point outside the plane.

4. The principal partition contains two sets with 2 elements each ($m = 4$, $n \geq 6$). Each point in p is in one of two fixed lines that are not coplanar. At least 3 distinct points are on each line.

5. The principal partition contains one set with 4 elements ($m = 4$, $n > 4$). The points are not in any of the above configurations.

[2]With $\mathcal{X}$ and $\mathcal{Z}$ sets, and E an equivalence relation on $\mathcal{X}$. A function $f : \mathcal{X} \to \mathcal{Z}$ is an *invariant* of E when $xEy \Rightarrow f(x) = f(y), \forall x, y \in \mathcal{X}$.

[3]A *partition of a set* $\mathcal{X}$ is any collection of disjoint subsets of $\mathcal{X}$ whose union is equal to $\mathcal{X}$.

Given a list $p \in (\mathbb{P}^3)^n$ and an integer $\tilde{n} \leq n$, the *leading $\tilde{n}$-sublist of p* is the list $\tilde{p}$ consisting of the first $\tilde{n}$ points in p. Since $\tilde{p}$ is itself a list in $(\mathbb{P}^3)^{\tilde{n}}$, one can talk about the dimension, principal coordinate system, and principal partition of $\tilde{p}$. Since all points in $\tilde{p}$ are points in p, $\dim \tilde{p} \leq \dim p$ and $\mathrm{PCS}[\tilde{p}] \subset \mathrm{PCS}[p]$. Moreover, every set in the principal partition of $\tilde{p}$ is a subset of some set in the principal partition of p.

9.3 Canonical Forms

Let p be a m-dimensional list in $(\mathbb{P}^3)^n$ and A a 4×4 nonsingular matrix that corresponds to a coordinate transformation that maps a particular set of vectors in $\mathbb{R}^4$ that represent the principal basis of p into the canonical basis of $\mathbb{R}^4$. Then, the principal basis of $q := \bar{\mathbf{A}}(f)$ is represented by the first m vectors of the canonical base of $\mathbb{R}^4$. A list for which this property holds is said to be in *upper triangular form*. One thus concludes that any list in $(\mathbb{P}^3)^n$ is projectively equivalent to some list in upper triangular form.

Given a list $p \in (\mathbb{P}^3)^n$ in upper triangular form, let $x = [x_1\ x_2\ x_3\ x_4]'$ be a vector in $\mathbb{R}^4$ that represents the last point of p, and Λ the diagonal matrix whose ith diagonal element is equal to

$$\lambda_i := \begin{cases} 1 & \text{if} \quad \forall k \in \mathcal{S}_{j_i}, \quad x_k = 0 \\ \dfrac{1}{x_{k_i}} & \text{if} \quad \exists k \in \mathcal{S}_{j_i}, \quad x_k \neq 0 \end{cases} \qquad i \in \{1, 2, \ldots, m\}$$

where $\mathcal{S}_{j_i}$ is the element of the principal partition of the leading $(n-1)$-sublist of p that contains i, and k_i is the largest index in $\mathcal{S}_{j_i}$ such that $x_{k_i} \neq 0$. Then, $q := \bar{\mathbf{\Lambda}}(f)$ is still in upper triangular form because Λ is diagonal. Since the last point in the list q can be represented by the vector $y := \Lambda x$, one concludes that the last point in the list q can be represented by a vector $y = [y_1\ y_2\ y_3\ y_4]' \in \mathbb{R}^4$ such that for each set $\mathcal{S}_j$ in the principal partition of the leading $(n-1)$-sublist of q,

$$y_{l_j} = 1 \tag{9.8}$$

where l_j is the largest index in $\mathcal{S}_j$ such that $y_{l_j} \neq 0$. A list q in upper triangular form whose last point can be represented by a vector y for which (9.8) holds for every $\mathcal{S}_j$ in the principal partition of the leading $(n-1)$-sublist of q, is said to have *the last element normalized.* We thus conclude that any list $p \in (\mathbb{P}^3)^n$ is projectively equivalent to a list q in upper triangular form with the last element normalized. The list q is said to be obtained from p by *upper triangularization and normalization of the last element.* Consider a list $p \in (\mathbb{P}^3)^n$ and let $\tilde{n}$ be some integer smaller than n.

Take now an arbitrary list $p \in (\mathbb{P}^3)^n$ and define

$$p^{(k)} := \begin{cases} p & k = 1 \\ \bar{\mathbf{A}}^{(k-1)}(p^{(k-1)}) & k > 1 \end{cases} \qquad k \in \{1, 2, \ldots, n\} \tag{9.9}$$

where $\mathbf{A}^{(k)}$ is the projective transformation that achieves the upper triangularization and normalization of the last element of the leading $(k+1)$-sublist of $p^{(k)}$. Since normalization of the last element of a list does not destroy normalization in the last elements of any of its leading sublists, the list $q := p^{(n)}$ has the property that all its leading sublists are in upper triangular form with the last element normalized. A list with this property is said to be in *normalized upper triangular form*. From the above construction one concludes that any list in $(\mathbb{P}^3)^n$ is projectively equivalent to another list in normalized upper triangular form.

