Supervisory Control of Families of Nonlinear Regulators*

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Abstract

A simply-structured high-level smart controller called a "supervisor", has recently been proposed [1] for the purpose of orchestrating the switching among a set of candidate controllers into feedback with an imprecisely modeled process so as to stabilize the process and make its output track a constant reference. In [1], the process's model is assumed to belong to the union of a finite number of sets of models, each set centered around a known nominal. The supervisor's unique feature, distinguishing it from other logics which might be used for the same purpose, is that controller selection is made by (i) continuously comparing in real time suitably defined normed output estimation errors or "performance signals" determined by the nominal models and (ii) by placing in the feedback-loop, from time to time, that candidate controller whose corresponding performance signal is the smallest. The paradigm is a manifestation of the idea of certainty equivalence from parameter adaptive control. The aim of this paper is to discuss some preliminary results dealing with the extension of these ideas to the supervisory control of families of nonlinear regulators.

1 Introduction

The process Σ_P to be controlled is assumed to be modeled by a 1-dimensional differential equation of the form

$$\dot{y} = a(y, \omega) + b(\omega)u \tag{1}$$

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where ω is an uncertain parameter taking values in a finite set $\mathcal{P} = \{p_1, \ldots, p_m\} \subset \mathbb{R}$. We will assume that $a(\cdot)$ and $b(\cdot)$ are given functions with the following properties:

(i) $a(y,p)$ is a locally Lipschitz continuous function of y that satisfies the *growth condition*

$$|a(y,p)| \leq g\left(|y| + y^2\right), \qquad \forall y \in \mathbb{R},\ p \in \mathcal{P} \tag{2}$$

(ii) $b(p)$ is nonzero for every $p \in \mathcal{P}$ and therefore there exists a constant h such that

$$\frac{|b(p)|}{|b(q)|} \leq h, \qquad \forall p, q \in \mathcal{P} \tag{3}$$

An example of such a pair of functions is

$$a(y,p) = p\, y^2 \sin(p\, y) \qquad \text{and} \qquad b(p) = \cos(p)$$

with $\mathcal{P}$ being a finite subset of $[0, 2\pi)$ not containing any zeros of $\cos(p)$. Despite its apparent simplicity, for some choices of $a(.)$ and $b(.)$, the problem of developing stabilizing controllers for such one-dimensional systems, seems to be beyond the scope of existing adaptive and/or robust control techniques.

In the sequel we will take as given, a family of candidate control signals of the form

$$u_p = -\frac{1}{b(p)}\Big(\mu y + a(y,p)\Big), \quad p \in \mathcal{P} \tag{4}$$

where μ is a positive number. Note that the control

$$u \triangleq u_\omega \tag{5}$$

is a "off-the-shelf" conventional feedback law which both stabilizes and linearizes the model of Σ_P defined by (1). In fact, with u so defined, y satisfies $\dot{y} = -\mu y$. Since the value of ω is not presumed to be know, implementation of (5) is not possible. What's to be used instead is the feedback control

$$u = u_\sigma$$

where σ is a suitably defined piecewise-constant *switching signal* taking values in $\mathcal{P}$. The problem addressed in this paper is that of developing a provably correct, "high level" controller called a *supervisor* (see Figure 1) which, without a-priori knowledge of ω, is capable of choosing σ so as to cause y to tend to zero provided $y(0)$ lies within a prescribed bounded set containing the origin.

The rest of this paper is organized as follows. Section 2 presents the formal description of the supervisor together with a brief discussion of some of its properties. The main results of the paper are stated in section 3, namely that the supervisor defined in section 2 stabilizes the process model. Section 4 contains the proof of the main theorem stated in section 3 and, finally, section 5 contains a brief discussion of the results achieved so far and some directions for the future research.

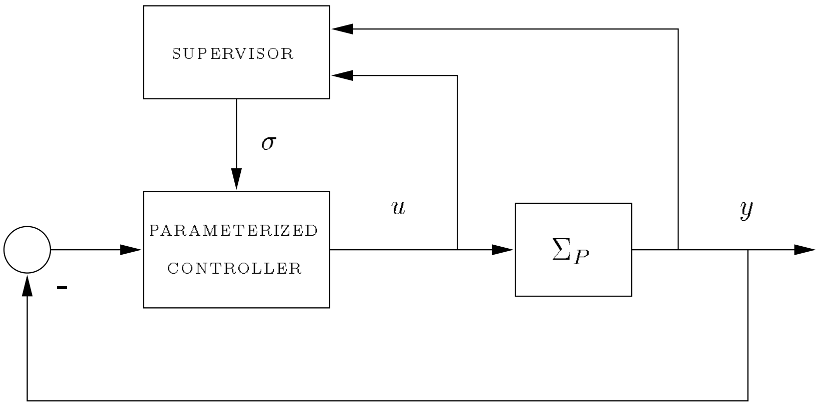

Figure 1: Supervisory Control.

