Switched seesaw control for the stabilization of underactuated vehicles ⋆

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Abstract

This paper addresses the stabilization of a class of nonlinear systems in the presence of disturbances, using switching controllers. To this effect we introduce two new classes of switched systems and provide conditions under which they are input-to-state practically stable (ISpS). By exploiting these results, a methodology for control systems design - called *switched seesaw control* - is obtained that allows for the development of nonlinear control laws yielding input-to-state stability. The range of applicability and the efficacy of the methodology proposed are illustrated via two non-trivial design examples. Namely, stabilization of the extended nonholonomic double integrator (ENDI) and stabilization of an underactuated autonomous underwater vehicle (AUV) in the presence of input disturbances and measurement noise.

Key words: Switched Systems, Hybrid Control, Stabilization, Nonholonomic Systems, Autonomous Underwater Vehicles.

1 Introduction

There has been increasing interest in hybrid control in recent years, in part due to its potential to overcome the basic limitations to nonlinear system stabilization introduced by Brockett's celebrated result in the area of nonholonomic systems control (Brockett, 1983). Hybrid controllers that combine time-driven with event-driven dynamics have been developed by a number of authors and their design is by now firmly rooted in a solid theoretical background. See for example (Kolmanovsky and McClamroch, 1996; Tomlin *et al.*, 1998; Morse, 1995; Hespanha, 1996; Liberzon, 2003) and the references therein.

Inspired by progress in the area, the first part of this paper offers a new design methodology for the stabilization of nonlinear systems in the presence of external disturbances by resorting to hybrid control. To this effect, two classes of switched systems are introduced: *unstable/stable switched systems* and *switched seesaw systems*. The first, as their name indicates, have the property of alternating between an unstable and a stable mode during consecutive periods of time. The latter can be viewed as the interconnection of two unstable/stable systems such that when one is stable the other is unstable, and vice-versa. Conditions are given under which the interconnection is input-to-state practically stable (ISpS). The results are then used to develop a control design framework called *switched seesaw control design* that allows for the solution of robust (in an appropriately defined sense) control problems using switching.

To illustrate the scope of the new design methodology proposed, the second part of the paper solves the challenging problems of stabilizing the so-called extended nonholonomic double integrator (ENDI) (Aguiar and Pascoal, 2000) and an underactuated autonomous underwater vehicle (AUV) *in the presence of input disturbances and measurement noise*. These examples were motivated by the problem of *point stabilization*, that is, the problem of steering an autonomous vehicle to a

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point with a desired orientation. The complexity of the point stabilization problem is highly dependent on the configuration of the vehicle under consideration. For underactuated vehicles, i.e., systems with fewer actuators than degrees-of-freedom, point-stabilization is particularly challenging because most of the vehicles exhibit second-order (acceleration) nonholonomic constraints. As pointed out by Brockett (Brockett, 1983), nonholonomic systems cannot be stabilized by continuously differentiable (or even simply continuous) time invariant static state feedback control laws. To overcome this basic limitation, a variety of approaches have been proposed in the literature. Among the proposed solutions are continuous smooth or almost smooth time-varying (periodic) controllers (Samson, 1995; Tell *et al.*, 1995; M'Closkey and Murray, 1997; Godhavn and Egeland, 1997; Morin and Samson, 2000; Dixon *et al.*, 2000; Morin and Samson, 2003), discontinuous or piecewise time-invariant smooth control laws (Canudas-de-Wit and Sørdalen, 1992; Bloch and Drakunov, 1994; Aicardi *et al.*, 1995; Astolfi, 1998; Aguiar and Pascoal, 2001), and hybrid controllers (Bloch *et al.*, 1992; Hespanha, 1996; Aguiar and Pascoal, 2000; Aguiar and Pascoal, 2002; Prieur and Astolfi, 2003; Lizárraga *et al.*, 2004).

From a practical point of view, the above problem has been the subject of much debate within the ground robotics community. However, it was only recently that the problem of point stabilization of underactuated autonomous underwater vehicles (AUVs) received special consideration in the literature (Leonard, 1995; Pettersen and Egeland, 1999; Pettersen and Fossen, 2000; Do *et al.*, 2004). Point stabilization of AUVs poses considerable challenges to control system designers because the dynamics of these vehicles are complicated due to the presence of complex, uncertain hydrodynamic terms.

One of the key contributions of the paper is the fact that the solution proposed for point stabilization of an AUV addresses explicitly the existence of external disturbances and measurement errors. In a general setting this topic has only been partially addressed in the literature and in many aspects *it still remains an open problem.* Noteworthy exceptions are e.g., (Morin and Samson, 2003), where smooth time-varying feedback control laws for practical stabilization of driftless nonlinear systems subjected to known or measured additive perturbations are derived by using the transverse function approach; (Prieur and Astolfi, 2003), where a hybrid control law is proposed for stabilization of nonholonomic chained systems that yields global exponential stability and global robustness against a class of small measurements errors; and (Lizárraga *et al.*, 2004) that addresses the point stabilization for the extended chained form in the presence of additive disturbances.

The paper is organized as follows. In Section 2 we introduce and analyze the stability of two new classes of switched systems: unstable/stable and seesaw switched systems. The results obtained are then used to derive a switched seesaw control design methodology that allows for the development of a new class of nonlinear control laws yielding input-to-state stability. In Section 3 we illustrate the applicability and the efficacy of the theoretical results derived in the previous section via two nontrivial design examples. Concluding remarks are given in Section 4.

Notation and definitions: $|\cdot|$ denotes the standard Euclidean norm of a vector in $\mathbb{R}^n$ and $\|u\|_I$ is the (essential) supremum norm of a signal $u : [0,\infty) \to \mathbb{R}^n$ on an interval $I \subset [0,\infty)$. Let $a \oplus b := \max\{a, b\}$ and denote by $M_{\mathcal{W}}$ the set of measurable, essentially bounded signals $w : [t_0,\infty) \to \mathcal{W}$, where $\mathcal{W} \subset \mathbb{R}^m$. A function $\gamma : [0,\infty) \to [0,\infty)$ is *of class* $\mathcal{K}$ ($\gamma \in \mathcal{K}$) if it is continuous, strictly increasing, and $\gamma(0) = 0$ and *of class* $\mathcal{K}_\infty$ if in addition it is unbounded. A function $\beta : [0,\infty) \times \mathbb{R} \to [0,\infty)$ is *of class* [1] $\mathcal{KL}$ if it is continuous, for each fixed $t \in \mathbb{R}$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for each fixed $r \geq 0$ the function $\beta(r, t)$ decreases with respect to t and $\beta(r, t) \to 0$ as $t \to \infty$. A class $\mathcal{KL}$ function $\beta(r, t)$ is called exponential if $\beta(r, t) \leq \hat{\beta} r e^{-\lambda t}$, $\hat{\beta} > 0$, $\lambda > 0$. We denote the identity function from $\mathbb{R}$ to $\mathbb{R}$ by **id**, and the composition of two functions $\gamma_i : \mathbb{R} \to \mathbb{R}; i = 1, 2$ in this order by $\gamma_2 \circ \gamma_1$. The acronym w.r.t. stands for "with respect to".

2 Dwell-time switching theorems and hybrid control

This section introduces and analyzes stability related results for two classes of systems that will be henceforth called unstable/stable and seesaw switched systems. The results obtained are key to the derivation of a new hybrid control methodology for nonlinear system stabilization in the presence of disturbances.

2.1 Unstable/stable switched system

Consider the switched system

$$\dot{x} = f_\sigma(x, w), \quad x(t_0) = x_0, \tag{1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ is the state, $w \in M_{\mathcal{W}}$ is a disturbance, and $\sigma : [t_0,\infty) \to \{1, 2\}$ is a piecewise constant switching signal that is continuous from the right and evolves according to

$$\sigma(t) = \begin{cases} 1, & t \in [t_{k-1}, t_k),\ k \text{ odd} \\ 2, & t \in [t_{k-1}, t_k),\ k \text{ even} \end{cases} \tag{2}$$

In (2), $\{t_k\} := \{t_1, t_2, t_3, \ldots\}$ is a sequence of strictly increasing infinite switching times in $[t_0,\infty)$ and t_0 is the

[1] Our definition of $\mathcal{KL}$ functions is slightly different from the standard one because the domain of the second argument has been extended from $[0,\infty)$ to $\mathbb{R}$. This will allow us to consider the case $\beta(r, -t)$ which may grow unbounded as $t \to \infty$.

initial time. We assume that both $f_i; i = 1, 2$ are locally Lipschitz w.r.t. (x, w) and that the solutions of (1) lie in $\mathcal{X}$ and are defined for all $t \geq t_0$.