We recall that a set of *canonical forms* for an equivalence relation E on a set $\mathcal{X}$ is a subset $\mathcal{Y}$ of $\mathcal{S}$ such that for each $x \in \mathcal{X}$ there is exactly one $y \in \mathcal{Y}$ for which xEy [125]. The following lemma can be stated:

Lemma 9.3. *The set of all lists in $(\mathbb{P}^3)^n$ in normalized upper triangular form is a set of canonical forms for projective equivalence.*

Proof of Lemma 9.3. It was shown above that any list $p \in (\mathbb{P}^3)^n$ is projectively equivalent to a list in normalized upper triangular form, so it remains to prove that p is projectively equivalent to a single list in normalized upper triangular form. Because of the transitivity of the equivalence relation, it is enough to show that any two projectively equivalent lists $p, q \in (\mathbb{P}^3)^n$ in normalized upper triangular form are equal.

Since p and q are projectively equivalent, there is a matrix $A \in \mathrm{GL}(4)$ such that

$$p_i = \mathbf{A}(q_i), \qquad\qquad i \in \{1, 2, \dots, n\} \tag{9.10}$$

where p_i and q_i denote the ith points in the lists p and q, respectively. Since p and q are projectively equivalent, the two lists have the same principal coordinate system. From this, the fact that (9.10) holds for each i in $\mathrm{PCS}[p] = \mathrm{PCS}[q]$, and the fact that p and q are in upper triangular form, one concludes that the first m columns of A must be zero everywhere except for nonzero diagonal elements λ_i, $i \in \{1, 2, \dots, m\}$.

Next we show that for each $\tilde{n} \leq n$

$$\lambda_i = \rho_k^{(\tilde{n})}, \qquad\qquad \forall i \in \mathcal{S}_k^{(\tilde{n})}, \quad k \in \{1, 2, \dots, s_{\tilde{n}}\} \tag{9.11}$$

where $\{\mathcal{S}_1^{(\tilde{n})}, \mathcal{S}_2^{(\tilde{n})}, \dots, \mathcal{S}_{s_{\tilde{n}}}^{(\tilde{n})}\}$ is the principal partition of the leading $\tilde{n}$-sublist of p, and $\{\rho_1^{(\tilde{n})}, \rho_2^{(\tilde{n})}, \dots, \rho_{s_{\tilde{n}}}^{(\tilde{n})}\}$ is an appropriately defined set of nonzero constants. Since the principal partition of the 1-sublist of p contains just the set $\{1\}$, (9.11) holds for $\tilde{n} = 1$ if one defines $\rho_1^{(1)} := \lambda_1$. Suppose now that (9.11) hold for some $\tilde{n} < n$ and pick some set $\mathcal{S}_k^{(\tilde{n}+1)}$ in the principal partition of the leading $(\tilde{n}+1)$-sublist of p. Since the leading $\tilde{n}$-sublist of p is a sublist of the $(\tilde{n}+1)$-sublist of p,

$$\mathcal{S}_k^{(\tilde{n}+1)} = \mathcal{S}_{k_1}^{(\tilde{n})} \cup \mathcal{S}_{k_2}^{(\tilde{n})} \cup \cdots \cup \mathcal{S}_{k_r}^{(\tilde{n})} \tag{9.12}$$

for appropriate indices $\{k_1, k_2, \dots, k_r\}$. Because the leading $(\tilde{n}+1)$-sublists of p and q have the last element normalized one concludes that the $(\tilde{n}+1)$th points of p and q can be represented by vectors x and y in $\mathbb{R}^4$, respectively, such that

$$x_{l_j} = 1, \qquad\qquad y_{l_j} = 1, \qquad\qquad j \in \{1, 2, \dots, r\}$$

where each l_j denotes the largest element in $\mathcal{S}_{k_j}^{(\tilde{n})}$ for which $x_{l_j} \neq 0$. Note that, because of (9.10) for $i = \tilde{n}+1$, and the fact that the first m rows of A are zero everywhere except for the diagonal, l_j is also the largest element in $\mathcal{S}_{k_j}^{(\tilde{n})}$ for which $y_{l_j} \neq 0$. But then, the rows $k_1, k_2, \dots, k_r$ of equation (9.10) with $i = \tilde{n}+1$, imply that

$$\lambda_{k_j} = \gamma, \qquad\qquad j \in \{1, 2, \dots, r\}$$

for some $\gamma \neq 0$. Thus, because of (9.11), one concludes that

$$\rho_{k_j}^{(\tilde{n})} = \gamma, \qquad\qquad j \in \{1, 2, \ldots, r\} \tag{9.13}$$

Defining $\rho_k^{(\tilde{n}+1)} := \gamma$, from (9.12) and (9.13) one concludes that

$$\lambda_i = \rho_k^{(\tilde{n}+1)}, \qquad\qquad \forall i \in \mathcal{S}_k^{(\tilde{n}+1)}, \quad k \in \{1, 2, \ldots, s_{\tilde{n}+1}\}$$

By induction, one concludes that (9.11) holds for every $\tilde{n} \leq n$.