2 Estimator-based Supervisor

The supervisor proposed in this paper is of the "estimator-based" type and its structure is shown in Figure 2. It consists of three subsystems:

- A *multi-estimator* Σ_E which is a dynamical system whose inputs are u and y and whose outputs $\{e_p : p \in \mathcal{P}\}$ are called *estimation errors*. Each such error is of the form $e_p = y_p - y$, where y_p is a suitably defined estimate of y which would be asymptotically correct if p were equal to ω.

- A *performance signal generator* Σ_Π which is a dynamical system whose inputs are the estimation errors and whose outputs $\{\pi_p : p \in \mathcal{P}\}$ are suitably normed values of the estimation errors called *performance signals*.

- A *switching logic* Σ_S which is a hybrid dynamical system whose inputs are the performance signals and whose output is σ.

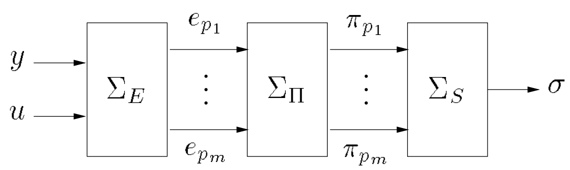

Figure 2: Estimator-based Supervisor

The rationale behind this structure can be explained as follows. Suppose that the multi-estimator is designed in such a way that in case ω is equal (or close) to some $p \in \mathcal{P}$ the estimation error e_p and its normed version π_p are "small". Since the control law $u \triangleq u_\omega$ stabilizes the system, the supervisor logic will try to keep σ as close as possible to the value $p \in \mathcal{P}$ that corresponds to the smallest π_p in the hope that this value is close to ω. In practice Σ_E and Σ_Π can be regarded as an estimator for ω and Σ_S a discontinuous tuner that works based on the certainty equivalence principle [2, 3]. In the sections that follow we will describe each of the subsystems that compose the supervisor in detail.

2.1 Multi-Estimator

For the problem at hand, the multi-estimator Σ_E is a dynamical system with state $x_E = [y_{p_1} \ \dots \ y_{p_m}]'$ consisting of a family of one-dimensional estimators of the form

$$\dot{y}_p = -\lambda(y_p - y) + a(y,p) + b(p)u, \qquad e_p = y_p - y, \qquad p \in \mathcal{P} \tag{6}$$

where λ is a positive number. This definition for Σ_E satisfies the basic requirement that for any control input u we have that

$$\dot{e}_\omega = -\lambda\, e_\omega \tag{7}$$

and therefore the estimation error associated with ω converges to zero. This is not the only requirement that Σ_E must meet. In the previous section it was said that the supervisor logic will try to keep σ equal to the value $p \in \mathcal{P}$ that corresponds to the smallest performance signal π_p, but in general there is no guarantee that this value will be ω. In order for this strategy to have a chance to work, one must guarantee that in case a particular estimation error e_q is zero (or very small) setting $u = u_q$ makes the state remain bounded. Behind this idea is the concept of detectability for nonlinear systems introduced in [4]. The following lemma is a manifestation of the fact that setting $u = u_q$, the multi-estimator defined by (6) is (locally) detectable through e_q.

Lemma 2.1. *Assuming that $\lambda < 2\mu$, given any positive constant $\mathcal{E}$ and time instants $t_0, t_1 \in \mathbb{R}^+$ ($t_0 < t_1$) such that*

$$u(t) = u_q(t) \qquad \text{and} \qquad y^2(t) + \|x_E(t)\|^2 \leq \mathcal{E}^2 \qquad \forall t \in [t_0, t_1)$$

the following inequality holds

$$\|x_E(t)\|^2 \leq \alpha\, e^{-\lambda(t-t_0)} \|x_E(t_0)\|^2 + \beta \int_{t_0}^{t} e^{-\lambda(t-\tau)} e_q^2(\tau)\, d\tau \qquad \forall t \in [t_0, t_1] \tag{8}$$

where

$$\alpha \triangleq \frac{\max\{1,\rho\}}{\min\{1,\rho\}} \qquad \beta \triangleq \frac{\rho^2|\mu-\lambda| + (2\mu-\lambda)\rho - |\mu-\lambda|}{\min\{1,\rho\}}$$

$$\rho \triangleq \frac{\lambda|\mu-\lambda| + 2(m-1)\Big(\lambda + h\mu + g(1+h)(1+\mathcal{E})\Big)^2}{\lambda(2\mu-\lambda)}$$