Let $\omega : \mathbb{R}^n \to [0, \infty)$ be a continuous nonnegative real function called a *measuring function*. For a given switching signal σ, system (1) is said to be *input-to-state practically stable* [2] (ISpS) on $\mathcal{X}$ w.r.t. ω if there exist functions $\beta \in \mathcal{KL}$, $\gamma^w \in \mathcal{K}$, and a nonnegative constant c such that for every initial condition $x(t_0)$ and every input $w \in M_{\mathcal{W}}$ such that the solution $x(t)$ of (1) lies entirely in $\mathcal{X}$, $x(t)$ satisfies

$$\omega(x(t)) \leq \beta(\omega(x(t_0)), t - t_0) \oplus \gamma^w(\|w\|_{[t_0,t]}) \oplus c. \quad (3)$$

for all $t \geq t_0$. When $\mathcal{X} = \mathbb{R}^n$, $\mathcal{W} = \mathbb{R}^m$, $\omega(x) = |x|$ and $c = 0$, ISpS is equivalent to the by now classical definition of *input-to-state stability* (ISS) (Sontag, 1989).

With respect to (1), assume the following conditions hold:

1. *Instability* ($\sigma = 1$). For $\dot{x} = f_1(x, w)$, there exist functions $\beta_1 \in \mathcal{KL}$, $\gamma_1^w \in \mathcal{K}$, and a nonnegative constant c_1 such that for every initial condition $x(t_0)$ and every input $w \in M_{\mathcal{W}}$ for which the solution $x(t)$ of (1) lies entirely in $\mathcal{X}$, $x(t)$ satisfies [3]

$$\omega(x(t)) \leq \beta_1\big(\omega(x(t_0)) \oplus \gamma_1^w(\|w\|_{[t_0,t]}) \oplus c_1, -(t - t_0)\big), \quad t \geq t_0. \quad (4)$$

   Notice how the negative term $-(t - t_0)$ in the second argument of β_1 captures the unstable characteristics of the system when $\sigma = 1$.

2. *Stability* ($\sigma = 2$). System $\dot{x} = f_2(x, w)$ is ISpS on $\mathcal{X}$ w.r.t. ω, that is, for every initial condition $x(t_0)$ and every input $w \in M_{\mathcal{W}}$ such that the solution $x(t)$ of system (1) lies entirely in $\mathcal{X}$, $x(t)$ satisfies

$$\omega(x(t)) \leq \beta_2(\omega(x(t_0)), t - t_0) \oplus \gamma_2^w(\|w\|_{[t_0,t]}) \oplus c_2, \quad t \geq t_0 \quad (5)$$

   where $\beta_2 \in \mathcal{KL}$, $\gamma_2^w \in \mathcal{K}$, $c_2 \geq 0$.

If conditions 1–2 above are met, we call (1)–(2) an *unstable/stable switched system on $\mathcal{X}$ w.r.t. ω*. The definition of a *stable/unstable switched* is done in the obvious manner.

The following result provides conditions under which an unstable/stable switched system is ISpS.

Lemma 1 *Consider an* unstable/stable switched system on $\mathcal{X}$ w.r.t. ω. *Let $t_i; i \in \mathbb{N}$ be a sequence of strictly increasing switching times $\{t_i\}$ such that the differences between consecutive instants of times $\Delta_i := t_i - t_{i-1}$ satisfy*

$$\beta_2\big(\beta_1(r, -\Delta_{k+1}), \Delta_{k+2}\big) \leq (\mathbf{id} - \alpha)(r), \quad \forall r \geq r_0 \quad (6)$$

for $k = 0, 2, 4, ...$, and for some class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ and $r_0 \geq 0$. Then, system (1)–(2) is ISpS at $t = t_k$, that is, $x(t)$ satisfies the ISpS condition (3) at $t = t_k$. Similarly, if

$$\beta_1\big(\beta_2(r, \Delta_k), -\Delta_{k+1}\big) \leq (\mathbf{id} - \alpha)(r), \quad \forall r \geq r_0 \quad (7)$$

for $k = 2, 4, 6, \ldots$, and for some class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ and $r_0 \geq 0$, then system (1)–(2) is ISpS at $t = t_{k+1}$. If either (6) or (7) hold and the piecewise continuous function that captures the differences between consecutive switching times $\mathbf{\Delta} : [t_0, \infty) \to [0, \infty)$ defined by $\mathbf{\Delta}(t) := \Delta_i$; $t \in [t_{i-1}, t_i), i \in \mathbb{N}$ is bounded, then system (1)–(2) is ISpS. □

Remark 2 *If $c_1 = c_2 = r_0 = 0$, $\omega(x) = |x|$, $\mathcal{X} = \mathbb{R}^n$, and $\mathcal{W} = \mathbb{R}^m$ and all the conditions of Lemma 1 are met, then system (1)–(2) is ISS.* □

Remark 3 *If (4)–(5) hold with exponential class $\mathcal{KL}$ functions, i.e., $\beta_i(r, t) \leq \hat{\beta}_i r e^{-\lambda_i t}$, $i = 1, 2$, and α can be taken as $\alpha(r) = \hat{\alpha} r; \hat{\alpha} \in (0, 1)$, then inequalities (6)–(7) become independent of r. In particular, (6) and (7) above degenerate into*

$$\Delta_{k+2} \geq \frac{\lambda_1}{\lambda_2}\Delta_{k+1} + \frac{1}{\lambda_2} \ln \frac{\hat{\beta}_1\hat{\beta}_2}{1 - \hat{\alpha}}; k = 0, 2, 4, ...$$

and

$$\Delta_{k+1} \leq \frac{\lambda_2}{\lambda_1}\Delta_k + \frac{1}{\lambda_1} \ln \frac{1 - \hat{\alpha}}{\hat{\beta}_1\hat{\beta}_2}; k = 2, 4, 6, ...,$$

respectively. Notice how the first condition sets lower bounds on the periods of time over which the switching system (1) is required to be stable. Similarly, the second condition enforces upper bounds on the periods of time over which the switching system may be unstable. □

Remark 4 *The results above can be extended to stable/unstable switched systems in the obvious manner.* □

[2] On a first reading, one can consider that $\mathcal{X} = \mathbb{R}^n$. In this case, the reference to the set $\mathcal{X}$ is omitted. However, we will need the more general setting when we consider applications to the stabilization of underactuated vehicles.

[3] Another alternative is to consider that $x(t)$ satisfies

$$\omega(x(t)) \leq \beta_1^x\big(\omega(x(t_0)), -(t-t_0)\big) \oplus \beta_1^w\big(\|w\|_{[t_0,t]}, -(t-t_0)\big) \oplus \beta_1^c\big(c_1, -(t - t_0)\big)$$

with $\beta_1^x, \beta_1^w, \beta_1^c \in \mathcal{KL}$. There is no loss of generality in considering (4), because one can always take $\beta_1(r, -t) = \beta_1^x(r, -t) \oplus \beta_1^w(r, -t) \oplus \beta_1^c(r, -t)$ with the advantage of introducing a less complicated notation. However, this may lead to more conservative estimates.

Proof of Lemma 1 Select some switching time t_k such that $\sigma(t) = 1$ for all $t \in [t_k, t_{k+1})$ (unstable mode). From (4), we conclude that

$$\omega(x(t_{k+1}^-)) \le \beta_1\big(\omega(x(t_k)) \oplus \gamma_1^w(\|w\|_{[t_k,t_{k+1})}) \oplus c_1, -\Delta_{k+1}\big),$$

where $x(t_{k+1}^-)$ denotes the limit from the left. Using (5) and the continuity of $x(t)$ it follows from the inequality $\beta_2(a \oplus b, c) \le \beta_2(b, c) \oplus \beta_2(a, c)$ that

$$\begin{aligned}\omega(x(t_{k+2})) \le\ & \beta_2\big(\beta_1(\omega(x(t_k)), -\Delta_{k+1}), \Delta_{k+2}\big)\\ & \oplus \beta_2\big(\beta_1(\gamma_1^w(\|w\|_{[t_k,t_{k+1})}), -\Delta_{k+1}), \Delta_{k+2}\big)\\ & \oplus \beta_2\big(\beta_1(c_1, -\Delta_{k+1}), \Delta_{k+2}\big)\\ & \oplus \gamma_2^w(\|w\|_{[t_{k+1},t_{k+2})}) \oplus c_2.\end{aligned}$$