Pick now an arbitrary point p_i in p. Since $\{\mathcal{S}_1^{(n)}, \mathcal{S}_2^{(n)}, \ldots, \mathcal{S}_{s_n}^{(n)}\}$ is the principal partition of p, all the coordinates of p_i on PCS[p] must be contained in some $\mathcal{S}_k^{(n)}$ with $k \in \{1, 2, \ldots, s_n\}$. Thus if x is a vector in $\mathbb{R}^4$ that represents p_i, from (9.11) with $\tilde{n} = n$, one concludes that

$$Ax = \rho_k^{(n)} x$$

and therefore $\mathbf{A}(p_i) = p_i$. From this and (9.10) one concludes that $p_i = q_i$. ∎

Since the set of all lists in $(\mathbb{P}^3)^n$ in normalized upper triangular form is a set of canonical forms for projective equivalence, there is a one-to-one correspondence between this set and the set $(\mathbb{P}^3)^n|\mathrm{GP}(3;n)$ of equivalence classes for projective equivalence. Thus, under the hypothesis of Lemma 9.2, there is a one-to-one correspondence between $\mathcal{F}|\mathrm{GP}(3;n)$ and the set of lists in $(\mathbb{P}^3)^n$ in normalized upper triangular form. Because of Lemma 9.1, the main result of the chapter follows:

Theorem 9.4. *Under the assumptions of Lemma 9.2 there is a one-to-one correspondence between tasks that are projectively invariant on $\mathcal{F}$ and subsets of $(\mathbb{P}^3)^n$ containing only lists in normalized upper triangular form.*

Table 9.1 shows lists of vectors in $\mathbb{R}^4$ that represent the normalized upper triangular forms of $(\mathbb{P}^3)^n$ for $n \leq 4$. To save space, this table does not contain representations for all the normalized upper triangular forms. However, it contains enough lists so that every list not represented is projectively equivalent to a list represented by a permutation of one set of vectors in the table.

$n=1$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$			
$n=2$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$		
$n=3$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$
$n=4$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\begin{bmatrix}\rho\\1\\0\\0\end{bmatrix}$ $\rho \in \mathbb{R}$
	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

Table 9.1: Lists of vectors that represent the normalized upper triangular forms in $(\mathbb{P}^3)^n$

From Table 9.1 one concludes that for $n = 1$ there is only one equivalence class of projective equivalence. This equivalence class is the whole set $\mathbb{P}^3$. Thus, for $n = 1$, there are only two decidable tasks. These tasks are specified by the two constant task functions:

$$T_{\text{always}}(f) = 0, \qquad T_{\text{never}}(f) = 1, \qquad \forall f \in \mathcal{F}$$

These tasks are *degenerate* in that they are accomplished for every feature list in $\mathcal{F}$, or for no feature list in $\mathcal{F}$.

For $n = 2$ there are two equivalence classes. Each equivalence class corresponds to an *elementary task* that is accomplished just on that equivalence class. The elementary task that corresponds to the equivalence class represented by the first list of vectors is the point-to-point task specified by (7.3). The elementary task that corresponds to the other equivalence class is the "complement" of the the point-to-point task, in that it is accomplished precisely at those feature lists at which the point-to-point task is not accomplished.

For $n = 3$ there are six equivalence classes. Four of them are represented by vectors in Table 9.1 and the other two can be represented by permutations of the second set of vectors in this table. The four elementary tasks corresponding to the equivalence classes represented in Table 9.1 are the following: the first one is accomplished just in case the 3 points in the feature list are equal; the second is accomplished when the 1st and 3rd points are equal to each other but different from the 2nd point; the third is accomplished when the 3 points are distinct but collinear; and the forth is accomplished when all the points are distinct and not collinear. The only task that is fundamentally different from the ones that are decidable for $n < 3$ is the third one, i.e., the one that is accomplished when the 3 points are distinct but collinear.