The proof of this lemma can be found in the appendix. To simplify the analysis, in the sequel it will be assumed that $\mu = \lambda$. In this case inequality (8) implies that

$$\|x_E(t)\|^2 \leq \left(\alpha(\lambda) + \alpha_1(\lambda)\mathcal{E}^2\right)\, e^{-\lambda(t-t_0)} \|x_E(t_0)\|^2 + \left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right) \int_{t_0}^{t} e^{-\lambda(t-\tau)} e_q^2\, d\tau \tag{9}$$

where

$$\alpha(\lambda) \triangleq 4(m-1)(1+h)^2 \left(1 + \frac{2g}{\lambda^2}\right) \qquad \beta(\lambda) \triangleq \lambda\, \alpha(\lambda)$$

$$\alpha_1(\lambda) \triangleq \frac{8(m-1)g^2(1+h)^2}{\lambda^2} \qquad \beta_1(\lambda) \triangleq \lambda\, \alpha_1(\lambda)$$

2.2 Performance Signals Generator

The performance signal generator Σ_Π to be considered consists of the following family of one-dimensional differential equations

$$\dot{\pi}_p = -\gamma \pi_p + e_p^2, \qquad \pi_p(0) = 0, \qquad p \in \mathcal{P} \tag{10}$$

where γ is a positive number strictly smaller than λ $(= \mu)$. This definition implies that for every $p \in \mathcal{P}$

$$\pi_p(t) = \int_0^t e^{-\gamma(t-\tau)} e_p^2(\tau)\, d\tau, \qquad t \in \mathbb{R}^+$$

which shows that the performance signal π_p is just an exponentially weighted $\mathcal{L}_2$-norm of the estimation error e_p.

2.3 Switching Logic

The switching logic Σ_S we shall consider is a simplified version of the dwell time switching logic introduced in [1]. Σ_S is a hybrid system whose inputs are the performance signals and whose state and output is the piecewise-constant switching signal σ whose value is held constant between successive event times. For the logic considered here event times are equally spaced by a pre-specified positive number τ_D called *dwell time*. To be more precise, event times are simply the set of times $\{k\,\tau_D : k \in \mathbb{N}\}$. The initial value of σ on the interval $[0, \tau_D)$ can be any point in $\mathcal{P}$ and for each $k \geq 1$, σ's value on $[k\,\tau_D, (k+1)\tau_D)$ is defined by the rule

$$\sigma = \begin{cases} \sigma' & \text{if } \pi_{\sigma'}(k\,\tau_D) \leq \pi_p(k\,\tau_D), \quad \forall p \in \mathcal{P} \\ \min_{p \in \mathcal{P}}\{\pi_p(k\,\tau_D)\} & \text{otherwise} \end{cases} \tag{11}$$

where σ' is the value of σ on the preceding interval $[(k-1)\tau_D, k\,\tau_D)$.

3 Main Results

The results presented in this paper are valid for the special case when the set $\mathcal{P}$ has only two elements, i.e. $\mathcal{P} = \{p_1, p_2\}$. Denoting by x be the composite state of the plant Σ_P and the multi-estimator Σ_E (i.e. $x \triangleq [y \quad y_{p_1} \quad y_{p_2}]'$), the system to be analyzed is given by (10), (11), and

$$\dot{x} = f(x, \sigma) \tag{12}$$

where the function $f : \mathbb{R}^3 \times \mathcal{P} \to \mathbb{R}^3$ is defined by (1) and (6). One can check that for all $p \in \mathcal{P}$

$$\|f(x,p)\| \leq a_1(\lambda)\|x\| + a_2\|x\|^2 \tag{13}$$

where

$$a_1(\lambda) \triangleq (m-1)\lambda + (m+1)(1+h)(\lambda+g) \qquad \text{and} \qquad a_2 \triangleq g(m+1)(1+h)$$

Theorem 3.1. *Let $\mathcal{P} = \{p_1, p_2\}$ and suppose that there exist 3 positive constants δ, ϵ and $\mathcal{E}$ such that*

$$e^{(a_1(\lambda)+\frac{\gamma}{2}+a_2\mathcal{E})\tau_D}\epsilon \leq \mathcal{E} \tag{C-1}$$

$$\alpha(\lambda) + \alpha_1(\lambda)\mathcal{E}^2 \leq e^{(\lambda-\gamma)\tau_D} \tag{C-2}$$

$$\left(5 + \frac{12\left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right)}{2\lambda - \gamma}\right)\delta^2 \leq \epsilon^2 \tag{C-3}$$

For every $x_0 \in \mathbb{R}^3$ such that $\|x_0\| \leq \delta$, the solution of the hybrid system defined by (10), (11), *and* (12) *with initial condition $x(0) = x_0$ exists globally and for every $t \in \mathbb{R}^+$*

$$\|x(t)\|^2 \leq e^{-\gamma t}\left(\left(1 + e^{(\lambda-\gamma)\tau_D}\right)\mathcal{E}^2 + \frac{2\left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right)}{2\lambda - \gamma}\delta^2\right) \tag{14}$$