Applying (6) it is straightforward to obtain

$$\begin{aligned}\omega(x(t_{k+2})) - \omega(x(t_k)) \le\ & -\alpha\big(\omega(x(t_k))\big)\\ & \oplus \hat{\gamma}_1^w(\|w\|_{[t_k,t_{k+2})}) \oplus \hat{c}_1,\end{aligned}$$

where $\hat{\gamma}_1^w(r) := (\mathbf{id} - \alpha) \circ \gamma_1^w(r) \oplus \gamma_2^w(r)$ and $\hat{c}_1 := (\mathbf{id} - \alpha)(c_1 \oplus \gamma_1^w(r_0) \oplus r_0) \oplus c_2$. It can now be shown that (3) is satisfied at $t = t_k; k = 0, 2, 4, ..$ by using the same arguments as in (Jiang and Wang, 2001, Lemma 3.5) and by viewing (with a slight abuse of terminology) $\omega(\cdot)$ as a discrete-time ISS-Lyapunov function. Estimates for γ^w and c in (3) can be derived by assuming without loss of generality that $\mathbf{id} - \alpha \in \mathcal{K}$ (cf. (Jiang and Wang, 2001, Lemma B.1)) and by choosing any $\rho \in \mathcal{K}_\infty$ such that $\mathbf{id} - \rho$ is of class $\mathcal{K}$. Then, (3) holds with $\gamma^w(r) := \alpha^{-1} \circ \rho^{-1} \circ \hat{\gamma}_1^w(r)$ and $c := \alpha^{-1} \circ \rho^{-1}(\hat{c}_1)$.
To prove (3) at $t = t_k; k = 2, 4, ...$ using (7) instead of (6), select a switching time t_{k-1} such that $\sigma(t) = 2$ for all $t \in [t_{k-1}, t_k)$ (stable mode). From (5), a bound on $x(t_k)$ can be written as

$$\omega(x(t_k^-)) \le \beta_2(\omega(x(t_{k-1})), \Delta_k) \oplus \gamma_2^w(\|w\|_{[t_{k-1},t_k)}) \oplus c_2.$$

Using the continuity of $x(t)$ and (4) yields

$$\begin{aligned}\omega(x(t_{k+1})) \le\ & \beta_1\big(\beta_2(\omega(x(t_{k-1})), \Delta_k), -\Delta_{k+1}\big)\\ & \oplus \beta_1\big(\gamma_2^w(\|w\|_{[t_{k-1},t_k)}) \oplus c_2\\ & \oplus \gamma_1^w(\|w\|_{[t_k,t_{k+1})}) \oplus c_1, -\Delta_{k+1}\big).\end{aligned}$$

Using (7) it follows that

$$\begin{aligned}\omega(x(t_{k+1})) - \omega(x(t_{k-1})) \le\ & -\alpha\big(\omega(x(t_{k-1}))\big)\\ & \oplus \hat{\gamma}_2^w(\|w\|_{[t_{k-1},t_{k+1})}) \oplus \hat{c}_2,\end{aligned}$$

where $\hat{\gamma}_2^w(r) := \beta_1(\gamma_2^w(r) \oplus \gamma_1^w(r), -\Delta_{k+1})$, and $\hat{c}_2 := \beta_1(c_2 \oplus c_1, -\Delta_{k+1}) \oplus (\mathbf{id} - \alpha)(r_0)$. Again, using the arguments advanced in (Jiang and Wang, 2001, Lemma 3.5) we conclude that (3) applies, possibly with different estimates for γ^w and c. The proof that system (1)–(2) is ISpS (at all times t) if either (6) or (7) hold and $\mathbf{\Delta}(t)$ is bounded, is straightforward and follows from simple algebra. □

Table 1
Temporal representation of the switched seesaw system

	Δ_1	Δ_2	Δ_3	Δ_4	$\cdots$
σ	1	2	1	2	$\cdots$
ω_{su}	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\cdots$
ω_{us}	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\cdots$

$\searrow$ Stable
$\nearrow$ Unstable

2.2 *Switched seesaw system*

This section introduces the concept of switched seesaw system. To this effect, consider the switched system (1)–(2). Given two measuring functions ω_{su}, ω_{us} and a set $\mathcal{X} \subset \mathbb{R}^n$ we call (1) a *switched seesaw system on* $\mathcal{X}$ w.r.t. $(\omega_{su}, \omega_{us})$ if the following conditions hold:

C1. For $\dot{x} = f_1(x, w)$, that is, $\sigma = 1$, there exist $\beta_{11}, \beta_{12} \in \mathcal{KL}$, $\gamma_{11}^{\omega_{us}}, \gamma_{12}^{\omega_{su}}, \gamma_{11}^w, \gamma_{12}^w \in \mathcal{K}$, $c_{11}, c_{12} \ge 0$ such that for every solution $x(\cdot) \in \mathcal{X}$

$$\begin{aligned}\omega_{su}(x(t)) \le\ & \beta_{11}(\omega_{su}(x(t_0)), t - t_0))\\ & \oplus \gamma_{11}^{\omega_{us}}(\|\omega_{us}(x)\|_{[t_0,t]}) \oplus \gamma_{11}^w(\|w\|_{[t_0,t]})\\ & \oplus c_{11}, \qquad (8)\\ \omega_{us}(x(t)) \le\ & \beta_{12}\big(\omega_{us}(x(t_0)) \oplus \gamma_{12}^{\omega_{su}}(\|\omega_{su}(x)\|_{[t_0,t]})\\ & \oplus \gamma_{12}^w(\|w\|_{[t_0,t]}) \oplus c_{12}, -(t - t_0)\big). \qquad (9)\end{aligned}$$

C2. For $\dot{x} = f_2(x, w)$, that is, $\sigma = 2$, there exist $\beta_{21}, \beta_{22} \in \mathcal{KL}$, $\gamma_{21}^{\omega_{us}}, \gamma_{22}^{\omega_{su}}, \gamma_{21}^w, \gamma_{22}^w \in \mathcal{K}$, $c_{21}, c_{22} \ge 0$ such that for every solution $x(\cdot) \in \mathcal{X}$

$$\begin{aligned}\omega_{su}(x(t)) \le\ & \beta_{21}\big(\omega_{su}(x(t_0)) \oplus \gamma_{21}^{\omega_{us}}(\|\omega_{us}(x)\|_{[t_0,t]})\\ & \oplus \gamma_{21}^w(\|w\|_{[t_0,t]}) \oplus c_{21}, -(t - t_0)\big), \qquad (10)\\ \omega_{us}(x(t)) \le\ & \beta_{22}(\omega_{us}(x(t_0)), t - t_0)\\ & \oplus \gamma_{22}^{\omega_{su}}(\|\omega_{su}(x)\|_{[t_0,t]}) \oplus \gamma_{22}^w(\|w\|_{[t_0,t]})\\ & \oplus c_{22}. \qquad (11)\end{aligned}$$

In view of the above, the switched seesaw system can be interpreted as a stable/unstable system w.r.t. ω_{su} when $\omega_{us}(x)$ and w are regarded as inputs, and an unstable/stable w.r.t. ω_{us} when $\omega_{su}(x)$ and w are regarded as inputs, see Table 1.

The following theorem gives conditions under which a switched seesaw system is ISpS.

Theorem 5 *Consider the switched seesaw system on* $\mathcal{X}$ *w.r.t.* $(\omega_{su}, \omega_{us})$. *Let* τ_1^{min}, τ_1^{max}, τ_2^{min}, τ_2^{max} *be positive constants, called* dwell time bounds, *such that* $\tau_1^{max} \ge \tau_1^{min} > 0$, $\tau_2^{max} \ge \tau_2^{min} > 0$, $\{t_k\}, k \in \mathbb{N}$ *a sequence of* *strictly increasing switching times, and* $\Delta_k = t_k - t_{k-1}$ *a sequence of intervals satisfying*

$$\Delta_i \in [\tau_1^{min}, \tau_1^{max}], \quad \Delta_{i+1} \in [\tau_2^{min}, \tau_2^{max}], \quad i = 1, 3, 5, \ldots$$

