

# Stochastic Programming Using Expected Value Bounds

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**Abstract**—We address the problem of minimizing an expected value with stochastic constraints, known in the literature as stochastic programming. Our approach is based on computing and optimizing bounds for the expected value that are obtained by solving a deterministic optimization problem that uses the probability density function to penalize unlikely values for the random variables. The sub-optimal solution obtained through this approach has performances guarantees with respect to the optimal one, while satisfying stochastic and deterministic constraints.

We illustrate this approach in the context of three different classes of optimization problems: finite horizon optimal stochastic control, with state or output feedback; parameter estimation with latent variables; and nonlinear Bayesian experiment design. By the means of several numerical examples, we show that our sub-optimal solution achieves results similar to those obtained with Monte Carlo methods with a fraction of the computational burden, highlighting the usefulness of this approach in real-time optimization problems.

## I. INTRODUCTION

Optimization of an expected value, also called stochastic programming, appears in countless areas of applied probability and engineering. In optimal stochastic control, a dynamical system is subject to stochastic disturbances, and one wants to find the control that minimizes the expected value of the trajectory tracking error. In maximum likelihood estimation, unobserved variables may need to be integrated out through an expected value, to obtain the likelihood of the observed variables. In machine learning, training a neural network means to find the weights that best classify the expected values of a random variable.

Stochastic Programming has been an active area of research for the last 60 years, therefore a complete overview of the literature is infeasible. Instead, we provide a brief overview of the most fundamental methods, some recent developments and how these relate to our work. We discuss separately three approaches: deterministic methods, stochastic methods, and methods based on robust optimization.

Deterministic methods rely on computing the expected value using a numerical integration method such as Gauss-Kronrod [1], [2]. Since numerical integration is computationally infeasible for large problems, deterministic approximations of the expected value are often used. Common approximations include minimizing the truncated Taylor expansion of the expected value [3]–[5] and the Laplace and saddlepoint approximations [6], [7]. A weakness of these methods is that they generally do not provide guarantees regarding how the solution found compares to the true optimum.

Stochastic optimization methods rely on some form of Monte Carlo sampling. These methods generally scale well and provide confidence intervals on the solutions. The most intuitive method is the Sample Average Approximation (SAA) (also known as Empirical Risk Minimization) [8]–[10], where the expected value is approximated by the empirical average obtained through sampling. A method that often converges faster than SAA is the Stochastic Gradient Descent (SGD) [11] and its many variants, most notably Adam [12]. The core idea of SGD is to directly draw samples of the gradient of the expected values, rather than using the gradient of the empirical mean to do a gradient descent. SGD’s main advantages are ease of implementation and versatility. Finally, in the case of chance constrained optimization, the Scenario Approach substitutes probabilistic constraints by constraints on Monte Carlos samples, while providing probabilistic guarantees on the result [13], [14].

In robust optimization one minimizes for the worst possible perturbation, while guaranteeing some base level of performance [15], [16]. Robust optimization has been gaining popularity in recent years, for examples in fields such as Model Predictive Control [17] and Machine Learning [18]–[20]. Some new developments have also been made in numerical aspects, notably in [21], where the authors provide first and second order optimality conditions for minmax when the criteria is nonconvex on the minimization variable and nonconcave on the maximization variable. Robust optimization was traditionally not regarded as an approach to solve stochastic programming problems, but in the last decade some articles have connected the two areas, for instance [22]–[24].

Given a scalar function  $V(\cdot)$  and a random vector  $D$ , the expected value of  $V(D)$  can be lower and upper bounded, respectively, by the minimum and maximum values that  $V(\cdot)$  takes over the support of  $D$ . Our first result in Section II shows how these very crude bounds can be improved by including information encoded in the probability density function (*pdf*) of  $D$ . In essence, we solve an optimization over the support of  $D$  that includes terms that penalize unlikely realizations for  $D$ . This means that we need to compute and solve optimality conditions — and therefore essentially compute derivatives and solve algebraic equations — rather than compute integrals.

The results in Section II actually define a family of bounds. Two instances of this family, which we call the additive and multiplicative bounds, are particularly useful. The first is more appropriate to problems where the cost function  $V(\cdot)$  is polynomial, while the second one is more appropriate when the cost function is exponential. Both the additive and

multiplicative bounds are parameterized by a scalar parameters  $\epsilon$ , which can itself be optimized. To guide the design of the bounds and select  $\epsilon$ , we develop necessary and sufficient conditions with respect to  $\epsilon$  that can be used to make sure that the additive and multiplicative bounds are finite.

Borrowing ideas from robust optimization, the bounds developed in Section II are used in Section III to compute approximate solutions to stochastic programming optimizations: Instead of minimizing an expected value subject to stochastic constraints, we minimize upper/lower bounds for the criterion subject to constraints on pessimistic/optimistic bounds for the stochastic constraints. For the lower bound, this leads to a minimization on an extended variable space; for the upper bound, it leads to a minmax problem. The minmax optimizations can be solved using the interior-point primal-dual algorithms proposed in [17] and implemented in `TensCalc`, a high performance numerical solver that combines symbolic differentiation with code generation [25].

In Section IV, we discuss three applications for these bounds. The first relates to finite-horizon stochastic optimal control, with either state feedback or output feedback. In the former case, the initial state is assumed known, but an expectation is needed over the realization of future disturbances. In the latter case, the initial state is unknown, and the expectation is taken with respect to a conditional distribution, given known realizations of past noisy measurements. Our approach can include stochastic constraints on the trajectory of the system, which we illustrate through a constraint on the final state.

The second application is related to Maximum Likelihood or Maximum a Posteriori estimation involving latent variables that cannot be measured [26]. These problems require the latent variables to be marginalized by an expectation that can be upper/lower bounded using the results from Section III.

The third application is in the area of Bayesian experiment design [11], [27], [28]. The goal is to optimize the values of experimental parameters to facilitate the estimation of unknown variables. Experiment design criteria typically involve taking expectations with respect to unknown variables, including the ones that need to be estimated. Also here, optimal experiment design can be performed by replacing expectations by bounds.

In the context of feedback control, all three applications discussed above typically need to be performed in real-time with limited computation, and benefit from the availability of bounds on how the approximate solution compares with the true optimum. It is in such scenarios that the approach proposed here is most attractive. In contrast, when computation is unlimited, Monte Carlo based methods can achieve arbitrarily accurate solutions to stochastic optimization problems as long as one uses a sufficiently large number of samples, and will thus eventually out-perform in accuracy the approach proposed here.

This paper expands on the work reported in the conference paper [29] in several ways: We introduce a family of bounds, while the earlier paper considered only additive bound; we generalize the stochastic programming problem to include stochastic inequalities constraints; and we include applications to estimation and experiment design.

*Notation:* Given an underlying probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , a random variable  $X$  and a scalar  $x \in \mathbb{R}$ , we denote by  $\mathbb{P}(X \leq x)$  the probability measure of the set  $\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$  and by  $\mathbb{E}[X]$  the expected value of  $X$ . The essential infimum and supremum are defined by

$$\begin{aligned} \text{ess inf } X &= \sup\{x \in \mathbb{R} : \mathbb{P}(X \geq x) = 1\} \\ \text{ess sup } X &= \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) = 1\}, \end{aligned}$$

respectively, where  $\inf$  and  $\sup$  refer to the usual infimum and supremum of a set. In an analogous way, given a measurable event  $E \in \mathcal{F}$ , we define conditional essential infimum and supremum by

$$\begin{aligned} \text{ess inf}[X \mid E] &= \sup\{x \in \mathbb{R} : \mathbb{P}(X \geq x \mid E) = 1\} \\ \text{ess sup}[X \mid E] &= \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x \mid E) = 1\}. \end{aligned}$$

The essential supremum and infimum relax the usual supremum and infimum by excluding sets of measure zero. One can informally think of them as  $\sup_{x \in \mathcal{X}} x$  and  $\inf_{x \in \mathcal{X}} x$  where  $\mathcal{X}$  is the support of  $X$ .

## II. BOUNDS ON AN EXPECTED VALUE

Given a random vector  $D$  taking values in  $\mathcal{D} \subset \mathbb{R}^M$  and a scalar measurable function  $V : \mathcal{D} \rightarrow \mathbb{R}$ , the monotonicity of the expected value  $\mathbb{E}[V(D)]$  provides the following basic bound

$$\text{ess inf } V(D) \leq \mathbb{E}[V(D)] \leq \text{ess sup } V(D). \quad (1)$$

The core idea of this section is to improve upon this crude bound by including information about  $D$ , for example, coming from its probability density function (*pdf*). To present our first result, we introduce the following terminology. Consider a right-ordered group  $G := (\mathcal{P}, \oplus)$  defined on a set  $\mathcal{P} \subset \mathbb{R}$  for which the group operation  $\oplus$  satisfies the usual group properties of closure, associativity, existence of an identity element, and existence of inverse elements (which we denote using  $\ominus$ ); as well as the right-ordered property

$$a \leq b \Rightarrow a \oplus c \leq b \oplus c, \quad \forall a, b, c \in \mathcal{P}$$

[30]. We say that  $G := (\mathcal{P}, \oplus)$  is *distributive with respect to integration* (or *E-distributive for short*) if it is right-ordered and, for every random variable  $X$  taking values on  $\mathcal{P}$ , we have that

$$a \oplus \mathbb{E}[X] = \mathbb{E}[a \oplus X], \quad \forall a \in \mathcal{P}.$$