For $n = 4$ most of the elementary tasks are similar to the ones described before, except for two: the ones corresponding to the equivalence classes represented by the fourth and seventh lists of vectors. The latter is accomplished when the 4 points are coplanar but no 3 of them are collinear, and the former is accomplished when the 4 points are on the same line (the first three of them being distinct) and the position along the line of the 4th point is specified with respect to the first three by the "projectively invariant coordinate" ρ. In fact, ρ determines the cross ratio of the four points [98].

9.4 Concluding Remarks

The set of all projectively invariant tasks was completely characterized using a set of canonical forms for the equivalence classes determined by projective equivalence. For feature spaces with up to 4 point features, geometric interpretations were given for a set of elementary tasks. Projectively invariant tasks defined on feature spaces with more then 5 points still need to be well understood, in particular as to their relationship with projectively invariant coordinate systems [86, 126].

Appendix A

Simultaneous Realizations

Lemma A.1. *Given two transfer matrices* N *and* K *with* N *strictly proper such that* K *stabilizes* N *with stability margin* λ*, there exist matrices* A_E*,* B_E*,* C_E*,* D_E*,* F_E*, and* G_E *with appropriate dimensions such that* $A_E + \lambda I$ *is a stability matrix, and* $\{A_E + D_E C_E, B_E, C_E\}$ *and* $\{A_E - B_E F_E, D_E - B_E G_E, F_E, G_E\}$ *are stabilizable and detectable realizations of* $\mathrm{N}_{\mathbb{P}}$ *and* $\mathrm{K}_{\mathbb{C}}$*, respectively, with stability margin* λ*.*

Proof of Lemma A.1. Let $\{A, B, C\}$ and $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ be minimal realizations of N and K respectively. Since K stabilizes N with stability margin λ, the poles of the matrix

$$\begin{bmatrix} A - B\bar{D}C & B\bar{C} \\ -\bar{B}C & \bar{A} \end{bmatrix} + \lambda I \tag{A.1}$$

have negative real parts. Let H and $\bar{H}$ be matrices such that $A + HC + \lambda I$ and $\bar{A} + \bar{H}\bar{C} + \lambda I$ are asymptotically stable. Defining

$$A_E := \begin{bmatrix} A+HC & 0 \\ 0 & \bar{A}+\bar{H}\bar{C} \end{bmatrix} \qquad B_E := \begin{bmatrix} B \\ -\bar{H} \end{bmatrix} \qquad D_E := \begin{bmatrix} -H \\ -\bar{B}-\bar{H}\bar{D} \end{bmatrix}$$

$$C_E := \begin{bmatrix} C & 0 \end{bmatrix} \qquad F_E := \begin{bmatrix} 0 & -\bar{C} \end{bmatrix} \qquad G_E := \bar{D}$$

the matrix $A_E + \lambda I$ is asymptotically stable and

$$\begin{aligned} &C_E(sI - A_E - D_E C_E)^{-1} B_E = \\ &\quad = \begin{bmatrix} C & 0 \end{bmatrix} \left(sI - \begin{bmatrix} A & 0 \\ -\bar{B}C - \bar{H}\bar{D}C & \bar{A}+\bar{H}\bar{C} \end{bmatrix}\right)^{-1} \begin{bmatrix} B \\ -\bar{H} \end{bmatrix} = \mathrm{N(s)} \\ &F_E(sI - A_E + B_E F_E)^{-1}(D_E - B_E G_E) + G_E = \\ &\quad = \begin{bmatrix} 0 & -\bar{C} \end{bmatrix} \left(sI - \begin{bmatrix} A+HC & B\bar{C} \\ 0 & \bar{A} \end{bmatrix}\right)^{-1} \begin{bmatrix} -H - B\bar{D} \\ -\bar{B} \end{bmatrix} + \bar{D} = \mathrm{K(s)} \end{aligned}$$

Stabilizability of $\{A_E + D_E C_E + \lambda I, B_E\}$ and $\{A_E - B_E F_E + \lambda I, D_E - B_E G_E\}$ is guaranteed by the fact that both $A_E + D_E C_E + \lambda I$ and $A_E - B_E F_E + \lambda I$ are a state feedback away from (A.1) which is a stability matrix. Detectability of $\{C_E, A_E + D_E C_E + \lambda I\}$ and $\{F_E, A_E - B_E F_E + \lambda I\}$ is guaranteed by the fact that both $A_E + D_E C_E + \lambda I$ and $A_E - B_E F_E + \lambda I$ are an output injection away from $A_E + \lambda I$ which is a stability matrix. ■

Remark A.2. When K is chosen to have the structure of an observer with state feedback, i.e., when $\mathrm{N}_{\mathbb{P}}$ and $\mathrm{K}_{\mathbb{C}}$ have realizations $\{A, B, C\}$ and $\{A + HC - BF, H, F\}$, respectively, one can pick $A_E = A + HC$, $B_E = B$, $C_E = C$, $D_E = -H$, $F_E = F$, and $G_E = 0$.

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