Assume there exist $\alpha_i \in \mathcal{K}_\infty; i = 1, 2$ *such that*

$$\beta_{21}\big(\beta_{11}(r, \tau_1^{min}), -\tau_2^{max}\big) \le (\mathbf{id} - \alpha_1)(r), \quad \forall r \ge r_0, \tag{12}$$

$$\beta_{22}\big(\beta_{12}(r, -\tau_1^{max}), \tau_2^{min}\big) \le (\mathbf{id} - \alpha_2)(r), \quad \forall r \ge r_0, \tag{13}$$

for some $r_0 \ge 0$ *and*

$$\bar{\gamma}_2^{\omega_{su}} \circ \bar{\gamma}_1^{\omega_{us}}(r) < r, \quad \forall r > \hat{r}_0, \tag{14}$$

$$\bar{\gamma}_1^{\omega_{us}} \circ \bar{\gamma}_2^{\omega_{su}}(r) < r, \quad \forall r > \hat{r}_0, \tag{15}$$

for some $\hat{r}_0 \ge 0$, *where*

$$\bar{\gamma}_1^{\omega_{us}}(r) := \alpha_1^{-1} \circ \rho_1^{-1} \circ \beta_{21}\big(\gamma_{11}^{\omega_{us}}(r) \oplus \gamma_{21}^{\omega_{us}}(r), -\tau_2^{max}\big), \tag{16}$$

$$\bar{\gamma}_2^{\omega_{su}}(r) := \alpha_2^{-1} \circ \rho_2^{-1} \circ \big[(\mathbf{id} - \alpha_2) \circ \gamma_{12}^{\omega_{su}}(r) \oplus \gamma_{22}^{\omega_{su}}(r)\big], \tag{17}$$

and $\rho_i \in \mathcal{K}_\infty; i = 1, 2$ *are arbitrary functions such that* $\mathbf{id} - \rho_i \in \mathcal{K}$. *Then, the seesaw switched system (1) is ISpS on* $\mathcal{X}$ *w.r.t. to* $\omega_{su} \oplus \omega_{us}$. □

Remark 6 *If the* $\mathcal{KL}$ *functions* β_{ij} *are exponential, that is, if* $\beta_{ij}(r, t) \le \hat{\beta}_{ij} r e^{-\lambda_{ij} t}$, *with* $\hat{\beta}_{ij} > 1$, *and the* α_i *can be taken as* $\alpha_i(r) = \hat{\alpha}_i r; \hat{\alpha} \in (0, 1)$, *then inequalities (12)–(13) with* $\tau_1 := \tau_1^{min} = \tau_1^{max}$, $\tau_2 := \tau_2^{min} = \tau_2^{max}$ *are equivalent to the linear matrix inequality (LMI)*

$$\Lambda \tau \ge b \tag{18}$$

where $\Lambda = \left[\begin{smallmatrix} \lambda_{11} & -\lambda_{21} \\ -\lambda_{12} & \lambda_{22} \end{smallmatrix}\right]$, $\tau = (\tau_1, \tau_2)'$, *and* $b = (b_1, b_2)'$ *is a positive vector, i.e.,* $b_1, b_2 > 0$. *This LMI together with the fact that* $\tau > 0$ *imply the necessary condition*

$$\frac{\lambda_{12}}{\lambda_{11}} \frac{\lambda_{21}}{\lambda_{22}} < 1.$$

The above expression sets an upper bound on the ratio of $\lambda_{12}\lambda_{21}$ *(product of the rates of explosion) versus* $\lambda_{11}\lambda_{22}$ *(product of the rates of implosion).* □

Proof of Theorem 5 We start to compute the evolution of $\omega_{us}(x(t_k)); k = 2, 4, 6, \ldots$ Condition (13) and Lemma 1 yield

$$\omega_{us}(x(t_k)) \le \bar{\beta}_2(\omega_{us}(x(t_0)), t_k - t_0) \oplus \bar{\gamma}_2^{\omega_{su}}(\|\omega_{su}(x)\|_{[t_0, t_k]}) \oplus \bar{\gamma}_2^{w}(\|w\|_{[t_0, t_k]}) \oplus \bar{c}_2 \tag{19}$$

where $\bar{\gamma}_2^{\omega_{su}}$ is defined in (17), $\bar{c}_2 := \alpha_2^{-1} \circ \rho_2^{-1}((\mathbf{id} - \alpha_2)(c_{12} \oplus \gamma_{12}^{\omega_{su}}(r_0) \oplus \gamma_{12}^{w}(r_0) \oplus r_0) \oplus c_{22})$, and $\bar{\beta}_2$ and $\bar{\gamma}_2^{w}$ are $\mathcal{KL}$ and $\mathcal{K}$ functions respectively, the form of which is not relevant. In a similar manner, consider the evolution of $\omega_{su}(x(t_k))$. Condition (12) and a straightforward reformulation of Lemma 1 for stable/unstable switched systems yield

$$\omega_{su}(x(t_k)) \le \bar{\beta}_1(\omega_{su}(x(t_0)), t_k - t_0) \oplus \bar{\gamma}_1^{\omega_{us}}(\|\omega_{us}(x)\|_{[t_0, t_k]}) \oplus \bar{\gamma}_1^{w}(\|w\|_{[t_0, t_k]}) \oplus \bar{c}_1 \tag{20}$$

where $\bar{\gamma}_2^{\omega_{us}}$ is defined in (16), $\bar{c}_1 := \alpha_1^{-1} \circ \rho_1^{-1}(\beta_{21}(c_{11} \oplus c_{21}, \tau_2^{max}) \oplus (\mathbf{id} - \alpha_2)(r_0))$, and $\bar{\beta}_2$ and $\bar{\gamma}_2^{w}$ are $\mathcal{KL}$ and $\mathcal{K}$ functions, respectively. Notice in (19) and (20) the existence of a cross-coupling term from $\omega_{su}(\cdot)$ to $\omega_{us}(\cdot)$. A straightforward application of the small-gain theorem (Jiang *et al.*, 1994; Jiang and Wang, 2001) implies that (1) is ISpS w.r.t. $\omega_{su} \oplus \omega_{us}$ at $t = t_k$, $k = 2, 4, \ldots$ if (14) is satisfied. The proof that (1) is ISpS w.r.t. $(\omega_{su}, \omega_{us})$ for all $t \ge t_0$ follows from the fact that $\mathbf{\Delta}(t)$ defined in Lemma 1 is uniformly bounded. □

2.3 Seesaw control systems design

Equipped with the mathematical results derived, this section proposes a new methodology for the design of stabilizing feedback control laws for nonlinear systems of the form

$$\dot{x} = f(x, u, w), \tag{21}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ is the state, $u \in \mathcal{U} \subset \mathbb{R}^m$ is the control input, and $w \in M_{\mathcal{W}}$, $\mathcal{W} \subset \mathbb{R}^{n_w}$ is a disturbance signal. In particular, we seek to derive a switching control law for u that will render the resulting closed-loop system ISpS w.r.t. $\omega(x) = |x|$.

The first step consist of finding two measuring functions $\omega_{su}(x)$, $\omega_{us}(x)$ that satisfy the following detectability property: if $\|\omega_{su}(x) \oplus \omega_{us}(x)\|$ and $\|w\|$ converge to zero as $t \to \infty$, then $|x(t)|$ tends also to zero as $t \to \infty$. More precisely, (21) associated with the outputs $\omega_{su}(x)$ and $\omega_{us}(x)$, must be *input-output-to-state stable* (IOSS) (Sontag and Wang, 1997) with respect to w, that is, there must exist a class $\mathcal{KL}$ function β and class $\mathcal{K}$ function γ such that [4]

$$|x(t)| \le \beta(|x(t_0)|, t - t_0) \oplus \gamma(\|\omega_{su}(x) \oplus \omega_{us}(x)\|_{[t_0, t]} \oplus \|w\|_{[t_0, t]}). \tag{22}$$

The choices of $\omega_{su}(x)$ and $\omega_{us}(x)$ are strongly motivated by the physics of the problem at hand, as the examples

[4] In fact, since we will only prove ISpS of the closed-loop system, it is sufficient that (21) be input-output-to-state practically stable (IOSpS) with respect to w, that is, that there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ, and a nonnegative constant c such that $|x(t)| \le \beta(|x(t_0)|, t - t_0) \oplus \gamma(\|\omega_{su}(x) \oplus \omega_{us}(x)\|_{[t_0, t]} \oplus \|w\|_{[t_0, t]}) \oplus c$.

in Section 3 reveal. It will be later seen that in general $\omega_{su}(x)$ and $\omega_{us}(x)$ are not functions of all the state x, but rather of disjoint, yet complementary collections of the elements of x.
The next step involves the design of two feedback laws $\alpha_1(x)$, $\alpha_2(x)$, such that (21) together with the switching controller

$$u = \alpha_\sigma(x), \quad \sigma \in \{1, 2\}$$

becomes a switched seesaw system w.r.t. $\omega_{su} \oplus \omega_{us}$. It is then easy to show that if $\sigma(t)$ is chosen such that the conditions of Theorem 5 hold and if the IOSS condition applies, then the closed-loop system

$$\dot{x} = f(x, \alpha_\sigma(x), w),$$

is ISpS w.r.t. $\omega(x) = |x|$.
The methodology proposed brings together tools from Lyapunov-stability and switching system analysis. Its rationale can be explained in simple terms, if one compares the strategy proposed against that of "classical" nonlinear controller design by resorting to a single Lyapunov function, a task that can be impossible or extremely difficult at best. Instead, the initial stabilization problem is somehow broken up into two separate, simpler problems. This is done by viewing the two measuring functions $\omega_{su}(x)$, $\omega_{us}(x)$ as candidate Lyapunov functions defined over two different collections of the elements of the original state. With a proper choice of these functions, the task of finding the Lyapunov-based control laws $\alpha_1(x)$, $\alpha_2(x)$ becomes simple, as the examples in Section 3 will show. Switching between the control laws is the final ingredient that will yield overall ISpS of the complete system w.r.t. $\omega(x) = |x|$.