*Theorem 1 (Bounds on an expected value):* Consider an E-distributive group  $G := (\mathcal{P}, \oplus)$ , a random vector  $D$  taking values in  $\mathcal{D} \subset \mathbb{R}^M$ , and measurable functions  $V, \alpha : \mathcal{D} \mapsto \mathcal{P}$ . If  $\mathbb{E}[V(D)]$  and  $\mathbb{E}[-\alpha(D)]$  are finite, then

$$\text{ess inf } J(D) \leq \mathbb{E}[V(D)] \leq \text{ess sup } J(D) \quad (2)$$

where the function  $J : \mathcal{D} \rightarrow \mathbb{R}$  is defined by

$$J(d) := V(d) \oplus \alpha(d) \oplus \mathbb{E}[-\alpha(D)]. \quad \square$$

*Proof.* We prove the upper bound, the proof for the lower bound can be obtained analogously. For every scalar  $v \geq \text{ess sup } V(D) \oplus \alpha(D)$ , we have that

$$\mathbb{P}(V(D) \leq v \oplus -\alpha(D)) = \mathbb{P}(V(D) \oplus \alpha(D) \leq v) = 1,$$

where the first equality is a consequence of the E-distributive property and the second of the definition of essential supremum. From the monotonicity of the expected value, we thus conclude that

$$\mathbb{E}[V(D) \oplus \alpha(D)] \leq v.$$

Since  $\mathbb{E}[-\alpha(D)]$  is finite, we can use the right-ordered property of  $(\mathcal{P}, \oplus)$  to conclude that

$$\mathbb{E}[V(D) \oplus \alpha(D)] \oplus \mathbb{E}[-\alpha(D)] \leq v \oplus \mathbb{E}[-\alpha(D)]$$

and then the E-distributed property to obtain

$$\mathbb{E}[V(D) \oplus \alpha(D) \oplus \mathbb{E}[-\alpha(D)]] = \mathbb{E}[J(D)] \leq v \oplus \mathbb{E}[-\alpha(D)].$$

The upper bound then follows by taking an infimum on the right-hand side over the set of such scalars  $v \geq \text{ess sup } V(D) \oplus \alpha(D)$ . ■

The key idea of Theorem 1 is to improve upon (1) by including in  $J(\cdot)$  terms that reduce the essential supremum and increase the essential infimum. To reduce the supremum, for example, one should select  $\alpha(d)$  so that it is strongly negative (in the sense that  $-\alpha(d)$  should be strongly positive) when  $V(d)$  is large and while keeping  $\mathbb{E}[-\alpha(D)]$  relatively small. In the remainder of the paper we mostly use two E-distributive groups  $G$  and associated functions  $\alpha$  that achieve this for our applications of interest. Both bounds assume that  $D$  has a probability density function (*pdf*) that we denote by  $p_D(\cdot)$ .

**Additive Bound:** The E-distributive group  $(\mathcal{P}, \oplus) = (\mathbb{R}, +)$  with the usual addition of reals, and  $\alpha(d) = \epsilon \log p_D(d)$  with  $\epsilon \in \mathbb{R}$ , leads to

$$J(d, \epsilon) := V(d) + \epsilon \log p_D(d) + \epsilon \mathcal{H}_D, \quad (3)$$

where  $\mathcal{H}_D := \mathbb{E}[-\log p_D(D)]$  is the differential entropy.

**Multiplicative Bound:** The E-distributive group  $(\mathcal{P}, \oplus) = (\mathbb{R}_{>0}, \times)$  with the usual multiplication of positive reals, and  $\alpha(d) = p_D(d)^\epsilon$  with  $\epsilon \in \mathbb{R}$ , leads to

$$J(d, \epsilon) := V(d) p_D(d)^\epsilon \mathcal{I}_D(\epsilon) \quad (4)$$

where  $\mathcal{I}_D(\epsilon) := \mathbb{E}[p_D(D)^{-\epsilon}]$ .

The functions  $J$  in (3) and (4) are not necessarily well defined on the measure zero set where  $p_D(D) = 0$ , but the value of  $J$  on such set is irrelevant, as it does not affect the value of the essential supremum or infimum in (2).

The key idea behind the additive bound is that unlikely values  $d$  for  $D$  will lead to a large negative value for  $\log p_D(d)$  and reduce the value of  $J(d)$ . These unlikely values will contribute with a strong positive value in  $-\log p_D(D)$ , but precisely because they are unlikely, they will not increase  $\mathcal{H}_D := \mathbb{E}[-\log p_D(D)]$  very much. Overall, this should thus decrease the supremum of  $J(d)$  over  $\mathcal{D}$  to create a tighter bound. A similar reason can be used to justify the function  $\alpha$  proposed for the multiplicative bound.

In Appendix A, we derive expressions for  $\log p_D(d) + \mathcal{H}_D$  and  $p_D(d)^\epsilon \mathcal{I}_D(\epsilon)$  for a few common distributions.

*Remark 1 (Bounds for conditional expectation):* Theorem 1 can also be stated for conditional expectations, provided that the E-distributive property holds for the conditional expectation with probability one. In this case, the additive and multiplicative bounds should involve conditional *pdf*. □

*Selection of bound and  $\epsilon$ :* It is possible to establish necessary and sufficient conditions such that the additive and multiplicative bounds lead to non-trivial results, which are presented in Appendix C. For now, we present a corollary with sufficient conditions which are the most relevant to the content of this section. We require the following definition to present the corollary.

Given  $\gamma \in (0, 1)$ , we say a measurable function  $f(\cdot)$  is  $\gamma$ -essentially bounded if  $\mathbb{P}(p_D(D) > \gamma) \in (0, 1)$  and  $\text{ess sup}(|f(D)| \mid p_D(D) > \gamma) < \infty$ . Being  $\gamma$ -essentially bounded is a mild requirement. Any bounded function is also  $\gamma$ -essentially bounded, but a function can be  $\gamma$ -essentially bounded without being bounded.

*Corollary 1 (Sufficient conditions for finite bounds):* Assume  $V(\cdot)$  and  $p_D(\cdot)$  are  $\gamma$ -essentially bounded. Then, for every finite  $\epsilon \in \mathbb{R}$ ,

$$\begin{aligned} \text{ess inf} \left[ \frac{-V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] &> \epsilon \\ \Rightarrow \text{ess inf} (V(D) + \epsilon \log p_D(D)) &> -\infty. \end{aligned}$$

and

$$\begin{aligned} \text{ess sup} \left[ \frac{-V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] &< \epsilon \\ \Rightarrow \text{ess sup} (V(D) + \epsilon \log p_D(D)) &< +\infty \end{aligned}$$

Alternatively, assume that  $\log V(\cdot)$  and  $p_D(\cdot)$  are  $\gamma$ -essentially bounded. Then, for every finite  $\epsilon \in \mathbb{R}$ ,  $\epsilon \in \mathbb{R}$ ,

$$\begin{aligned} \text{ess inf} \left[ \frac{-\log V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] &> \epsilon \\ \Rightarrow \text{ess inf} (\log V(D) + \epsilon \log p_D(D)) &> -\infty. \end{aligned}$$

and

$$\begin{aligned} \text{ess sup} \left[ \frac{-\log V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] &< \epsilon \\ \Rightarrow \text{ess sup} (\log V(D) + \epsilon \log p_D(D)) &< +\infty \quad \square \end{aligned}$$

Corollary 1 establishes that for the additive and multiplicative bounds to be non trivial, it suffices to pick an  $\epsilon$  such that  $\log p_D(D)$  dominates either  $V(D)$  or  $\log V(D)$ . Therefore, which bound to use essentially depends on the rates of growth of  $V(\cdot)$ ,  $\log V(\cdot)$ , and  $\log p_D(\cdot)$ . In the case where both bounds have a finite value, we have heuristically observed that the approximation is better when  $V(\cdot)$  (or  $\log V(\cdot)$ ) has roughly the same magnitude as  $\log p_D(\cdot)$ . This observation could motivate different  $\alpha(\cdot)$  and E-distributive groups in Theorem 1 for applications other than those that will be presented in this journal.

Among the values of  $\epsilon$  that lead to a finite upper bound, the conservativeness of the bound can be minimized by selecting the value of  $\epsilon \in \mathbb{R}$  that minimizes

$$\min_{\epsilon \in \mathbb{R}} J^*(\epsilon), \quad J^*(\epsilon) := \text{ess sup } J(D, \epsilon) \quad (5)$$

with  $J(d, \epsilon)$  as in (3) or (4). It turns out that such minimization over the scalar parameters  $\epsilon$  is well-behaved as the function  $J^*(\epsilon)$  in (5) has appropriate convexity properties, as noted in the following result:

*Proposition 1 (Optimization over  $\epsilon$ ):* The function  $J^*(\epsilon)$  in (5) is convex for  $J(d, \epsilon)$  in (3) and log-convex for  $J(d, \epsilon)$  in (4). Moreover,  $J^*(\epsilon)$  is finite on a convex set.  $\square$

*Proof.* See Appendix B.  $\blacksquare$

*Remark 2 (Finiteness of  $\mathcal{I}_D(\epsilon)$ ):* With respect to the multiplicative bound, as the quantity  $\mathcal{I}_D(\epsilon) = \mathbb{E}[p_D(D)^{-\epsilon}]$  is not as widely used as the differential entropy  $\mathcal{H}_D$ , we state explicit conditions under which it takes a finite value: Assume that  $\text{ess sup } p_D(D) < +\infty$ . If  $D$  has finite support in the sense that  $\int_{\mathcal{D}} 1 dx < +\infty$ , then  $\mathcal{I}_D(\epsilon)$  is always finite. Otherwise,  $\mathcal{I}_D(\epsilon)$  is finite for  $\epsilon < 1$ .  $\square$

*Proof.* See Appendix B.  $\blacksquare$

### III. STOCHASTIC PROGRAMMING

We define the following *stochastic programming problem* with a single scalar constraint, but the approach proposed can easily be extended to multiple constraints: Let  $D$  be a random vector taking values in  $\mathcal{D} \subset \mathbb{R}^M$ . Given measurable functions  $V : \mathcal{U} \times \mathcal{D} \mapsto \mathbb{R}$  and  $G : \mathcal{U} \times \mathcal{D} \mapsto \mathbb{R}$ , with  $\mathcal{U} \subset \mathbb{R}^N$  we want to solve

$$V^* := \inf \left\{ \mathbb{E}[V(u, D)] : u \in \mathcal{U}, \mathbb{E}[G(u, D)] \leq 0 \right\}. \quad (6)$$