Before proving theorem 3.1 we will discuss its implication. Suppose we are given values for τ_D, λ, and γ such that

$$\gamma < \lambda \qquad \text{and} \qquad \alpha(\lambda) < e^{(\lambda-\gamma)\tau_D}$$

It is always possible to find a positive constant $\mathcal{E}$ small enough so that inequality (C-2) holds and, for this particular value of $\mathcal{E}$, one can find a positive value for ϵ such that inequality (C-1) also holds. Furthermore, for any given $\epsilon > 0$ and $\mathcal{E} > 0$ it is possible to find a positive value for δ such that (C-3) holds. This proves the following corollary.

Corollary 3.2 (Local Stability). *Assuming that $\mathcal{P} = \{p_1, p_2\}$ and that τ_D, λ and γ are such that*

$$\gamma < \lambda \qquad \textit{and} \qquad \alpha(\lambda) < e^{(\lambda-\gamma)\tau_D}$$

there exists a positive constant δ such that for any $x_0 \in \mathbb{R}^3$ such that $\|x_0\| \leq \delta$, the solution of the hybrid system defined by (10), (11), *and* (12) *with initial condition $x(0) = x_0$ exists globally and converges to zero exponentially fast.*

The previous corollary guarantees exponential stability for certain values of the dwell time and the design parameters provided that the initial conditions are inside a sufficiently small neighborhood of the origin. The next corollary guarantee that this neighborhood can be made arbitrarily large by choosing suitable values for the dwell time and the design parameters.

Corollary 3.3 (Semi-global Stability). *Assuming that $\mathcal{P} = \{p_1, p_2\}$, for any given positive constant δ, one can find values for the dwell time τ_D and the design parameters λ and γ such that for any $x_0 \in \mathbb{R}^3$ such that $\|x_0\| \leq \delta$, the solution of the hybrid system defined by* (10), (11), *and* (12) *with initial condition $x(0) = x_0$ exists globally and converges to zero exponentially fast.*

Proof of Corollary 3.3: We will show that for any given positive constant δ, it is possible to find values for τ_D, λ, γ, ϵ, and $\mathcal{E}$ such that the three inequalities (C-1), (C-2), and (C-3) hold. In order to reduce the number of parameters, we will set

$$\gamma = \frac{\lambda}{2} \qquad \text{and} \qquad \mathcal{E} = \frac{1}{\tau_D}$$

Introducing the auxiliary variable $\rho \triangleq \lambda\tau_D$, the problem reduces to find values for λ, ρ, and ϵ such that

$$e^{a_2+\rho\left(\frac{a_1(\lambda)}{\lambda}+\frac{1}{4}\right)}\epsilon \leq \frac{\lambda}{\rho} \tag{C-1a}$$

$$\alpha(\lambda) + \frac{\lambda^2\alpha_1(\lambda)}{\rho^2} \leq e^{\frac{\rho}{2}} \tag{C-2a}$$

$$\left(5 + \frac{24}{3}\frac{\beta(\lambda)}{\lambda} + \frac{24}{3}\frac{\lambda\beta_1(\lambda)}{\rho^2}\right)\delta^2 \leq \epsilon^2 \tag{C-3a}$$

Since $\beta(\lambda)/\lambda$ is bounded for $\lambda \geq 1$, one can find a positive value for ϵ such that

$$\left(5 + \frac{24}{3}\frac{\beta(\lambda)}{\lambda}\right)\delta^2 < \epsilon^2 \qquad \forall\lambda \geq 1$$

Since $\alpha(\lambda)$, $\lambda^2\alpha_1(\lambda)$, and $\lambda\beta_1(\lambda)$ are bounded for $\lambda \geq 1$, it is possible to find a large enough value for ρ such that (C-2a) and (C-3a) hold for every $\lambda \geq 1$. Finally, since $a_1(\lambda)/\lambda$ is bounded for $\lambda \geq 1$ one can pick λ large enough such that (C-1a) also holds. ∎

4 Proof of Theorem 3.1

Since the function f in (12) is not globally Lipschitz, one of the questions that has to be addressed in the proof of theorem 3.1 is global existence of solution. The following technical lemma guarantees existence and boundness of the solution of (12) in a finite interval.