3 Stabilization of underactuated vehicles

The present section focuses on control system design and provides the insight that goes into the choice of the switching control laws referred to before. This is done by addressing the non-trivial problem of underactuated underwater vehicle stabilization in the presence of disturbances and measurement noise. For the sake of clarity, the illustrative example proceeds in two steps. First, the techniques are applied to the stabilization of so-called extended nonholonomic double integrator (ENDI), which captures the kinematic and dynamic equations of a wheeled robot. The methodology adopted is then extended to deal with an underwater vehicle by showing that its dynamics can be cast in a form similar to (but more complex than) that of the ENDI.

3.1 The Extended nonholonomic double integrator

The nonholonomic integrator introduced in (Brockett, 1983) captures (under suitable state and control transformations) the kinematics of a wheeled robot, displays all basic properties of nonholonomic systems, and is often quoted in the literature as a benchmark for control system design, e.g., (Bloch and Drakunov, 1994; Hespanha, 1996; Astolfi, 1998). However, to tackle the realistic case where both the kinematics and dynamics of a wheeled robot must be taken into account, the nonholonomic integrator model must be extended. In (Aguiar and Pascoal, 2000), it is shown that the dynamic equations of motion of a mobile robot of the unicycle type can be transformed into the system

$$\ddot{x}_1 = u_1, \quad \ddot{x}_2 = u_2, \quad \dot{x}_3 = x_1\dot{x}_2 - x_2\dot{x}_1, \qquad (23)$$

where $x := (x_1, x_2, x_3, \dot{x}_1, \dot{x}_2)' \in \mathbb{R}^5$ is the state vector and $u := (u_1, u_2)' \in \mathbb{R}^2$ is a two-dimensional control vector. System (23) will be referred to as *the extended nonholonomic double integrator* (ENDI).
The ENDI falls into the class of control affine nonlinear systems with drift and cannot be stabilizable via a time-invariant continuously differentiable feedback law [cf., e.g., (Aguiar, 2002)].

3.1.1 Seesaw control design

We now solve the problem of practical stabilization of the ENDI system (23) subject to input disturbances $v \in M_{\mathcal{V}}$, $\mathcal{V} := \{v \in \mathbb{R}^2 : \|v\|_{[0,\infty)} \le \bar{v}\}$ and measurement noise $n \in M_{\mathcal{N}}$, $\mathcal{N} := \{n \in \mathbb{R}^5 : \|n\| \le \bar{n}\}$, where $\bar{v}$ and $\bar{n}$ are finite but otherwise arbitrary. To this effect, the dynamics of (23) are first extended to

$$\ddot{x}_1 = u_1 + v_1, \quad \ddot{x}_2 = u_2 + v_2, \quad \dot{x}_3 = x_1\dot{x}_2 - x_2\dot{x}_1, \qquad (24)$$

$$y = x + n \qquad (25)$$

where $y \in \mathbb{R}^5$ is the vector of state measurements corrupted by noise n. Following the procedure described in Section 2.3 we first introduce the measuring functions

$$\omega_{su} := z^2, \quad z := \dot{x}_3 + \lambda_1 x_3, \ \lambda_1 > 0 \qquad (26)$$
$$\omega_{us} := x_1^2 + \dot{x}_1^2 + x_2^2 + \dot{x}_2^2. \qquad (27)$$

and the feedback laws

$$\alpha_1(x) := \begin{bmatrix} -k_2\dot{x}_1 \\ -k_2\dot{x}_2 - \frac{k_3}{x_1}z \end{bmatrix}, \ \alpha_2(x) := \begin{bmatrix} -k_2\dot{x}_1 - k_1(x_1 - \kappa) \\ -k_2\dot{x}_2 - k_1 x_2 \end{bmatrix}, \qquad (28)$$

where $\kappa, k_1, k_2, k_3 > 0$. Notice that in order for the first control law to be well defined, x_1 must be bounded away from 0. This justifies the need to required that all trajectories lie in some specific set $\mathcal{X} \subset \{x \in \mathbb{R}^5 : |x_1| \ge \delta\}$ for some $\delta > 0$, as explained later. To provide some insight into (26)–(28), observe that ω_{su} and ω_{us} can be viewed as positive semi-definite Lyapunov functions of z and $(x_1, \dot{x}_1, x_2, \dot{x}_2)'$, respectively, the time-derivatives of which are given by

$$\dot{\omega}_{su} = 2z\big[x_1(u_2 + v_2 + k_2\dot{x}_2) - x_2(u_1 + v_1 + k_2\dot{x}_1)\big], \tag{29}$$

$$\dot{\omega}_{us} = 2\dot{x}_1(x_1 + u_1 + v_1) + 2\dot{x}_2(x_2 + u_2 + v_2). \tag{30}$$

In the absence of input disturbances and measurement noise, it is straightforward to conclude that with the control law $u = \alpha_1(x)$, the measuring function ω_{su} satisfies $\dot{\omega}_{su} = -2k_3\omega_{su}$ as long as $x_1 \neq 0$. This in turn implies that ω_{su} converges exponentially fast to zero during the intervals of time in which $u = \alpha_1(x)$ is applied. In a similar vein, consider the evolution of ω_{us} under the influence of the control law $u = \alpha_2(x)$. Simple computations show that

$$\ddot{x}_1 = -k_2\dot{x}_1 - k_1(x_1 - \kappa), \quad \ddot{x}_2 = -k_2\dot{x}_2 - k_1 x_2$$

and therefore ω_{us} converges exponentially fast to κ^2 during the intervals of time in which $u = \alpha_2(x)$ is applied. We now proceed with the seesaw control design as explained in Section 2.3. For clarity of exposition all the proofs related to this example are at the end of this section. Following the procedure described in Section 2.3, the first step is to show that the measuring functions satisfy the IOSS detectability property.

Proposition 7 *The ENDI system together with the measuring functions $\omega_{su}(x)$ and $\omega_{us}(x)$ of* (26) *and* (27)*, respectively as outputs, is IOSS.* □

The next step consists in showing that the closed-loop system described by the ENDI system and the control law

$$u = \alpha_\sigma(x + n) \tag{31}$$

defined in (28) verifies the seesaw conditions C1 and C2.

Proposition 8 *Consider the ENDI system subject to input disturbances and measurement noise and the control law* (31)*. For every $\delta > \bar{n} \geq 0$, there are control gains such that the closed-loop system*

$$\dot{x} = f(x, \alpha_\sigma(x + n), w),$$

with $w = (v, n)$ verifies the seesaw conditions C1 and C2 w.r.t. $\omega_{su} \oplus \omega_{us}$ on $\mathcal{X} \subset \{x \in \mathbb{R}^5 : |x_1| \geq \delta\}$. □

It is now easy to conclude that if the switched seesaw controller (31) is applied to the ENDI system and a suitable selection of the dwell times τ_1, τ_2 is made such that conditions (12)–(15) hold, then the resulting closed-loop system is ISpS as long as $|x_1(t)| \geq \delta$. It remains to state conditions under which $|x_1|$ is indeed bounded away from zero.

Proposition 9 *Consider the closed-loop system that consists of* (24)–(25) *and the feedback control law* (28), (31)*. Given any $\delta > \bar{n} \geq 0$, there exists $\mu > 0$ such that under a suitable choice of the controller gains, for every initial condition $x(t_0) \in S_0 := \{x \in \mathbb{R}^5 : |(x_1 - \kappa, \dot{x}_1)| \leq \mu\}$, the resulting solution $x(\cdot)$ lies in $\mathcal{X} \subset \{x \in \mathbb{R}^5 : |x_1| \geq \delta\}$.* □

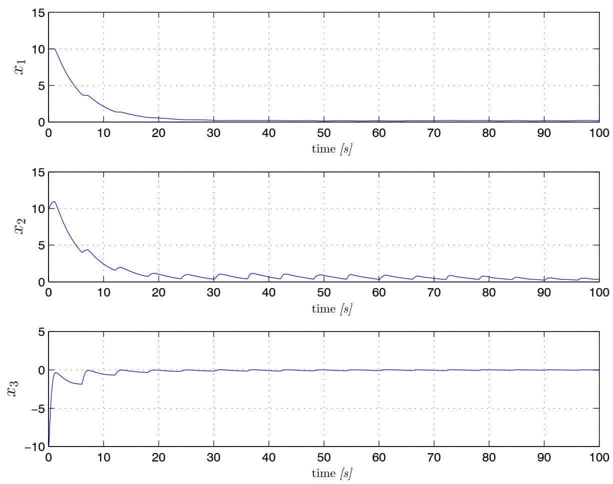

Fig. 1. Time evolution of state variables $x_1(t)$, $x_2(t)$, and $x_3(t)$.