*Theorem 2 (Bounds to Stochastic Programming):* Consider two E-distributive groups  $(\mathcal{P}_V, \oplus_V)$  and  $(\mathcal{P}_G, \oplus_G)$ , functions  $\alpha_V : \mathcal{D} \mapsto \mathcal{P}_V$  and  $\alpha_G : \mathcal{D} \mapsto \mathcal{P}_G$ , and define

$$\begin{aligned} J_V(u, d) &:= V(u, d) \oplus_V \alpha_V(d) \oplus_V \mathbb{E}[-\alpha_V(D)] \\ J_G(u, d) &:= G(u, d) \oplus_G \alpha_G(d) \oplus_G \mathbb{E}[-\alpha_G(D)]. \end{aligned}$$

If  $\mathbb{E}[-\alpha_V(D)]$  and  $\mathbb{E}[-\alpha_G(D)]$  are finite, then the values  $V^\nabla$  and  $V^\Delta$  obtained from solving

$$V^\nabla := \inf \left\{ \text{ess inf } J_V(u, D) : u \in \mathcal{U}, \text{ess inf } J_G(u, D) \leq 0 \right\} \quad (7)$$

and

$$V^\Delta := \inf \left\{ \text{ess sup } J_V(u, D) : u \in \mathcal{U}, \text{ess sup } J_G(u, D) \leq 0 \right\} \quad (8)$$

satisfy  $V^\nabla \leq V^* \leq V^\Delta$ . Furthermore, if any  $u^\Delta$  achieves  $V^\Delta$  and is feasible for (8), then it is also feasible for (6).  $\square$

In essence, Theorem 2 guarantees that any solution  $u^\Delta$  to the optimization (8) is feasible for the original stochastic program in (6). It also provides performance guarantees for  $u^\Delta$ , in the sense that the expected value  $\mathbb{E}[V(u^\Delta, D)]$  obtained using  $u^\Delta$  will be away from the optimal  $V^*$  by no more than  $V^\Delta - V^\nabla$ , which can be computed by solving the optimizations (7)–(8).

It is important to note that  $\mathbb{E}[-\alpha_V(D)]$  is a constant that does not depend on either  $u$  nor  $d$ , and therefore does not affect the optimizations in (7)–(8). This means that if one is not able to determine analytically  $\mathbb{E}[-\alpha_V(D)]$ , any errors in estimating this quantity will not introduce errors in determining  $u^\Delta$ .

As it was the case for Theorem 1, relevant choices for  $\alpha(\cdot)$  and  $\mathcal{P}$  include the additive (3) and multiplicative (4) bounds. In these cases, one can also include an optimization with respect to  $\epsilon$  in Theorem 2 which would have the appropriate convexity properties as it was shown in Proposition 1.

*Proof of Theorem 2.* Starting with the sets, from Theorem 1, we know that  $\text{ess inf } J_G(u, D) \leq \mathbb{E}[G(u, D)] \leq \text{ess sup } J_G(u, D)$ , which implies the following set relation  $\{u \in \mathcal{U} : \text{ess inf } J_G(u, D) \leq 0\} \supset \{u \in \mathcal{U} : \mathbb{E}[G(u, D)] \leq 0\} \supset \{u \in \mathcal{U} : \text{ess sup } J_G(u, D) \leq 0\}$ . Therefore,

$$\inf \left\{ \mathbb{E}[V(u, D)] : u \in \mathcal{U}, \text{ess inf } J_G(u, D) \leq 0 \right\} \quad (9)$$

$$\leq \inf \left\{ \mathbb{E}[V(u, D)] : u \in \mathcal{U}, \mathbb{E}[G(u, D)] \leq 0 \right\} \quad (10)$$

$$\leq \inf \left\{ \mathbb{E}[V(u, D)] : u \in \mathcal{U}, \text{ess sup } J_G(u, D) \leq 0 \right\} \quad (11)$$

Applying the lower bound of Theorem 1 to  $\mathbb{E}[V(u, D)]$  in (9) and the upper bound to  $\mathbb{E}[V(u, D)]$  in (11) finishes the proof.  $\blacksquare$

*Remark 3 (Numerically computing the bounds):* When  $J_V(\cdot)$  and  $J_G(\cdot)$  are continuous and the problem satisfies the conditions of the Weirstrass Theorem (Proposition A.8 from [31]), for instance if  $\mathcal{D}$  and  $\mathcal{U}$  are compact, then (7) and (8) are respectively equivalent to

$$V^\nabla := \min \left\{ \min_{d \in \mathcal{D}} J_V(u, d) : u \in \mathcal{U}, \exists \tilde{d} \in \mathcal{D} : J_G(u, \tilde{d}) \leq 0 \right\} \quad (12)$$

$$V^\Delta := \min \left\{ \max_{d \in \mathcal{D}} J_V(u, d) : u \in \mathcal{U}, \forall \tilde{d} \in \mathcal{D} : J_G(u, \tilde{d}) \leq 0 \right\}. \quad (13)$$

In the case where (6) does not have stochastic constraints (*i.e.* no  $G(\cdot)$ ), then (12) reduces to a – possibly nonconvex – minimization problem. Likewise, (13) reduces to a – possibly nonconvex-nonconcave – minmax problem. In this case, one can use algorithms such as the one presented in [17] and implemented in `TensCalc` [25], which find stationary points, and then use the conditions presented in [21] to verify whether the points are minmax. In the other case, when there are stochastic constraints, we refer to [15], [32], [33] for a detailed exposition on robust optimization.  $\square$

*Remark 4 (Improving upon the bounds by combining with Monte Carlo methods):* While solving the optimization in (6) may be challenging, it is generally straightforward to estimate the expected values  $\mathbb{E}[V(u^\Delta, D)]$  and  $\mathbb{E}[G(u^\Delta, D)]$  for a specific candidate solutions  $u^\Delta$  using a numerical method, such as Monte Carlo integration [6]. This allows one to improve the bounds of Theorem 2 to  $V^\nabla \leq V^* \leq \mathbb{E}[V(u^\Delta, D)]$ .

Likewise, in the case where the constraint  $\text{ess sup } J_G(u, D) \leq 0$  in (8) is too conservative, one can relax it – either by including offsets or using Lagrangian relaxation – and use numerical integration to verify whether the minimum obtained using relaxed constraints still satisfies the stochastic constraint of the original problem (6).  $\square$

## IV. SELECTED APPLICATIONS

### A. Stochastic control

Consider the dynamical system

$$x_{t+1} = f(x_t, \theta, u_t, d_t) \quad (14a)$$

$$y_t = h(x_t) + n_t, \quad (14b)$$

where  $x_t$  denotes the state of the system at time  $t$ ,  $u_t$  the controlled input,  $d_t$  a random disturbance input,  $y_t$  the measured output,  $n_t$  measurement noise, and  $\theta$  a random vector of parameters.

Our goal to select control inputs  $u_0, \dots, u_{T-1}$  to minimize a finite-horizon criterion of the form

$$\mathbb{E}[W(x_1, \dots, x_T, u_0, \dots, u_{T-1})], \quad (15)$$

subject to a constraint of the form

$$\mathbb{E}[U(x_1, \dots, x_T, u_0, \dots, u_{T-1})] \leq 0. \quad (16)$$

We consider two versions of this problem: First a *state-feedback* scenario in which the initial state  $x_0$  is known and the expectation (15) is with regard to the random parameters  $\theta$  and the disturbances  $d_0, \dots, d_{T-1}$ . We then consider an *output-feedback* scenario in which the initial state is not known, but one has available past measurements  $y_{-K}, \dots, y_0$ . In this case, the expectation in (15) is conditioned to these past measurements and it regards the measurement noise  $n_{-K}, \dots, n_0$ , the initial state  $x_{-K}$ , and the past disturbances  $d_{-K}, \dots, d_{-1}$ .

a) *State Feedback*: The state-feedback control problem can be viewed as an instance of (6), with the following associations

$$\begin{aligned} u &:= (u_0, \dots, u_{T-1}), \\ D &:= (\theta, d_0, \dots, d_{T-1}), \\ V(u, D) &:= W(x_1, \dots, x_T, u_0, \dots, u_{T-1}), \\ G(u, D) &:= U(x_1, \dots, x_T, u_0, \dots, u_{T-1}), \end{aligned}$$

with the understanding that the states  $x_1, \dots, x_T$  that appear in the definitions of  $V(u, D)$  and  $G(u, D)$  are obtained along solutions to (14a) for the control input in  $u$  and the parameters and input disturbances in  $D$ .