Lemma 4.1. *Given a vector $x_0 \in \mathbb{R}^3$, two time instants $t_0, t_1 \in \mathbb{R}^+$ ($t_0 < t_1$) and a positive constant $\mathcal{E}$ such that*

$$e^{(a_1(\lambda)+a_2\mathcal{E})(t_1-t_0)}\|x_0\| \leq \mathcal{E}$$

the solution of the differential equation (12) *with initial condition $x(t_0) = x_0$ exists until t_1 and*

$$\|x(t)\| \leq e^{(a_1(\lambda)+a_2\mathcal{E})(t-t_0)}\|x_0\| \qquad \forall t \in [t_0, t_1] \tag{15}$$

The proof of this lemma can be found in the appendix. Following the notation introduced in the proof of the previous lemma, we will denote by $[0, T)$ the maximal interval of existence of solution of (12) with initial condition $x(0) = x_0 \in \mathbb{R}^3$. The following technical lemma is used in the proof of Theorem 3.1 and is a simple iteration of inequality (9).

Lemma 4.2. *Let $t < T$ by an arbitrary event time. Assuming that (i) there exists a positive constant $\mathcal{E}$ such that $\|x(\tau)\| \leq \mathcal{E}$ for all $\tau \in [0, t)$ and (ii) for any pair of consecutive event times t_0 and t_1 such that $t_0 < t_1 \leq t$, the following inequality holds*

$$\alpha(\lambda) + \alpha_1(\lambda)\mathcal{E}^2 \leq e^{(\lambda-\gamma)(t_1-t_0)} \tag{16}$$

then

$$\|x(t)\|^2 \leq 5e^{-\gamma t}\|x(0)\|^2 + 3\left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right)\int_0^t e^{-\gamma(t-\tau)} e_\sigma^2 \, d\tau \tag{17}$$

The proof of this lemma can be found in the appendix.

Remark 4.3. The coefficients 3 and 5 in inequality (17) are in some sense arbitrary and it is possible to derive similar inequalities with different values for these coefficients. However, these coefficients will always be larger than 1.

Proof of Theorem 3.1: Let x_0 be any vector in $\mathbb{R}^3$ such that $\|x_0\| < \delta$, and $x(t)$ the solution of the hybrid system defined by (10), (11), and (12) with initial condition $x(0) = x_0$. It will be shown that either (i) for every $k \in \mathbb{N}$

$$e^{\frac{\gamma}{2}t}\|x(k\tau_D)\| \leq \epsilon \tag{18}$$

$$e^{\frac{\gamma}{2}t}\|x(t)\| \leq \mathcal{E} \qquad \forall t \in (k\tau_D - \tau_D, k\tau_D] \tag{19}$$

which proves the theorem *a fortiori* or (ii) after some finite time the switching signal σ will become constant equal to ω and inequality (14) holds for all time. The simplest way to prove this theorem is to use mathematical induction on k to prove (18) and (19). We will see that the mathematic induction may fail but in this case switching must stop and inequality (14) still holds.

The base of induction trivially holds since (C-1) and (C-3) imply that that $\delta \leq \epsilon \leq \mathcal{E}$. Proceeding to the induction step, let us assume that (18) and (19) hold up to a certain event time t_0. In order to make sure that the solution of (12) exists until the following event time $t_1 = t_0 + \tau_D$ we will use Lemma 4.1. By the induction hypothesis, $\|x(t_0)\| \leq e^{-\frac{\gamma}{2}t_0}\epsilon$, therefore a sufficient condition for this lemma to apply is that

$$e^{(a_1(\lambda)+a_2\mathcal{E})\tau_D} e^{-\frac{\gamma}{2}t_0}\epsilon \leq \mathcal{E}$$

and in this case the lemma guarantees that

$$e^{\frac{\gamma}{2}t}\|x(t)\| \leq e^{(a_1(\lambda)+a_2\mathcal{E})\tau_D} e^{\frac{\gamma}{2}(t-t_0)}\epsilon \qquad \forall t \in [t_0, t_1] \tag{20}$$

Therefore, imposing that

$$e^{(a_1(\lambda)+\frac{\gamma}{2}+a_2\mathcal{E})\tau_D}\epsilon \leq \mathcal{E} \tag{C-1}$$

guarantees both existence of solution until t_1 and the induction step for (19). The induction step for (18) will be checked using Lemma 4.2. By the induction hypothesis, the norm of the state at any time prior to t_1 is no larger than $\mathcal{E}$, therefore a sufficient condition for this lemma to apply is that

$$\alpha(\lambda) + \alpha_1(\lambda)\mathcal{E}^2 \leq e^{(\lambda-\gamma)\tau_D} \tag{C-2}$$

If this inequality holds then

$$\begin{aligned}
e^{\gamma t_1}\|x(t_1)\|^2 &\leq 5\delta^2 + 3\left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right)\int_0^{t_1} e^{\gamma\tau} e_\sigma^2 \, d\tau \\
&\leq 5\delta^2 + 3\left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right)\left(\int_0^{t_1} e^{\gamma\tau} e_{p_1}^2 \, d\tau + \int_0^{t_1} e^{\gamma\tau} e_{p_2}^2 \, d\tau\right) \\
&= 5\delta^2 + 3\left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right)\left(e^{\gamma t_1}\pi_{p_1}(t_1) + e^{\gamma t_1}\pi_{p_2}(t_1)\right)
\end{aligned} \tag{21}$$