From Propositions 7–9 and Theorem 5 we finally conclude

Theorem 10 *Consider the ENDI system subject to input disturbances and measurement noise, together with the switching control law* (28), (31)*. Assume the conditions of Theorem 5 hold and let the initial conditions of the closed-loop system be in S_0, defined in Proposition 9. Then, the switching controller stabilizes the state around a neighborhood of the origin, that is, it achieves ISpS of the closed-loop system on $\mathcal{X}$ w.r.t. $\omega(x) = |x|$.*

Remark 11 *It is always possible to make sure that x starts in S_0 by applying $u = \alpha_2(y)$ during a finite amount of time before the normal switching takes over. In fact, from* (24),(28) *it is clear that with $u = \alpha_2(y)$, $(x_1, \dot{x}_1)$ reaches in finite time S_0. From the particular evolution of $(x_1, \dot{x}_1)$, ω_{us}, ω_{su} during this time interval, Proposition 7, and Theorem 10 we can also conclude that with the procedure adopted, the resulting switching controller achieves ISpS of the closed-loop system on $\mathbb{R}^n$ w.r.t. $\omega(x) = |x|$.* □

Remark 12 *We have eschewed the general problem of deriving a procedure to choose the dwell time bounds (if they exist) that satisfy the conditions of Theorem 5. For the example considered, however, this turns out to be simple because all the $\mathcal{KL}$ functions are exponential and the $\mathcal{K}$ functions are linear (see also Remark 6).* □

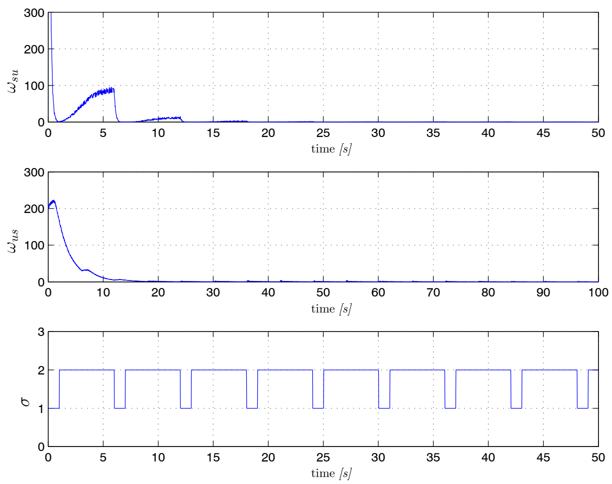

Fig. 2. Time evolution of measuring functions $\omega_{su}(t)$, $\omega_{us}(t)$, and the switching signal $\sigma(t)$.

3.1.2 Simulation results

Numerical simulations were done to illustrate the performance of the switching controller proposed, when applied to the ENDI. Figures 1 and 2 show the time evolution of state variables x_1, x_2, x_3 and signals $\omega_{su}, \omega_{us}, \sigma$, respectively in the presence of measurement noise and input disturbances. In the simulations, the measurement noise is a zero mean uniform random noise with amplitude 0.1, and the input disturbances are $v_1 = 0.1\sin(t)$ and $v_2 = 0.1\sin(t+\pi/2)$, in the appropriate units. With the dwell-time constants set to $\tau_1 = \tau_1^{min} = \tau_1^{max} = 1.0\,s$ and $\tau_2 = \tau_2^{min} = \tau_2^{max} = 5.0\,s$, the assumptions of Theorem 10 were verified to hold. Notice how the state variables converge to a small neighborhood of the origin. Fig. 2 shows clearly, during the first switching intervals, how the behavior of ω_{su} and ω_{us} capture the successive "stable/unstable" and "unstable/stable" cycles, respectively.

3.1.3 Proofs

Proof of Proposition 7 Let $x = (x_1, \dot{x}_1, x_2, \dot{x}_2, x_3)'$. From (26) using the variation of constant formula and taking norms, x_3 can be bounded as

$$|x_3(t)| \le e^{-k_2(t-t_0)}|x_3(t_0)| + \frac{1}{k_2}\sqrt{\|\omega_{su}(x)\|_{[t_0,t)}}.$$

Therefore,

$$\begin{aligned}|x(t)| &= \sqrt{\omega_{us}(x(t)) + x_3^2(t)} \le \sqrt{\omega_{us}(x(t))} + |x_3(t)| \\ &\le \beta(|x(t_0)|, t-t_0) \oplus \gamma(\|\omega_{su}(x)\|_{[t_0,t]} \oplus \|\omega_{us}(x)\|_{[t_0,t]})\end{aligned}$$

with $\beta(r,t) := 2e^{-k_2 t}r$ and $\gamma(r) := 2(\frac{1}{k_2}+1)\sqrt{r}$ thus satisfying (22). □

Proof of Proposition 8 We start by showing that C1 is observed when $\sigma = 1$. In the presence of measurement noise, the control input $u = \alpha_1(x+n)$ is given by

$$\begin{aligned}u_1 &= -k_2(\dot{x}_1 + n_4), \\ u_2 &= -k_2(\dot{x}_2 + n_5) - \frac{k_3}{x_1 + n_1}(z + n_z),\end{aligned}$$

where $n_z = x_1 n_5 + n_1 \dot{x}_2 + n_1 n_5 - x_2 n_4 - n_2 \dot{x}_1 - n_2 n_4 + k_2 n_3$ which, from the fact that $\|n\|_{[0,\infty)} \le \bar{n}$, satisfies $|n_z| \le 4\bar{n}\sqrt{\omega_{us}} + 2\bar{n}^2 + \lambda\bar{n}$. For $|x_1| > \delta > \bar{n}$, a bound for ω_{su} is determined by computing the time-derivative of ω_{su} as

$$\begin{aligned}\dot{\omega}_{su} &= -2\frac{k_3}{1+\frac{n_1}{x_1}}z^2 + 2zx_1[-k_2 n_5 - \frac{k_3}{x_1+n_1}n_z + v_2] \\ &\quad - 2zx_2[-k_2 n_4 + v_1] \\ &\le -\lambda_{11}\omega_{su} + \frac{\lambda_{11}}{3}\hat{\gamma}_{11}^{\omega_{us}}\omega_{us} + \frac{\lambda_{11}}{3}c_{11}; \quad \theta_1, \theta_2 > 0\end{aligned}$$

where $\lambda_{11} = 2\left[\frac{k_3}{1+\frac{\bar{n}}{\delta}} - \frac{k_3}{1-\frac{\bar{n}}{\delta}}\frac{\theta_1}{2} - \bar{n}k_2 - \theta_2\bar{v}\right]$, $\frac{\lambda_{11}}{3}\hat{\gamma}_{11}^{\omega_{us}} = 2\frac{k_3}{1-\frac{\bar{n}}{\delta}}\frac{16}{\theta_1}\bar{n}^2 + \bar{n}k_2 + \frac{1}{\theta_2}\bar{v}$, and $\frac{\lambda_{11}}{3}c_{11} = 2\frac{k_3}{1-\frac{\bar{n}}{\delta}}\frac{1}{\theta_1}\bar{n}^2(2\bar{n}+\lambda)^2$. Therefore,[5]

$$\omega_{su}(t) \le 3\omega_{su}(t_0)e^{-\lambda_{11}(t-t_0)} \oplus \hat{\gamma}_{11}^{\omega_{us}}\|\omega_{us}\|_{[t_0,t]} \oplus c_{11}. \tag{32}$$