Assuming that the disturbances  $d_t$  are independent and identically distributed with *pdf*  $p_d(\cdot)$  and differential entropy  $\mathcal{H}_d$ , and that the parameter  $\theta$  has *pdf*  $p_\theta(\cdot)$  and differential entropy  $\mathcal{H}_\theta$ , we have that

$$\mathcal{H}_D = \mathcal{H}_\theta + T\mathcal{H}_d, \quad \log p_D(\theta, d) = \log p_\theta(\theta) + \sum_{t=0}^{T-1} \log p_d(d_t),$$

and the optimization in (8) with additive upper bounds for  $\oplus_G$  and  $\oplus_V$  takes the form

$$\begin{aligned} V^\Delta = \min_{u \in \mathcal{U}} \left\{ X(u) : U(\bar{x}_1, \dots, \bar{x}_T, u_0, \dots, u_{T-1}) + \bar{\epsilon} \mathcal{H}_D + \right. \\ \left. + \bar{\epsilon} \log p_D(\bar{\theta}, \bar{d}) \leq 0, \forall \bar{\theta}, \bar{d} \right\} \\ X(u) := \max_{\theta \in \Theta, d \in \mathcal{D}} W(x_1, \dots, x_T, u_0, \dots, u_{T-1}) + \epsilon \mathcal{H}_D + \\ + \epsilon \log p_D(\theta, d), \end{aligned}$$

where  $\mathcal{U}$  denotes the set of admissible controls;  $\Theta$  and  $\mathcal{D}$  the supports of the distributions for the random parameter and disturbance, respectively;  $\bar{x}_1, \dots, \bar{x}_T$  the solution to (14a) for the control  $u := (u_0, \dots, u_{T-1})$ , parameter  $\theta$  and disturbance  $\bar{d} := (\bar{d}_0, \dots, \bar{d}_{T-1})$ ;  $x_1, \dots, x_T$  the solution to (14a) for the same control  $u := (u_0, \dots, u_{T-1})$ , but parameter  $\theta$  and disturbance  $d := (d_0, \dots, d_{T-1})$ ; and  $\epsilon, \bar{\epsilon}$  the scalar parameters associated with additive upper bounds used for  $\oplus_G$  and  $\oplus_V$ , respectively. An equivalent formulation of the optimization in (7) gives  $V^\nabla$ .

b) *Output Feedback*: The output-feedback problem can also be viewed as an instance of (6), but now with the following associations

$$\begin{aligned} u &:= (u_0, \dots, u_{T-1}), \\ D &:= (\theta, x_{-K}, d_{-K}, \dots, d_{T-1}), \\ V(u, D) &:= W(x_1, \dots, x_T, u_0, \dots, u_{T-1}), \\ G(u, D) &:= U(x_1, \dots, x_T, u_0, \dots, u_{T-1}), \end{aligned}$$

with the understanding that the states  $x_1, \dots, x_T$  that appear in the definition of  $V(u, D)$  and  $G(u, D)$  are obtained along solutions to (14a) for the control input in  $u$  and the parameters, initial state, and input disturbances in  $D$ . In addition, the expectation in (6) is now a conditional expectation, given measurements  $Y = (y_{-K}, \dots, y_0)$  defined by (14b).

In this case, the optimization in (8) with additive upper bounds for  $\oplus_V$  and  $\oplus_G$  takes the form

$$\begin{aligned} V^\Delta = \min_{u \in \mathcal{U}} \left\{ X(u) : U(\bar{x}_1, \dots, \bar{x}_T, u_0, \dots, u_{T-1}) + \right. \\ \left. \bar{\epsilon} \log p_{D|Y}(\bar{\theta}, \bar{d}, \bar{x}_k) + \bar{\epsilon} \mathcal{H}_{D|Y}(y_{-K}, \dots, y_0) \leq 0, \forall \bar{\theta}, \bar{d}, \bar{x}_k \right\} \\ X(u) := \max_{\theta \in \Theta, d \in \mathcal{D}, x_{-K} \in \mathcal{X}_{-K}} W(x_1, \dots, x_T, u_0, \dots, u_{T-1}) + \\ \epsilon \log p_{D|Y}(\theta, d, x_{-K}) + \epsilon \mathcal{H}_{D|Y}(y_{-K}, \dots, y_0), \quad (17) \end{aligned}$$

where we use the version of the bounds for conditional expectation mentioned in Remark 1. The conditional *pdf* that appears in (17) can be computed using the following result.

*Lemma 1 (Conditional pdf of a dynamical system)*: In addition to the assumptions made for the state feedback case, also assume that the observation noises  $n_t$  are independent and identically distributed with *pdf*  $p_n(\cdot)$  and that the initial state  $x_{-K}$  has *pdf*  $p_{x_{-K}}(\cdot)$ . If  $p_Y(y_{-K}, \dots, y_0) \neq 0$ , the conditional probability density function  $p_{D|Y}(\cdot)$  is given by

$$\frac{\prod_{t=-K}^0 p_n(y_t - h(x_t)) \prod_{t=-K}^{T-1} p_d(d_t) p_{x_{-K}}(x_{-K}) p_\theta(\theta)}{p_Y(y_{-K}, \dots, y_0)}$$

with the understanding that  $x_t$  is obtained along the solutions to (14a).  $\square$

*Proof of Lemma 1.* Using the independence of  $n_t$ , one deduces that the observations  $y_t$  are conditionally independent:

$$p_{Y|D}(y_{-K}, \dots, y_0 | x_{-K}, \dots, x_0) = \prod_{t=-K}^0 p_{Y_t|D}(y_t | x_t).$$

As the noise  $n_t$  is additive in (14b), a change of variable gives  $p_{Y_t|D}(y_t | x_t) = p_{n_t}(y_t - h(x_t))$ . Using Bayes' theorem and the independence of  $d_t$ ,  $\theta$ , and  $x_{-K}$  finishes the proof.  $\blacksquare$

When, the differential entropy  $\mathcal{H}_{D|Y}(y_{-K}, \dots, y_0)$  that appears in (17) does not have a closed form expression, one can use a numerical integration methods to obtain it, as mentioned in Remark 4. In particular, if the disturbances  $d_t$  are additive, one can use a Monte Carlo integrator with the sampling method described in [34].

*Example 1 (Linear system with unknown dynamics)*: Consider a linear system, *i.e.*, a system with dynamics

$$\begin{aligned} x_{t+1} &= A x_t + B u_t + d_t \\ y_t &= C x_t + n_t \end{aligned}$$

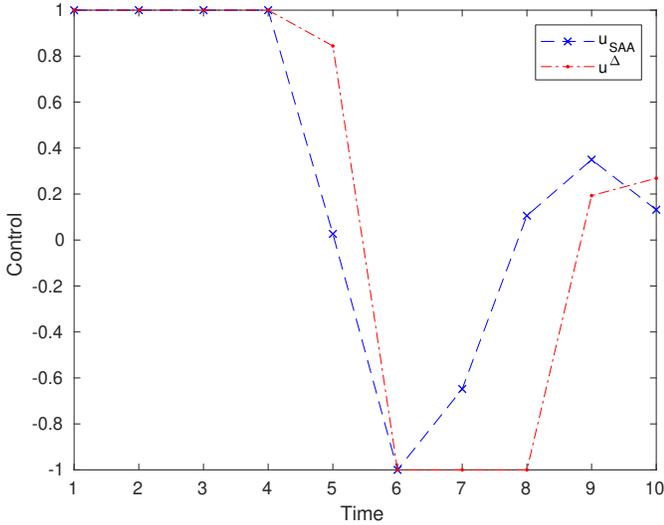


Fig. 1: Linear system with unknown dynamics, comparison of the controls  $u^\Delta$  obtained from Theorem 2 and  $u_{SAA}$  obtained using Sample Average Approximation. Using Monte Carlo integration, we obtain that  $\mathbb{E}[V(u^\Delta, D)] = 1.79 \times 10^3$  and  $\mathbb{E}[V(u_{SAA}, D)] = 1.62 \times 10^3$ .

with  $d_t$  and  $n_t$  independent zero mean standard Gaussian processes. The system is time-invariant,  $C$  is an identity matrix, but the matrices  $A$  and  $B$  are unknown stochastic parameters of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ B_{31} \end{bmatrix},$$

where  $A_{11}, A_{12}, A_{22}, A_{23}, A_{33}, B_{31}$  are independent Gaussian random variable with mean 1 and standard deviation 0.25. We chose a quadratic cost

$$W(u_0, \dots, u_{T-1}, x_0 \dots x_T) = \sum_{t=0}^{T-1} 0.5 \|u_t\|_2^2 + 0.5 \|x_t\|_2^2 + 0.5 \|x_T\|_2^2$$

with a future horizon  $T = 10$  and constraints on the control that  $\|u\|_\infty \leq 1$ . We suppose access to past measurements  $y_{-K}, \dots, y_0$  with  $K = 20$ .

The value of the upper bound  $V^\Delta$  is  $5.04 \times 10^5$  and the value of the lower bound  $V^\nabla$  is 28. We compare our results with an approximate solution obtained using Sample Average Approximation (SAA) (*i.e.*, minimizing an empirical mean of the cost). Solving the upper bound and lower bound optimizations (Theorem 2) takes about 0.1, while solving the Sample Average Approximation takes about 5 minutes. In Figure 1 one can see that the controls match each other fairly closely until  $t = 6$ , when they start to slightly diverge. We also use Monte Carlo integration, as mentioned in Remark 4, to estimate the expected value of the cost for the two controls, obtaining that they differ by about 10%.

*Example 2 (Dubins vehicle):* Consider a discrete time Dubins vehicle [35], [36] with dynamics

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \omega_{t+1} \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \\ \omega_t \end{bmatrix} + T_s \begin{bmatrix} v \cos(\omega_t) \\ v \sin(\omega_t) \\ u_t \end{bmatrix} + \frac{T_s^2}{2} \begin{bmatrix} -v \sin(\omega_t) u_t \\ v \cos(\omega_t) u_t \\ 0 \end{bmatrix} + d_t$$

where  $T_s = 0.1$  is the sampling period,  $v = 1$  is a constant forward speed. The initial state is known to be  $[x_0, y_0, \omega_0]^\top = [0, 0, 0]$ , and we want to optimize for a future horizon  $T = 50$ . The controls are constrained such that  $\|u\|_\infty \leq \pi/2$ . The disturbance  $d_t = [d_t^{(x)}, d_t^{(y)}, d_t^{(\omega)}]^\top$  is such that  $d_t^{(x)}, d_t^{(y)}$  are zero mean Gaussian random variable with variance  $T_s$ , and  $d_t^{(\omega)}$  is a von Mises random variable, with probability density function  $e^{\kappa \cos(x)} / (2\pi I_0(\kappa))$  with  $\kappa = 5/T_s$  and where  $I_0(\kappa)$  is the modified Bessel function of order 0. The cost function is

$$W(u_0, \dots, u_{T-1}, x_0 \dots x_T) = \sum_{t=0}^{T-1} 0.5 \|u_t\|_2^2 + \sum_{t=0}^T 0.5 \|x_t\|_2^2 + 0.5 \|y_t\|_2^2 + 10^{-3} \|\omega_t\|_2^2$$

We present two cases, one with no constraints on the states and one with a constraint on the final state. For both of them, we use the additive bounds.