Without loss of generality let us assume that $\omega = p_1$. By virtue of (7) and since $\gamma < \lambda$,

$$e^{\gamma t}\pi_{p_1}(t) = \int_0^t e^{\gamma\tau} e_{p_1}^2 \, d\tau \leq \frac{e_{p_1}(0)^2}{2\lambda - \gamma} \qquad \forall t \in [0, T)$$

But $e_{p_1}(0) = y(0) - y_1(0)$ therefore $e_{p_1}(0)^2 \leq 2\|x(0)\|^2 \leq 2\delta^2$ which proves that

$$e^{\gamma t}\pi_{p_1}(t) \leq \frac{2\delta^2}{2\lambda - \gamma} \qquad \forall t \in [0, T)$$

At time t_1 two situations may occur:

1. If $e^{\gamma t_1}\pi_{p_2}(t_1) \leq \frac{2\delta^2}{2\lambda-\gamma}$ then, by virtue of (21), we conclude that

   $$e^{\gamma t_1}\|x(t_1)\|^2 \leq 5\delta^2 + \frac{12\left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right)}{2\lambda - \gamma}\delta^2$$

   This means that the following condition is sufficient for the induction step for (18) to hold

   $$\left(5 + \frac{12\left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right)}{2\lambda - \gamma}\right)\delta^2 \leq \epsilon^2 \tag{C-3}$$

2. If $e^{\gamma t_1}\pi_{p_2}(t_1) > \frac{2\delta^2}{2\lambda-\gamma}$ then, for all $t \in [t_1, T)$

   $$e^{\gamma t}\pi_{p_2}(t) = \int_0^t e^{\gamma\tau} e_{p_2}^2 \, d\tau \geq \int_0^{t_1} e^{\gamma\tau} e_{p_2}^2 \, d\tau = e^{\gamma t_1}\pi_{p_2}(t_1) > \frac{2\delta^2}{2\lambda - \gamma} \geq e^{\gamma t}\pi_{p_1}(t)$$

   and therefore $\pi_{p_2}(t) > \pi_{p_1}(t)$. In this case the switching logic guarantees that $\sigma(t) = p_1$ for all $t \geq t_1$. In practice this means that switching stops in the controller that is known to stabilize the plant exponentially and the induction can stop right away since exponential convergence is assured. This does not mean that inequalities (18) and (19) will necessarily hold at time t_1 and after, but since there is no more switching after

t_1, by virtue of (9), (C-1), (C-2) and inequality (20) at time t_1, one concludes that for every $t \in [t_1, T)$

$$\begin{aligned} \|x_E(t)\|^2 \leq & \left(\alpha(\lambda) + \alpha_1(\lambda)\mathcal{E}^2\right) e^{-\lambda(t-t_1)}\|x_E(t_1)\|^2 \\ & + \left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right) \int_{t_1}^t e^{-\gamma(t-\tau)} e_{p_1}^2 \, d\tau \\ \leq & \, e^{(\lambda-\gamma)\tau_D} e^{-\gamma(t-t_1)} e^{-\gamma t_1} \mathcal{E}^2 + \left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right) \int_{t_1}^t e^{-\gamma(t-\tau)} e_{p_1}^2 \, d\tau \\ \leq & \, e^{(\lambda-\gamma)\tau_D} e^{-\gamma t} \mathcal{E}^2 + \left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right) \pi_{p_1}(t) \\ \leq & \, e^{-\gamma t} \left(e^{(\lambda-\gamma)\tau_D} \mathcal{E}^2 + \frac{2\left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right)}{2\lambda - \gamma} \delta^2 \right) \end{aligned} \tag{22}$$

On the other hand for every $t \in [t_1, T)$, $\dot{y} = -\mu y = -\lambda y$, therefore

$$y^2(t) = e^{-\lambda(t-t_1)} y^2(t_1) \leq e^{-\gamma(t-t_1)} e^{-\gamma t_1} \mathcal{E}^2 = e^{-\gamma t} \mathcal{E}^2 \tag{23}$$

Combining inequalities (22) and (23) one concludes that for every $t \in [t_1, T)$

$$e^{\gamma t} \|x(t)\|^2 = e^{\gamma t} \left(y^2(t) + \|x_E(t)\|^2 \right) \leq \left(1 + e^{(\lambda-\gamma)\tau_D} \right) \mathcal{E}^2 + \frac{2\left(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\right)}{2\lambda - \gamma} \delta^2$$

In fact, by the induction hypothesis one concludes that the above inequality holds for any $t \in [0, T)$. Since this gives a finite bound for the state of the differential equation (12) on the maximal interval of existence, we conclude that $T = +\infty$. ■

5 Concluding Remarks

The main shortcoming of the approach we've outlined, when compared with existing adaptive techniques, is that the derived results do not hold globally in the space in which y and the $\{y_p : p \in \mathcal{P}\}$ take values. Although local, the results are of a "semi-global" nature in the sense that the neighborhood of the origin for which asymptotic stability is guaranteed can be made arbitrarily large by suitable choice of the design parameters. Just how much of a shortcoming this actually is, in more realistic situations, remains to be seen.