Notice the absence of the term γ_{11}^{w} due to the fact that the disturbances and noise are assumed to be bounded and their bounds are known in advance[6]. We now establish a bound for ω_{us}. Computing its time-derivative yields[7]

$$\begin{aligned}\dot{\omega}_{us} &= 2(\dot{x}_1 x_1 + \dot{x}_2 x_2) - 2k_2(\dot{x}_1^2 + \dot{x}_2^2) + 2\dot{x}_1(-k_2 n_4 + v_1) \\ &\quad + 2\dot{x}_2(-k_2 n_5 + v_2) - 2\dot{x}_2\frac{k_3}{x_1+n_1}(z+n_z) \\ &\le \lambda_{12}\omega_{us} + \lambda_{12}\hat{\gamma}_{12}^{\omega_{su}}\omega_{su} + \lambda_{12}\gamma_{12}^{v}(|v|) + \lambda_{12}c_{12}\end{aligned}$$

where $\lambda_{12} = 2 + \frac{k_3}{\delta-\bar{n}} + 4\bar{n} + \frac{\theta_3}{2}$, $\lambda_{12}\hat{\gamma}_{12}^{\omega_{su}} = \frac{k_3}{\delta-\bar{n}}$, $\lambda_{12}\gamma_{12}^{v}(r) = 2r^2$, and $\lambda_{12}c_{12} = 4k_2^2\bar{n}^2 + \frac{k_3\bar{n}^2}{\delta-\bar{n}}\frac{(2\bar{n}+k_2)^2}{2\theta_3}$. Therefore, ω_{us} satisfies

$$\omega_{us}(t) \le 4\big(\omega_{us}(t_0) \oplus \hat{\gamma}_{12}^{\omega_{su}}\|\omega_{su}\|_{[t_0,t]}$$

---

[5] We exploit the fact that for every class $\mathcal{K}$ function α and arbitrary positive numbers $r_1, r_2, \ldots, r_k$ we have $\alpha(r_1 + \cdots + r_k) \le \alpha(kr_1) + \cdots + \alpha(kr_n)$.

[6] To simplify the control algorithm, we use explicitly in advance the fact that the disturbances and noise are bounded by $\|v\|_{[0,\infty)} \le \bar{v}$ and $\|n\|_{[0,\infty)} \le \bar{n}$, respectively. It is possible to avoid this at the cost of introducing the $\mathcal{K}$ function $\gamma_{11}^{v}(r)$ and making $\gamma_{12}^{\omega_{su}}(r)$ a quadratic function.

[7] We have used the fact that $|\dot{x}_2||n_z| \le 4\bar{n}\omega_{us} + \frac{\bar{n}^2(2\bar{n}+k_2)^2}{2\theta_3} + \omega_{us}\frac{\theta_3}{2}$, $\theta_3 > 0$.

$$\oplus\, \gamma_{12}^{v}(\|v\|_{[t_0,t]}) \oplus c_{12}\big)e^{\lambda_{12}(t-t_0)}. \quad (33)$$

From (32) and (33), we can now conclude that condition C1 holds by identifying w in C1 with the input disturbance v.
Similarly, we check that condition C2 is satisfied when $\sigma = 2$. In this case, the control input $u = \alpha_2(x+n)$ is given by

$$u_1 = -k_2(\dot{x}_1 + n_4) - k_1(x_1 + n_1 - \kappa),$$
$$u_2 = -k_2(\dot{x}_2 + n_5) - k_1(x_2 + n_2).$$

Substituting the above equations into (29) yields

$$\begin{aligned}\dot{\omega}_{su} &= 2z\big[x_1(-k_2n_5 - k_1n_2 + v_2)\\ &\quad - x_2(-k_2n_4 - k_1n_1 + k_1\kappa + v_1)\big]\\ &\le \lambda_{21}\omega_{su} + \lambda_{21}\hat{\gamma}_{21}^{\omega_{us}}\omega_{us}, \quad \theta_4, \theta_5 > 0\end{aligned}$$

where $\lambda_{21} = \frac{2(k_2+k_1)+k_1\kappa}{\theta_4} + 2\frac{\bar{v}}{\theta_5}$ and $\lambda_{21}\hat{\gamma}_{21}^{\omega_{us}} = \theta_4[(k_2 + k_1)\bar{n} + k_1\kappa] + \theta_5\bar{v}$. Therefore,

$$\omega_{su} \le 2\big(\omega_{su}(t_0) \oplus \hat{\gamma}_{21}^{\omega_{us}}\|\omega_{us}\|_{[t_0,t]}\big)e^{\lambda_{21}(t-t_0)}. \quad (34)$$

To compute a bound for $\omega_{us}(t)$, we first observe that

$$\omega_{us} = |\chi_1|^2 + |\chi_2|^2, \quad (35)$$

where $\chi_1 := [x_1, \dot{x}_1]'$, $\chi_2 := [x_2, \dot{x}_2]'$, and χ_1, χ_2 satisfy

$$\dot{\chi}_i = A\chi_i + Bd_i, \quad i = 1,2 \quad (36)$$

with $A := \left[\begin{smallmatrix} 0 & 1 \\ -k_1 & -k_2 \end{smallmatrix}\right]$, $B = [0,1]'$, $d_1 = -k_2n_4 - k_1n_1 + \kappa + v_1$, and $d_2 = -k_2n_5 - k_1n_2 + v_2$. Let $\lambda > 0$ be an arbitrary constant such that $(A + \frac{\lambda}{2}I)$ is Hurwitz. Further, let $P > 0$ satisfy

$$\big(A + \frac{\lambda}{2}I\big)P + P\big(A + \frac{\lambda}{2}I\big)' + BB' \le 0. \quad (37)$$

Define $V_i := \chi_i'P^{-1}\chi_i$ and compute $\dot{V}_i$ to obtain

$$\begin{aligned}\dot{V}_i &= \chi_i'(P^{-1}A + A'P^{-1})\chi_i + 2\chi'P^{-1}Bd_i\\ &\le -(\lambda - \theta_6)V_i, \quad V_i \ge \frac{|d_i|^2}{\theta_6}, \quad \theta_6 \in (0,\lambda)\end{aligned}$$

From the above, it follows that $V_i(t) \le V_i(t_0)e^{-(\lambda-\theta_6)(t-t_0)} \oplus \frac{|d_i|^2}{\theta_6}$, and therefore

$$\omega_{us} \le \hat{\beta}_{22}\omega_{us}(t_0)e^{-\lambda_{22}(t-t_0)} \oplus \gamma_{22}^{v}(\|v\|_{[t_0,t]}) \oplus c_{22},$$

where $\hat{\beta}_{22} = 3\frac{\lambda_{max}(P)}{\lambda_{min}(P)}$, $\lambda_{22} = (\lambda - \theta_6)$, $\gamma_{22}^{v}(r) = 6r^2$, and $c_{22} = 6\big[((k_2+k_1)\bar{n} + k_1\kappa)^2 + (k_2+k_1)^2\bar{n}^2\big]$. □

Proof of Proposition 9 Let $\xi := (\xi_1, \xi_2)' := (x_1 - \kappa, \dot{x}_1)'$. For $\sigma = i$; $i \in 1,2$, consider the dynamics

$$\dot{\xi} = A_i\xi + Bd_i, \quad (38)$$

where $A_1 := \left[\begin{smallmatrix} 0 & 1 \\ 0 & -k_2 \end{smallmatrix}\right]$, $A_2 := \left[\begin{smallmatrix} 0 & 1 \\ -k_1 & -k_2 \end{smallmatrix}\right]$, $B = [0,1]'$, $|d_1| \le k_2\bar{n} + |v|$, and $|d_2| \le (k_2+k_1)\bar{n} + |v|$. Let $\lambda_{\xi_1}, \lambda_{\xi_2} > 0$ be arbitrary constants such that $(A_1 - \frac{\lambda_{\xi_1}}{2}I)$, and $(A_2 + \frac{\lambda_{\xi_2}}{2}I)$ are Hurwitz. Further let $P_i > 0; i = 1,2$ satisfy

$$\big(A_1 - \frac{\lambda_{\xi_1}}{2}I\big)P_1 + P_1\big(A_1 - \frac{\lambda_{\xi_1}}{2}I\big)' + BB' \le 0,$$
$$\big(A_2 + \frac{\lambda_{\xi_2}}{2}I\big)P_2 + P_2\big(A_2 + \frac{\lambda_{\xi_2}}{2}I\big)' + BB' \le 0,$$

and define $V_i = \xi'P_i^{-1}\xi$. A reasoning similar to the one used in the last part of the proof of Proposition (8) shows that

$$V_1(t) \le \big(V_1(t_0) + \frac{|d_1|^2}{\lambda_{\xi_1}}\big)e^{\lambda_{\xi_1}(t-t_0)},$$
$$V_2(t) \le V_2(t_0)e^{-(\lambda_{\xi_2}-\theta_7)(t-t_0)} \oplus \frac{|d_2|^2}{\theta_7}, \quad \theta_7 \in (0,\lambda_{\xi_2}).$$