The first case, without constraints, takes about 0.01 seconds to solve, the value of the upper bound  $V^\Delta$  is  $6.25 \times 10^7$  the lower bound  $V^\nabla$  only provides the trivial value of 0. However, using a Monte Carlo integration we compute the expected value of the cost given the control and obtain  $2.39 \times 10^6$ . Using Monte Carlo integration we are able to verify that

$$\frac{1}{T} \left\| \frac{d\mathbb{E}[V(u^\Delta, D)]}{du} \right\|_2 = 0.0071$$

suggesting that  $u^\Delta$  approximately satisfies the first order optimality condition.

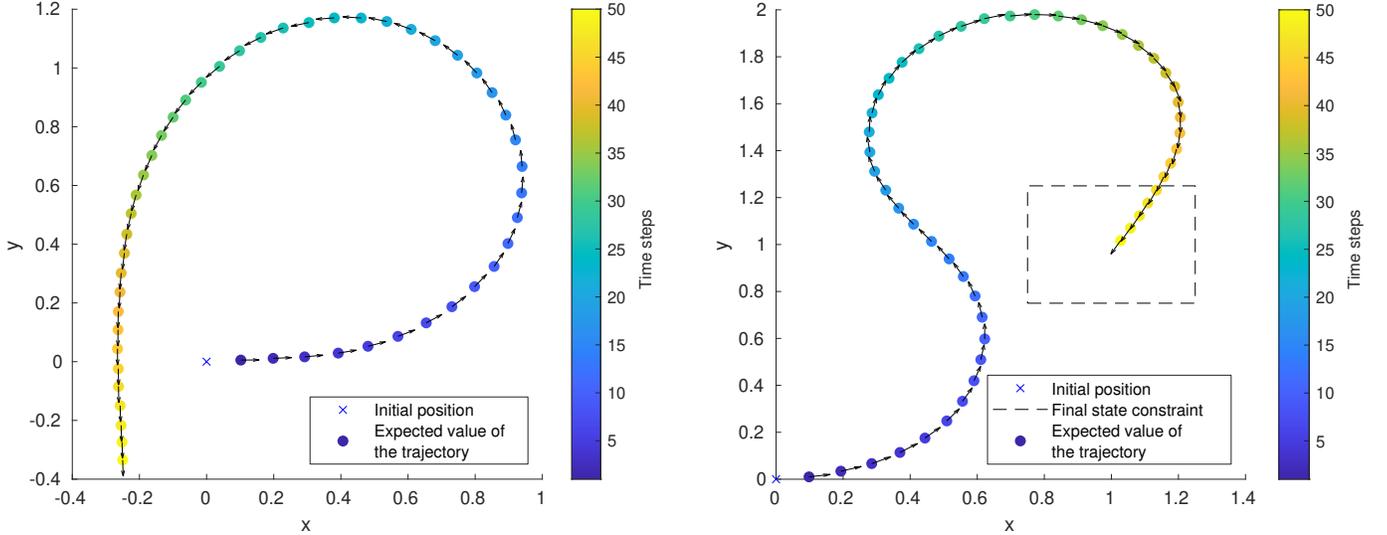
For the second case, we include the constraint

$$\mathbb{E} \left[ \left\| \begin{bmatrix} x_T \\ y_T \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_\infty \right] \leq 0.25 \quad (18)$$

*i.e.*, we want to find a control such that the expectation of the final value of the trajectories of  $(x, y)$  be in a box of length 0.5 around the point  $(1, 1)$  (look at Figure 2b for a visualization of the constraints). Applying the upper bound to (18) leads to

$$\max_d \left\| \begin{bmatrix} x_T \\ y_T \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_\infty + \epsilon \log p_D(d) + \epsilon \mathcal{H}_D \leq 0.25. \quad (19)$$

However the bound (19) of (18) is too conservative, and using (19) as constraint renders the problem unfeasible. Instead, we relax this constraint using Lagrangian relaxation, *i.e.*, we minimize the Lagrangian function of the constraint. As mentioned in Remark 4 we can solve the relaxed problem, and then use Monte Carlo integration to verify whether the solution satisfies the constraint (18). It takes about 30 seconds to solve the optimization for which we obtain a value for the upper bound  $V^\Delta$  of  $1.25 \times 10^8$  and the lower bound  $V^\nabla$  provides only the trivial value of 0. However, using Monte Carlo integration, we obtain that the expected value of the cost is  $3.06 \times 10^6$ .



(a) Expected value of the trajectory without constraints on the final state; the expected value of the cost for this control is  $\mathbb{E}(V(u^\Delta, D)) = 2.39 \times 10^6$ .

(b) Expected value of the trajectory with constraints on the final state; the expected value of the cost for this control is  $\mathbb{E}(V(u^\Delta, D)) = 3.06 \times 10^6$ .

Fig. 2: Expected value of the trajectory of the Dubins vehicle given two different controls estimated using Monte Carlo integration. Without constraints (a), the control brings the expected value of the trajectory back to near the origin. With the inclusion of constraints (b), the control drives the expected value of the final state towards the correct region.

### B. Maximum Likelihood and Maximum a Posteriori with latent variables

Consider an observation  $x$  of a random vector  $X$  taking values in  $\mathbb{R}^M$  whose distribution depends on an unknown parameter  $\theta \in \mathbb{R}^P$  that one wants to estimate. The Maximum Likelihood Estimation (MLE) [26] of  $\theta$  is a vector  $\theta^* \in \mathbb{R}^P$  such that

$$\theta^* \in \arg \max_{\theta} p_X(x; \theta). \quad (20)$$

where the *pdf* of  $X$  is  $p_X(x; \theta)$ . The Maximum a Posteriori (MAP) is the analogous of the MLE in Bayesian estimation, *i.e.*, when one regards  $\theta$  as a realization of a random variable  $\Theta$ , called the prior, which has *pdf*  $p_{\Theta}(\cdot)$ . In this case, the MAP estimation of  $\theta$  is a vector  $\theta^* \in \mathbb{R}^P$  such that

$$\theta^* \in \arg \max_{\theta} p_{X|\Theta}(x | \theta) p_{\Theta}(\theta). \quad (21)$$

In many cases, constructing the model requires including latent variables that cannot be directly observed. This means that one does not know  $p_X(x; \theta)$  but does know  $p_{X|D}(x | d; \theta) p_D(d)$ , where  $D$  is a "latent" random vector taking values in  $\mathbb{R}^N$ . In this case, the MLE  $\theta^*$  is given by

$$\begin{aligned} \theta^* \in \arg \max_{\theta} p_X(x; \theta) &= \int_{\mathcal{D}} p_{X|D}(x | d; \theta) p_D(d) dd \\ &= \arg \max_{\theta} \mathbb{E}[p_{X|D}(x | D; \theta)]. \end{aligned} \quad (22)$$

For the MAP, the analogous deduction leads to

$$\theta^* \in \arg \max_{\theta} \mathbb{E}[p_{X|D}(x | D; \theta)] p_{\Theta}(\theta). \quad (23)$$

Computing the expected values in (22) or in (23) is normally intractable. The standard approach is to use the Expectation Maximization (EM) algorithm [37]. An issue with EM, in

addition to a rate of convergence that might be very slow, is that it requires computing in closed form the expected value  $\mathbb{E}_{D|X; \hat{\theta}}[\log p_{X,D}(X, D; \theta)]$ , which is often not possible. In some cases, one can use Monte Carlo EM [37] to compute it, but with rates of convergence even slower.

The MLE optimization (22) can be viewed as an unconstrained form of (6), which using the multiplicative upper bound in (8) leads to

$$\theta^\nabla \in \arg \max_{\theta} \min_d p_{X|D}(x | d; \theta) p_D(d)^\epsilon \mathcal{I}_D(\epsilon) \quad (24)$$

or equivalently,

$$\theta^\nabla \in \arg \max_{\theta} \min_d \log(p_{X|D}(x | d; \theta) p_D(d)^\epsilon \mathcal{I}_D(\epsilon)), \quad (25)$$

which is numerically more stable. For the MAP, one would add  $\log p_{\Theta}(\theta)$  to the right hand side of (25). The multiplicative bound because is more amenable for the optimization than the additive as it allows to solve (24) in its logarithmic form (25).

*Example 3 (Linear measurements with additive Gaussian noise):* Let  $D \sim \mathcal{N}(0, \sigma_D)$ ,  $N \sim \mathcal{N}(0, \sigma_N)$ . Consider  $T$  observations of the random variable  $X_t = \theta + D_t + N_t$  where  $\theta$  is the parameter to be estimated. This problem has a closed form solution, which is the empirical average of  $x_t$ . Applied to this problem, equation (25) reduces to <sup>1</sup>

$$\begin{aligned} \theta^\nabla \in \arg \max_{\theta} \min_{d_{1:T}} \sum_{t=1}^T -\|x_t - d_t - \theta\|_2^2 \sigma_N^{-1} - \epsilon \|d_t\|_2^2 \sigma_D^{-1} \\ - T \log(1 - \epsilon) - T \log(2\pi\sigma_N), \end{aligned} \quad (26)$$

<sup>1</sup>We refer the reader to Appendix A for the deduction of the penalizing term.

$\theta$ (actual value)	$\theta^\nabla$	naive MAP	MC MMSE	MC MAP
$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1.95 \\ 2.37 \end{bmatrix}$	$\begin{bmatrix} 2.8 \\ 0.21 \end{bmatrix}$	$\begin{bmatrix} 1.82 \\ 1.62 \end{bmatrix}$	$\begin{bmatrix} 2.13 \\ 1.70 \end{bmatrix}$

TABLE I: Comparison between the real value of  $\theta$ , of  $\theta^\nabla$  obtained from (27) and three other estimators.

where we use the notation  $\|v\|_Q^2 := v'Qv$ . If we take any  $\epsilon$  such that  $\epsilon < -\sigma_N^{-1}/\sigma_D^{-1}$ , then the solution is  $\frac{1}{T} \sum_{t=1}^T x_t$  which is the same as the exact solution.