On the other hand, on the basis of what is known so far about supervisory control of *linear* regulators {e.g., see [4]}, it is reasonable to expect that the approach will prove to have advantages over conventional adaptive techniques. Among these are the following.

1. Unlike conventional adaptive control, supervisory control can deal with nonlinearly parameterized models.

2. Since supervisory control does not necessarily involve nonlinear "zero" cancelling, the approach is not restricted to "minimum phase" systems.

3. Supervisory control makes use of off-the-shelf {non-adaptive} controllers rather than highly specialized controllers tailored to the specific needs of continuously-tuned conventional adaptive algorithms.

4. Supervisory control is almost certainly inherently robust to small modeling errors whereas conventional adaptive techniques typically are not unless tuner modifications are introduced.

5. It is almost certainly true that supervisory control can be shown to perform satisfactory in the face of small disturbance and measurements signals whereas, without substantial modification, more familiar adaptive techniques cannot.

The results presented in this paper are still preliminary. Current research involves extending them to higher dimensional classes of nonlinear systems which are state-feedback equivalent to linear systems.

Appendix

Proof of Lemma 2.1: For any interval $[t_0, t_1)$ on which $u = u_q$, we have

$$\dot{y} = a(y, \omega) - \frac{b(\omega)}{b(q)}\Big(\mu y + a(y, q)\Big) \tag{24}$$

$$\dot{y}_p = -\lambda(y_p - y) + a(y, p) - \frac{b(p)}{b(q)}\Big(\mu y + a(y, q)\Big) \qquad p \in \mathcal{P} \tag{25}$$

Equations (25) can be re-written as

$$\begin{aligned} \dot{y}_q &= -\lambda(y_q - y) - \mu\, y = -\mu\, y_q + (\mu - \lambda) e_q \\ \dot{y}_p &= -\lambda y_p + N_{pq}(y) \qquad\qquad\qquad\qquad p \in \mathcal{P} \setminus \{q\} \end{aligned}$$

where

$$N_{pq}(y) \triangleq \left(\lambda - \frac{b(p)}{b(q)}\mu\right) y + a(y, p) - \frac{b(p)}{b(q)} a(y, q) \qquad p, q \in \mathcal{P}$$

Since in the interval in question, $|y| \leq \mathcal{E}$, using (2) and (3), one concludes that

$$|N_{pq}(y)| \leq \Big(\lambda + h\mu + g(1 + h)(1 + \mathcal{E})\Big)|y|$$

and since $y^2 = (y_q - e_q)^2 \leq 2\, y_q^2 + 2\, e_q^2$

$$N_{pq}(y)^2 \leq 2\Big(\lambda + h\mu + g(1 + h)(1 + \mathcal{E})\Big)^2 \Big(y_q^2 + e_q^2\Big)$$

Defining

$$V(x_E) = \rho\, y_q^2 + \sum_{p\in\mathcal{P}\setminus\{q\}} y_p^2$$

we have that for all $t \in [t_0, t_1)$

$$\dot V = -2\rho\mu\, y_q^2 + 2\rho(\mu-\lambda) y_q e_q - 2\lambda \sum_{p\in\mathcal{P}\setminus\{q\}} y_p^2 + 2\sum_{p\in\mathcal{P}\setminus\{q\}} y_p N_{pq}(y)$$

therefore, for any positive constants ϵ_1 and ϵ_2 we have

$$\begin{aligned}\dot V \le &- \left(2\rho\mu - \epsilon_1\rho|\mu-\lambda| - 2\epsilon_2^{-1}(m-1)\Big(\lambda + h\mu + g(1+h)(1+\mathcal{E})\Big)^2\right) y_q^2\\ &- (2\lambda - \epsilon_2)\sum_{p\in\mathcal{P}\setminus\{q\}} y_p^2\\ &+ \left(\epsilon_1^{-1}\rho|\mu-\lambda| + 2\epsilon_2^{-1}(m-1)\Big(\lambda + h\mu + g(1+h)(1+\mathcal{E})\Big)^2\right) e_q^2\end{aligned}$$