*Example 4 (Norm measurements with Gaussian disturbances and noise):* We have  $T$  observations of the random variable  $X_i = \|\theta + D_i\|_2 + N_i$  where  $\theta$  is the parameter to be estimated,  $D \sim \mathcal{N}(0, \Sigma_D)$  and  $N \sim \mathcal{N}(0, \sigma_N)$ . We also have a prior distribution  $\Theta \sim \mathcal{N}(\bar{\theta}, \Sigma_\Theta)$  on  $\theta$ . Applied to this problem, equation (25) reduces to

$$\theta^\nabla \in \arg \max_{\theta} \min_{d_{1:T}} \sum_{t=1}^T -\left\|x_t - \|\theta + d_t\|_2\right\|_{\sigma_N^{-1}}^2 - \epsilon \|d_t\|_{\Sigma_D^{-1}}^2 - \|\theta - \bar{\theta}\|_{\Sigma_\Theta^{-1}}^2 - 2T \log(1 - \epsilon) - T \log(2\pi\sigma_N). \quad (27)$$

We take the numerical values  $T = 20$ ,  $\Sigma_D = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ ,  $\sigma_N = 1$ ,  $\bar{\theta} = \begin{bmatrix} 1.8 \\ 1.8 \end{bmatrix}$ ,  $\Sigma_\Theta$  the identity matrix.

The result of (27) is shown in TABLE I where we compare it with three other estimators. The first one is what we call naive MAP, where one treats  $D_1, \dots, D_T$  not as a latent variable, but as a regular variable that one wants to estimate, *i.e.*,

$$\arg \max_{\theta, d_{1:T}} \sum_{t=1}^T -\left\|x_t - \|\theta + d_t\|_2\right\|_{\sigma_N^{-1}}^2 - \|d_t\|_{\Sigma_D^{-1}}^2 - \|\theta - \bar{\theta}\|_{\Sigma_\Theta^{-1}}^2.$$

The second and third are Monte Carlo methods, where we use a Markov Chain Monte Carlo to obtain  $10^6$  samples from  $\Theta | X$ , which takes about 30 minutes. Using these sample, the second estimator is the Monte Carlo estimate of the Minimum Mean Square Error (MC MMSE) estimator (*i.e.*, the empirical average of the samples). The third estimator, we use the sample based estimator of the mode described in [38] to compute a Monte Carlo estimate of the MAP (MC MAP).

Our estimator  $\theta^\nabla$  is significantly closer to real  $\theta$  and to the MCMC estimate of the MAP than the naive MAP.  $\theta^\nabla$  is also approximately as distant to the true value of  $\theta$  as the MMSE estimate. Although none of them is the real MAP, these results suggest that  $\theta^\nabla$  accurately captures the estimation problem and provides a better result than naively trying to estimate  $d_{1:T}$  as in the naive MAP.

### C. Bayesian Optimal Experiment Design

The goal in experiment design is to find inputs for an estimation problem that will yield samples that provide "more information per sample". Consider a random vector  $X$  with pdf  $p_X(x; u, \theta)$  where  $\theta$  is a vector of unknown parameters and  $u$  a vector of control decision taking values in  $\mathcal{U} \subset \mathbb{R}^N$ . The Fisher Information Matrix is

$$\mathcal{FI}(u, \theta) = \mathbb{E} \left[ \frac{d \log p_X(X; u, \theta)}{d\theta} \frac{d \log p_X(X; u, \theta)'}{d\theta} \right],$$

where the expected value is taken with respect to  $X$  and where we use the denominator-layout notation for the derivatives (producing column vectors). The Cramer-Rao lower bound states that, given any unbiased estimator  $\hat{\theta}(u, X)$  of  $\theta$ , its covariance  $\mathbb{E}[(\hat{\theta}(u, X) - \theta)(\hat{\theta}(u, X) - \theta)']$  is lower bounded (in the positive definite matrix sense) by  $\mathcal{FI}(u, \theta)^{-1}$ . Therefore, if one minimizes (according to some criteria)  $\mathcal{FI}(u, \theta)^{-1}$ , one will decrease the covariance of any estimator achieving the Cramer-Rao bound.

In Bayesian optimal experiment design one assumes that  $\theta$  is a realization of an underlying random vector  $\Theta$ , with pdf  $p_\Theta(\cdot)$ , and select  $u^*$  to minimize the Bayesian D-optimality, criteria:

$$u^* \in \arg \min_{u \in \mathcal{U}} \mathbb{E}[\log \det(\mathcal{FI}(u, \Theta)^{-1})], \quad (28)$$

where the expected value is taken with respect to  $\Theta$ . It is shown in [28] that (28) optimizes the gain in the Shannon information of the experiment when  $\hat{\theta}(u, X)$  is a Gaussian distribution with mean  $\theta$  and covariance  $\mathcal{FI}(u, \theta)^{-1}$ . In other words, it designs an experiment that brings more information on average. Alternative Bayesian criteria include A-optimality, where one wants to find a  $u^*$  such that

$$u^* \in \arg \min_{u \in \mathcal{U}} \text{tr}(\mathbb{E}[\mathcal{FI}(u, \Theta)^{-1}]). \quad (29)$$

In this case, (29) minimizes the mean square error of any estimator  $\hat{\theta}(u, X)$  that is unbiased and achieves the Cramer-Rao bound.

The experiment design in (28) and (29) is an unconstrained form of (6). Using the additive upper bound in (8) leads to

$$\begin{aligned} V^\Delta &= \min_{u \in \mathcal{U}, \epsilon} \max_{\theta \in \Omega} -\log \det(\mathcal{FI}(u, \theta)) + \epsilon \log p_\Theta(\theta) + \epsilon \mathcal{H}_\Theta \\ V^\nabla &= \max_{\epsilon} \min_{u \in \mathcal{U}, \theta \in \Omega} -\log \det(\mathcal{FI}(u, \theta)) + \epsilon \log p_\Theta(\theta) + \epsilon \mathcal{H}_\Theta. \end{aligned} \quad (30)$$

For Bayesian A-optimality (29), we obtain

$$\begin{aligned} V^\Delta &= \min_{u \in \mathcal{U}, \epsilon} \max_{\theta \in \Omega} \text{tr}(\mathcal{FI}(u, \theta)) + \epsilon \log p_\Theta(\theta) + \epsilon \mathcal{H}_\Theta \\ V^\nabla &= \max_{\epsilon} \min_{u \in \mathcal{U}, \theta \in \Omega} \text{tr}(\mathcal{FI}(u, \theta)) + \epsilon \log p_\Theta(\theta) + \epsilon \mathcal{H}_\Theta. \end{aligned} \quad (31)$$

*Example 5 (Optimal trajectories for thermal air wind detection):*

A glider is an air vehicle that flies without propellers, using only wind forces to change its altitude. In order to move up, a glider needs to estimate the location and intensity of the thermal vertical wind that would push it [39]–[41].

Given an air column, a common model for the intensity of the vertical wind speed at position  $z = (x, y)$  is

$$w(\bar{w}, \gamma, \bar{z}, z) = \bar{w} e^{-\gamma \|z - \bar{z}\|_2^2}$$

where  $\bar{z} = (\bar{x}, \bar{y})$  denotes the position of the thermal center,  $\bar{w}$  the wind speed at the thermal center and  $1/\gamma$  the thermal radius. Our goal is to estimate the thermal parameters  $\theta = (\bar{w}, \gamma, \bar{z})$  based on noisy measurements of the vertical air speed of the form

$$V_t = w(\bar{w}, \gamma, \bar{z}, z_t) + N_t$$

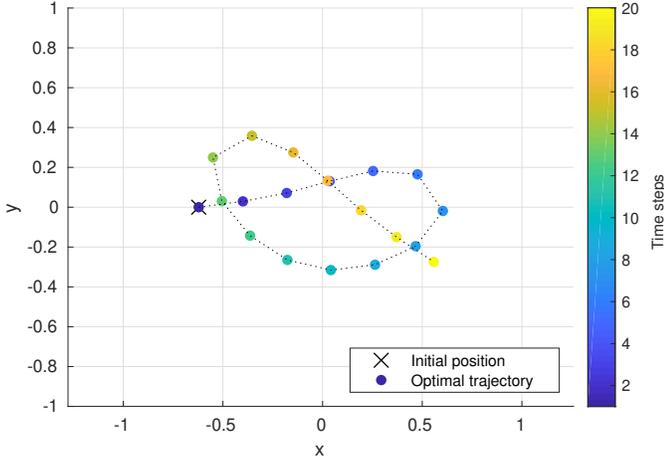


Fig. 3: Optimal trajectory for the Bayesian experiment design for detecting the parameters of a vertical thermal air flow

where  $z_t$  is that location where the measurement is taken and  $N_t$  are independent zero mean Gaussian distribution with variance  $\sigma^2$ . The probability density function for  $T$  measurements  $v = (v_1, \dots, v_T)$  is given by

$$p_V(v; \theta) = \frac{1}{(2\pi)^{T/2} \sigma^T} e^{-\frac{\sum_{t=1}^T (v_t - w_t)^2}{2\sigma^2}}$$

where  $w_t = w(\bar{w}, \gamma, \bar{z}, z_t)$ . The Fisher Information matrix associated to the estimation of  $\theta$  is given by,

$$\begin{aligned} \mathcal{FI}(z_{1:T}, \theta) &= \mathbb{E} \left[ \frac{d \log p_V(V; \theta)}{d\theta} \frac{d \log p_V(V; \theta)'}{d\theta} \right] \\ &= \frac{1}{\sigma^4} \mathbb{E} \left[ \sum_{t=1}^T \sum_{l=1}^T (V_t - w_t)(V_l - w_l) \frac{dw_t}{d\theta} \frac{dw_l'}{d\theta} \right] \\ &= \frac{1}{\sigma^2} \sum_{t=1}^T \frac{dw_t}{d\theta} \frac{dw_t'}{d\theta} \end{aligned}$$

where

$$\begin{aligned} \frac{dw_t}{d\theta} &= \left[ \frac{\partial w_t}{\partial \bar{w}}, \frac{\partial w_t}{\partial \gamma}, \frac{\partial w_t}{\partial \bar{z}} \right]' \\ &= e^{-\gamma \|z_t - \bar{z}\|^2} \left[ 1, -\bar{w} \|z_t - \bar{z}\|^2, -\bar{w} \gamma (z_t - \bar{z})' \right]'. \end{aligned}$$

Given prior distributions on  $\bar{w}$ ,  $\gamma$  and  $\bar{z}$ , we want to find the measurement points  $z_1, z_2, \dots, z_T$  that minimize (28) subject to the constraint that the distance between two consecutive  $z_t$  should be no larger than  $\Delta z$ . As the problem is rotationally symmetric, we fix the  $y$  coordinate of the first point to be 0.

We assign the following prior distributions. Both  $\bar{w}$  and  $\gamma$  follow a Gamma distribution with parameters respectively  $(\alpha_{\bar{w}}, \beta_{\bar{w}})$  and  $(\alpha_{\gamma}, \beta_{\gamma})$  and the thermal center  $\bar{z}$  follows a zero mean Gaussian distribution with covariance  $\Sigma_{\bar{z}}$ .

We take the following numerical values. The number of measurements is  $T = 20$ . The parameters of the priors are  $\alpha_{\bar{w}} = \alpha_{\gamma} = 1.25$ ,  $\beta_{\bar{w}} = \beta_{\gamma} = 0.25$  and  $\Sigma_{\bar{z}} = 0.1I$ . The maximum displacement between two sampling points is  $\Delta z = 0.05$ . The problem is highly nonconvex, requiring multiple initializations. For the lower bound, it takes about

6.56 seconds to run 100 optimizations with a random walk initialization, obtaining the lower bound  $V^\nabla = -14.27$ . For the upper bound it takes about 8.92 seconds to run 100 optimizations with random walk initialization, obtaining the upper bound  $V^\Delta = 95.41$ . Using Monte Carlo integration (Remark 4) we obtain that the expected value of the log determinant of the Fisher Information Matrix given the trajectory is 1.203. The optimal trajectory can be seen in Figure 3.

## V. CONCLUSIONS AND FUTURE WORK

We presented a general method to bound the expected value of any random variable with known probability density function. Stochastic programming is the main application of the bounds, where they can be used to determine an optimizer which has performance guarantees and satisfies inequality constraints. We illustrate the results with applications to finite-horizon stochastic control, estimation with latent variables and experiment design. The numerical results in these applications show that optimizing the bound lead to solutions close to the optimal.

There are many future work directions to be considered. On the bounds themselves, most of the properties were determined for the additive and multiplicative bound, but other versions of the bounds could unlock other applications. In terms of stochastic control, an evident extension would be stochastic model predictive control. In the estimation section, it would be interesting to study the asymptotic properties of the bound. As for new applications, machine learning is an area of significant potential. In particular, this method could either be used to accelerate the training of Neural Networks when there is a partial knowledge of the underlying model or in to adversarial training.

## APPENDIX

### A. Penalization term for common distributions

a) *Gaussian distribution*: The probability density function of a Gaussian Distribution with mean  $\mu$  and covariance matrix  $\Sigma$  is

$$p_D(d) = \det(2\pi\Sigma)^{-1/2} \exp\left(-\frac{1}{2} \|d - \mu\|_{\Sigma^{-1}}^2\right)$$

where we use the notation  $\|v\|_Q^2 := v'Qv$ .

For the additive bound,  $\mathcal{H}_D := \mathbb{E}[-\log p_D(D)] = \frac{1}{2} \log \det(2\pi e\Sigma)$ , therefore the penalization term simplifies to

$$\log p_D(d) + \mathcal{H}_D = -\frac{1}{2} \|d - \mu\|_{\Sigma^{-1}}^2 + \frac{1}{2} M$$

where  $M$  is the dimension of  $D$ .

For the multiplicative bound,  $\mathcal{I}_D(\epsilon) := \mathbb{E}[p_D(D)^{-\epsilon}] = \det(2\pi\Sigma)^{\epsilon/2} (1-\epsilon)^{-M/2}$  if  $\epsilon < 1$  and  $+\infty$  otherwise, therefore for  $\epsilon < 1$  the penalization terms simplifies to

$$p_D(d)^\epsilon \mathcal{I}_D(\epsilon) = \exp\left(-\frac{1}{2} \|d - \mu\|_{\Sigma^{-1}}^2\right) (1-\epsilon)^{-M/2}$$

b) *Uniform distribution:* If  $D$  is a Uniform distribution over a bounded support  $\mathcal{D}$ , its pdf is  $p_D(d) = \mathcal{V}_D^{-1} \mathbb{1}_{\mathcal{D}}(d)$  where  $\mathbb{1}_{\mathcal{D}}(\cdot)$  is the indicator function of  $\mathcal{D}$  and  $\mathcal{V}_D = \mathbb{E}[\mathbb{1}_{\mathcal{D}}(D)]$  is the volume of  $\mathcal{D}$ .

For the additive bound,  $\mathcal{H}_D = \mathbb{E}[-\log(\mathbb{1}_{\mathcal{D}}(D))] + \log(\mathcal{V}_D) = \mathbb{E}[0] + \log(\mathcal{V}_D)$ , therefore the penalization terms simplifies to  $\log p_D(d) + \mathcal{H}_D = 0 \forall d \in \mathcal{D}$ .

For the multiplicative bound,  $\mathcal{I}_D(\epsilon) = \mathbb{E}[(\mathcal{V}_D)^\epsilon \mathbb{1}_{\mathcal{D}}(D)^{-\epsilon}] = (\mathcal{V}_D)^\epsilon$ , therefore the penalization term simplifies to  $p_D(d)^\epsilon \mathcal{I}_D(\epsilon) = 1 \forall d \in \mathcal{D}$ .

## B. Proofs of Section II

**Notation:** Given two random variables  $X, Y$  we use the notation  $X \stackrel{\text{wpo}}{\leq} Y$  when  $\mathbb{P}(X \leq Y) = 1$  and analogously for  $\stackrel{\text{wpo}}{\geq}, \stackrel{\text{wpo}}{<}, \stackrel{\text{wpo}}{>}$ .

In order to prove some of the proposition, we need the following properties of the essential supremum and infimum which we state without a proof.

*Lemma 2:* Given two random variables  $X$  and  $Y$  then

$$\begin{aligned} X \stackrel{\text{wpo}}{\geq} Y &\Rightarrow \text{ess inf } X \geq \text{ess inf } Y \\ X \stackrel{\text{wpo}}{\leq} Y &\Rightarrow \text{ess sup } X \leq \text{ess sup } Y \\ \text{ess inf}(X + Y) &\geq \text{ess inf } X + \text{ess inf } Y \\ \text{ess sup}(X + Y) &\leq \text{ess sup } X + \text{ess sup } Y \quad \square \end{aligned}$$

*Proof of Proposition 1.* Take  $\epsilon_1, \epsilon_2 \in \mathbb{R}$  such that  $J^*(\epsilon_1), J^*(\epsilon_2) < +\infty$  and  $\lambda \in [0, 1]$

$$\begin{aligned} &J^*(\lambda \epsilon_1 + (1 - \lambda) \epsilon_2) \\ &= \text{ess sup } V(D) + (\epsilon_1 \lambda + \epsilon_2 (1 - \lambda)) \log p_D(D) \\ &= \text{ess sup}(\lambda + 1 - \lambda) V(D) + (\epsilon_1 \lambda + \epsilon_2 (1 - \lambda)) \log p_D(D) \\ &\leq \lambda \text{ess sup } V(D) + \epsilon_1 \log p_D(D) \\ &\quad + (1 - \lambda) \text{ess sup } V(D) + \epsilon_2 \log p_D(D) < \infty \end{aligned}$$

where the inequality follows from Lemma 2. This establishes that the additive upper bound is convex in  $\epsilon$  and that  $J^*(\epsilon)$  is finite on a convex set.

For the multiplicative bound, it remains to show that  $\mathcal{I}_D(\epsilon)$  is log convex: take  $\epsilon_1, \epsilon_2 \in \mathbb{R}$  such that  $\mathcal{I}_D(\epsilon_1), \mathcal{I}_D(\epsilon_2)$  are finite and  $\lambda \in [0, 1]$ . By applying Hölder's inequality we obtain

$$\begin{aligned} &\mathbb{E}[p_D(D)^{-\lambda \epsilon_1} p_D(D)^{-(1-\lambda) \epsilon_2}] \\ &\leq \left( \mathbb{E}[p_D(D)^{-\lambda \epsilon_1 / \lambda}] \right)^\lambda \left( \mathbb{E}[p_D(D)^{-(1-\lambda) \epsilon_2 / (1-\lambda)}] \right)^{(1-\lambda)} \\ &= \left( \mathbb{E}[p_D(D)^{-\epsilon_1}] \right)^\lambda \left( \mathbb{E}[p_D(D)^{-\epsilon_2}] \right)^{(1-\lambda)} \end{aligned}$$

which establishes log convexity.  $\blacksquare$

*Proof of Remark 2.* By assumption  $\text{ess sup } p_D(D)$  is finite, therefore we can define the normalized probability density function  $\tilde{p}_D(d) := (\text{ess sup } p_D(D))^{-1} p_D(d)$ . There is no loss of generality in analyzing the normalized pdf as

$$V(d) p_D(d)^\epsilon \mathbb{E}[p_D(D)^{-\epsilon}] = V(d) \tilde{p}_D(d)^\epsilon \mathbb{E}[\tilde{p}_D(D)^{-\epsilon}] \forall d \in \mathcal{D}.$$