Setting, $\epsilon_1 = 1/\rho$, and $\epsilon_2 = \lambda$ one concludes that

$$\lambda\rho = 2\rho\mu - \epsilon_1\rho|\mu-\lambda| - 2\epsilon_2^{-1}(m-1)\Big(\lambda + h\mu + g(1+h)(1+\mathcal{E})\Big)^2$$

and therefore $\dot V \le -\lambda V + \min\{1,\rho\}\beta\, e_q^2$. This means that for all $t\in[t_0,t_1)$

$$V(t) \le e^{-\lambda(t-t_0)}V(t_0) + \min\{1,\rho\}\beta \int_{t_0}^{t} e^{-\lambda(t-\tau)} e_q^2\, d\tau$$

On the other hand

$$\min\{1,\rho\}\|x_E(t)\|^2 \le V(t) \le \max\{1,\rho\}\|x_E(t)\|^2$$

therefore (8) holds. ∎

Proof of Lemma 4.1: Since f is locally Lipschitz with respect to x, continuous with respect to σ, and σ is a measurable function of time, the solution of (12) with the given initial condition exists in an interval of nonzero length. Let us denote by $[t_0, T)$ ($T \in (t_0, +\infty]$) the maximal interval of existence of solution. By contradiction, let us assume that inequality (15) fails to hold for some time instant in $[t_0, t_1] \cap [t_0, T)$. In this case

$$\bar t = \inf\left\{t \in [t_0,t_1]\cap[t_0,T) : \|x(t)\| \ge e^{(a_1(\lambda)+a_2\mathcal{E})(t-t_0)}\|x_0\|\right\}$$

Since $\|x(t)\|$ is a continuous function of t, the infimum is actually a minimum and therefore

$$\|x(\bar t)\| = e^{(a_1(\lambda)+a_2\mathcal{E})(\bar t-t_0)}\|x_0\| \qquad \text{and} \qquad \|x(t)\| < \|x(\bar t)\| \;\big(<\mathcal{E}\big) \quad \forall t\in[t_0,\bar t) \tag{26}$$

Since $\bar{t} < T$, for any $t \in [t_0, \bar{t}]$ we have that

$$x(t) = x_0 + \int_{t_0}^{t} f\big(x(\tau), \sigma(\tau)\big)\, d\tau$$

and therefore, by virtue of (13),

$$\|x(t)\| \leq \|x_0\| + \int_{t_0}^{t} (a_1(\lambda) + a_2\|x(\tau)\|)\|x(\tau)\|\, d\tau$$

Using the Bellman-Gronwall Lemma [5, p. 346] on the previous inequality, one concludes that

$$\|x(\bar{t})\| \leq e^{\int_{t_0}^{\bar{t}} (a_1(\lambda) + a_2\|x(\tau)\|)\, d\tau}\|x_0\| < e^{\int_{t_0}^{\bar{t}} (a_1(\lambda)+a_2\mathcal{E})\, d\tau}\|x_0\| = e^{(a_1(\lambda)+a_2\mathcal{E})(\bar{t}-t_0)}\|x_0\|$$

which contradicts (26). ∎

Proof of Lemma 4.2: Given any pair of consecutive event times t_0 and t_1 such that $t_0 < t_1 \leq t$, by virtue of (9), (16), and the fact that $\gamma < \lambda$,

$$\|x_E(t_1)\|^2 \leq e^{-\gamma(t_1-t_0)}\|x_E(t_0)\|^2 + \big(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\big)\int_{t_0}^{t_1} e^{-\gamma(t_1-\tau)} e_\sigma^2\, d\tau$$

Iterating this inequality from 0 until the switching time t one gets

$$\|x_E(t)\|^2 \leq e^{-\gamma t}\|x_E(0)\|^2 + \big(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\big)\int_{0}^{t} e^{-\gamma(t-\tau)} e_\sigma^2\, d\tau \tag{27}$$

On the other hand,

$$\|x(t)\|^2 = y(t)^2 + \|x_E(t)\|^2 = \Big(y_\omega(t) - e_\omega(t)\Big)^2 + \|x_E(t)\|^2 \leq 2e_\omega(t)^2 + 3\|x_E(t)\|^2 \tag{28}$$

By virtue of (7), $e_\omega(t) = e^{-\lambda t}e_\omega(0)$, therefore combining (27) and (28) yields

$$\|x(t)\|^2 \leq 2e^{-2\lambda t}e_\omega(0)^2 + 3e^{-\gamma t}\|x_E(0)\|^2 + 3\big(\beta(\lambda) + \beta_1(\lambda)\mathcal{E}^2\big)\int_{0}^{t} e^{-\gamma(t-\tau)} e_\sigma^2\, d\tau$$

Since $\gamma < \lambda < 2\lambda$ one concludes that inequality (17) must hold. ∎

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