For the finite support case, it suffices to see that for every finite  $\epsilon$  we can upper bound the expected value by

$$\mathbb{E}[\tilde{p}_D(D)^{-\epsilon}] \leq \left( \text{ess sup } \tilde{p}_D(D)^{-\epsilon} \tilde{p}_D(D) \right) \int_{\mathcal{D}} 1 dx < +\infty.$$

For the non-finite support case, the proof is based on applying the Monotone Convergence Theorem (MCT) for Lebesgue integration (Section 5.2.3 of [42]). For all  $d \in \mathcal{D}$ ,  $\tilde{p}_D(d)^{-\epsilon}$  is increasing in  $\epsilon$ , this is obtained from the derivative

$$\frac{d}{d\epsilon} \tilde{p}_D(d)^{-\epsilon} = -\log(\tilde{p}_D(d)) \tilde{p}_D(d)^{-\epsilon}$$

being non negative. Looking at the asymptotic behavior towards  $-\infty$ , we obtain that

$$\lim_{\epsilon \rightarrow -\infty} \tilde{p}_D(d) = \begin{cases} 1 & \text{if } \tilde{p}_D(d) = 1 \\ 0 & \text{otherwise} \end{cases}$$

which, using the MCT implies that  $\lim_{\epsilon \rightarrow -\infty} \mathbb{E}[\tilde{p}_D(D)^{-\epsilon}] = 0$  Using the MCT to take the limit towards  $\epsilon = 1$  we obtain

$$\lim_{\epsilon \nearrow 1} \mathbb{E}[\tilde{p}_D(D)^{-\epsilon}] = \mathbb{E}[\lim_{\epsilon \nearrow 1} \tilde{p}_D(D)^{-\epsilon}] = \int_{\mathcal{D}} 1 dx = +\infty \quad \blacksquare$$

## C. Necessary and sufficient conditions for finite bounds

Starting by slightly generalizing the definition of  $\gamma$ -essentially bounded. Given  $\gamma \in (0, 1)$ , we say a measurable function  $f(\cdot)$  is  $\gamma$ -essentially upper bounded (resp. lower) if  $\mathbb{P}(p_D(D) > \gamma) \in (0, 1)$  and  $\text{ess sup}(|f(D)| \mid p_D(D) > \gamma) < \infty$  (resp.  $\text{ess inf}(|f(D)| \mid p_D(D) > \gamma) > -\infty$ ).

*Theorem 3 (Finite additive bounds):* For all cases, assume that  $p_D(\cdot)$  is  $\gamma$ -essentially bounded.

When considering the additive bound: For the lower bound, assume  $V(\cdot)$   $\gamma$ -essentially lower bounded, then, for every  $\epsilon \in \mathbb{R}$ .

$$\begin{aligned} \text{ess inf} \left[ \frac{-V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] &> \epsilon \\ \Rightarrow \text{ess inf} (V(D) + \epsilon \log p_D(D)) &> -\infty. \quad (32) \end{aligned}$$

Conversely, if  $\text{ess inf } p_D(D) = 0$  then, for every  $\epsilon \in \mathbb{R}$ ,

$$\begin{aligned} \text{ess inf} (V(D) + \epsilon \log p_D(D)) &> -\infty \\ \Rightarrow \exists L > 0, \forall \gamma > 0 : \\ \text{ess inf} \left[ \frac{-V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] &\geq \epsilon + \frac{L}{\log \gamma}. \quad (33) \end{aligned}$$

For the upper bound, assume  $V(\cdot)$   $\gamma$ -essentially upper bounded, then, for every finite  $\epsilon \in \mathbb{R}$ .

$$\begin{aligned} \text{ess sup} \left[ \frac{-V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] &< \epsilon \\ \Rightarrow \text{ess sup} (V(D) + \epsilon \log p_D(D)) &< +\infty \quad (34) \end{aligned}$$

Conversely, if  $\text{ess inf } p_D(D) = 0$ , then

$$\begin{aligned} \text{ess sup} (V(D) + \epsilon \log p_D(D)) &< \infty \\ \Rightarrow \exists L > 0, \forall \gamma > 0 : \\ \text{ess sup} \left[ \frac{-V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] &\leq \epsilon - \frac{L}{\log \gamma}. \quad (35) \end{aligned}$$

When considering the multiplicative bound: For the lower bound, assume  $\log V(\cdot)$   $\gamma$ -essentially lower bounded, then, for every  $\epsilon \in \mathbb{R}$ .

$$\begin{aligned} \text{ess inf} \left[ \frac{-\log V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] &> \epsilon \\ \Rightarrow \text{ess inf} (\log V(D) + \epsilon \log p_D(D)) &> -\infty. \end{aligned}$$

Conversely, if  $\text{ess inf } p_D(D) = 0$  then, for every  $\epsilon \in \mathbb{R}$ ,

$$\begin{aligned} \text{ess inf} (\log V(D) + \epsilon \log p_D(D)) &> -\infty \\ \Rightarrow \exists L > 0, \forall \gamma > 0 : \\ \text{ess inf} \left[ \frac{-\log V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] &\geq \epsilon + \frac{L}{\log \gamma}. \quad (36) \end{aligned}$$

For the upper bound, assume  $\log V(\cdot)$   $\gamma$ -essentially upper bounded, then, for every finite  $\epsilon \in \mathbb{R}$ .

$$\begin{aligned} \text{ess sup} \left[ \frac{-\log V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] &< \epsilon \\ \Rightarrow \text{ess sup} (\log V(D) + \epsilon \log p_D(D)) &< +\infty \end{aligned}$$

Conversely, if  $\text{ess inf } p_D(D) = 0$ , then

$$\begin{aligned} \text{ess sup} (\log V(D) + \epsilon \log p_D(D)) &< \infty \\ \Rightarrow \exists L > 0, \forall \gamma > 0 : \\ \text{ess sup} \left[ \frac{-\log V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] &\leq \epsilon - \frac{L}{\log \gamma}. \quad (37) \end{aligned}$$

□

*Proof.* We will prove the theorem for the additive lower bound (i.e., (32) and (33)). The proof for the other bounds can be obtained in an analogous way.

To prove (32), we use the fact that

$$\text{ess inf} \left[ \frac{-V(D)}{\log p_D(D)} \mid p_D(D) \leq \gamma \right] > \epsilon$$

implies that

$$\mathbb{P} \left( \frac{-V(D)}{\log p_D(D)} \geq \epsilon \mid p_D(D) \leq \gamma \right) = 1,$$

or equivalently,

$$\mathbb{P} (V(D) + \epsilon \log p_D(D) \geq 0 \mid p_D(D) \leq \gamma) = 1. \quad (38)$$

In addition, as  $V(\cdot)$  is  $\gamma$ -essentially lower bounded, we conclude that there exist finite constants  $L_1, L_2$  such that

$$\mathbb{P} (V(D) \geq -L_1, p_D(D) \leq L_2 \mid p_D(D) > \gamma) = 1. \quad (39)$$

Since

$$\begin{aligned} p_D(D) \stackrel{\text{wpo}}{\leq} L_2, p_D(D) \stackrel{\text{wpo}}{>} \gamma \\ \Rightarrow |\log p_D(D)| \stackrel{\text{wpo}}{\leq} \max\{|\log \gamma|, |\log L_2|\} \end{aligned}$$

we have that

$$\begin{aligned} V(D) \stackrel{\text{wpo}}{\geq} -L_1, p_D(D) \stackrel{\text{wpo}}{\leq} L_2, p_D(D) \stackrel{\text{wpo}}{>} \gamma \\ \Rightarrow V(D) + \epsilon \log p_D(D) \stackrel{\text{wpo}}{\geq} -L^* \end{aligned}$$

with  $L^* := L_1 + |\epsilon| \max\{|\log \gamma|, |\log L_2|\} < \infty$ . We thus conclude from (39) that

$$\mathbb{P} (V(D) + \epsilon \log p_D(D) \geq -L^* \mid p_D(D) > \gamma) = 1 \quad (40)$$

Combining (38) and (40), we conclude that the corresponding unconditional probability satisfies

$$\mathbb{P} (V(D) + \epsilon \log p_D(D) \geq -\max\{0, L^*\}) = 1,$$

from which (32) follows.

To prove (33), we use the fact that  $\text{ess inf} (V(D) + \epsilon \log p_D(D)) > -\infty$  implies that there exists some finite scalar  $L > 0$ , for which

$$\mathbb{P}(V(D) + \epsilon \log p_D(D) \geq -L) = 1. \quad (41)$$

Since the assumption  $\text{ess inf } p_D(D) = 0$  implies that

$$\mathbb{P}(p_D(D) < \gamma) > 0, \quad \forall \gamma > 0,$$

we conclude from (41) that

$$\mathbb{P} (V(D) + \epsilon \log p_D(D) \geq -L \mid p_D(D) < \gamma) = 1,$$

or equivalently

$$\mathbb{P} \left( \frac{-V(D)}{\log p_D(D)} \geq \epsilon + \frac{L}{\log p_D(D)} \mid p_D(D) < \gamma \right) = 1, \quad (42)$$

$\forall \gamma > 0$ . We thus conclude using Lemma 2 that

$$\begin{aligned} \text{ess inf} \left[ \frac{-V(D)}{\log p_D(D)} \mid p_D(D) < \gamma \right] \\ \geq \epsilon + \text{ess inf} \left[ \frac{L}{\log p_D(D)} \mid p_D(D) < \gamma \right] \geq \epsilon + \frac{L}{\log \gamma}, \end{aligned}$$

where we used the fact that  $p_D(D) < \gamma \in (0, 1)$  implies that  $\frac{L}{\log p_D(D)} > \frac{L}{\log \gamma}$ . This completes the proof of the implication in (33). ■

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