Therefore, for $\sigma = 1$,

$$|\xi(t)| \le 2\left[\frac{\lambda_{max}^{1/2}(P_1)}{\lambda_{min}^{1/2}(P_1)}|\xi(t_0)| \oplus \lambda_{max}^{1/2}(P_1)\|d_1\|_{[t_0,t]}\right]e^{\frac{\lambda_{\xi_1}}{2}(t-t_0)}$$

and for $\sigma = 2$

$$|\xi(t)| \le \frac{\lambda_{max}^{1/2}(P_2)}{\lambda_{min}^{1/2}(P_2)}|\xi(t_0)|e^{-\frac{\lambda_{\xi_2}-\theta_7}{2}(t-t_0)} \oplus \lambda_{max}^{1/2}(P_2)\frac{\|d_2\|_{[t_0,t]}}{\theta_7}$$

Consider now the switched system with state ξ and external input $d = (d_1, d_2)'$ satisfying the two inequalities above. Since the unstable mode λ_{ξ_1} can be made arbitrarily close to zero, simple but lengthy computations show that there is always a choice of controller gains k_2 and k_3 such that the conditions of Lemma 1 are met. Therefore, the switched system (38) is ISpS. In particular, $\xi(t)$ satisfies $|\xi(t)| \le \beta(|\xi(t_0)|, 0) \oplus \gamma(\|v\|_{[t_0,t)}) \oplus c \le \beta(\mu, 0) \oplus \gamma(\bar{v}) \oplus c$, for some $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, and $c \ge 0$. Therefore, choosing κ large enough, with $\kappa > \beta(\mu,0) \oplus \gamma(\bar{v}) \oplus c \oplus \delta$, it follows that $|x_1(t)| > \delta$ for all $t \ge t_0$. □

3.2 *The underactuated autonomous underwater vehicle*

This section addresses the problem of stabilizing an underactuated autonomous underwater vehicle (AUV) in the horizontal plane to a point, with a desired orientation. The AUV has no side thruster, and its control inputs are the thruster surge force τ_u and the thruster yaw torque τ_r. The AUV model is a second-order nonholonomic system, falls into the class of control affine nonlinear systems with drift, and there is no time-invariant continuously differentiable feedback law that asymptotically stabilizes the closed-loop system to an equilibrium point (Aguiar and Pascoal, 2001; Aguiar and Pascoal, 2002; Aguiar, 2002).

3.2.1 Vehicle Modeling

In the horizontal plane, the kinematic equations of motion of the vehicle can be written as

$$\dot{\mathbf{x}} = \mathbf{u}\cos\psi - \mathbf{v}\sin\psi, \quad \dot{\mathbf{y}} = \mathbf{u}\sin\psi + \mathbf{v}\cos\psi, \quad \dot{\psi} = \mathbf{r},$$

where, following standard notation, $\mathbf{u}$ (surge speed) and $\mathbf{v}$ (sway speed) are the body fixed frame components of the vehicle's velocity, $\mathbf{x}$ and $\mathbf{y}$ are the cartesian coordinates of its center of mass, ψ defines its orientation, and $\mathbf{r}$ is the vehicle's angular speed. Neglecting the motions in heave, roll, and pitch the simplified dynamic equations of motion in the horizontal plane for surge, sway and heading yield (Aguiar, 2002)

$$m_u\dot{\mathbf{u}} - m_v\mathbf{v}\mathbf{r} + d_u\mathbf{u} = \tau_u, \tag{39}$$
$$m_v\dot{\mathbf{v}} + m_u\mathbf{u}\mathbf{r} + d_v\mathbf{v} = 0, \tag{40}$$
$$m_r\dot{\mathbf{r}} - m_{uv}\mathbf{u}\mathbf{v} + d_r\mathbf{r} = \tau_r, \tag{41}$$

where the positive constants $m_u = m - X_{\dot{u}}$, $m_v = m - Y_{\dot{v}}$, $m_r = I_z - N_{\dot{r}}$, and $m_{uv} = m_u - m_v$ capture the effect of mass and hydrodynamic added mass terms, and $d_u = -X_u - X_{|u|u}|u|$, $d_v = -Y_v - Y_{|v|v}|v|$, and $d_r = -N_r - N_{|r|r}|r|$ capture hydrodynamic damping effects. The symbols τ_u and τ_r denote the external force in surge and the external torque about the z axis of the vehicle, respectively. Since there is no thruster capable of imparting a direct thrust on sway, the vehicle is underactuated.

3.2.2 Coordinate Transformation

Consider the global diffeomorphism given by the state and control coordinate transformation (Aguiar, 2002)

$$\begin{aligned}
x_1 &= \psi \\
x_2 &= \mathbf{x}\cos\psi + \mathbf{y}\sin\psi \\
x_3 &= -2\big(\mathbf{x}\sin\psi - \mathbf{y}\cos\psi\big) + \psi\big(\mathbf{x}\cos\psi + \mathbf{y}\sin\psi\big) \\
u_1 &= \frac{1}{m_r}\tau_r + \frac{m_{uv}}{m_r}\mathbf{u}\mathbf{v} - \frac{d_r}{m_r}\mathbf{r} \\
u_2 &= \frac{m_v}{m_u}\mathbf{v}\mathbf{r} - \frac{d_u}{m_u}\mathbf{u} + \frac{1}{m_u}\tau_u - u_1\frac{x_1x_2 - x_3}{2} + \mathbf{v}\mathbf{r} - \mathbf{r}^2 z_2
\end{aligned}$$

that yields

$$\ddot{x}_1 = u_1, \quad \ddot{x}_2 = u_2, \quad \dot{x}_3 = x_1\dot{x}_2 - x_2\dot{x}_1 + 2\mathbf{v}, \tag{42}$$

and transforms the second order constraint (40) for the sway velocity into

$$m_v\dot{\mathbf{v}} + m_u\left(\dot{x}_2 + \dot{x}_1\frac{x_1x_2 - x_3}{2}\right)\dot{x}_1 + d_v\mathbf{v} = 0. \tag{43}$$

Throughout the paper, $q := \mathrm{col}(x, \mathbf{v})$, $x := (x_1, x_2, x_3, \dot{x}_1, \dot{x}_2)'$ and $u = (u_1, u_2)'$ denote the state vector and the input vector of (42)–(43), respectively.

3.2.3 Seesaw control design

We now design a switching feedback control law for system (42)–(43) so as to stabilize (in an ISpS sense) the state q around a small neighborhood of the origin. We omit many of the details, because the methodology adopted for control system design follows closely that adopted for the ENDI. A comparison of (42)–(43) with the ENDI system (23) shows the presence of an extra state variable $\mathbf{v}$ that is not in the span of the input vector field but enters as an input perturbation in the x_3 dynamics. We also note that since $\frac{d_v}{m_v} > 0$, (43) is ISS when x is regarded as input. Motivated by these observations, we select for measuring functions $\omega_{su}(\cdot)$, $\omega_{us}(\cdot)$ the ones given in (26)–(27). Using Proposition 7 and the fact that $\mathbf{v}$ satisfies

$$|\mathbf{v}(t)| \le \hat{\beta}_v|\mathbf{v}(t_0)|e^{-\lambda_v(t-t_0)} \oplus \gamma_v(\|\omega_{su}\|_{[t_0,t]} \oplus \|\omega_{us}\|_{[t_0,t]})$$

for some $\hat{\beta}_v, \lambda_v > 0$, and $\gamma_v(r) \in \mathcal{K}$ we conclude that system (42)–(43) with ω_{su} and ω_{us} as outputs is IOSS. Before we define the feedback laws $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ we compute the time-derivatives of ω_{su} and ω_{us} to obtain

$$\begin{aligned}
\dot{\omega}_{su} &= 2z\big[x_1(u_2 + v_2 + k_2\dot{x}_2) - x_2(u_1 + v_1 + k_2\dot{x}_1 + 2\nu)\big] \\
\dot{\omega}_{us} &= 2\dot{x}_1(x_1 + u_1 + v_1) + 2\dot{x}_2(x_2 + u_2 + v_2),
\end{aligned}$$

where $\nu := \dot{\mathbf{v}} + k_2\mathbf{v}$ satisfies the linear bound

$$|\nu| \le \hat{\gamma}_v|\mathbf{v}| + \hat{\gamma}_{\omega_{us}}|\omega_{us}| + \hat{\gamma}_z|z|, \tag{44}$$

for some positive constants $\hat{\gamma}_v$, $\hat{\gamma}_{\omega_{us}}$, $\hat{\gamma}_z$, and assuming that $\|(x_1 - \kappa, \dot{x}_1)\|_{[0,\infty)} \le \mu$, for a given $\mu > 0$. Comparing $\dot{\omega}_{su}, \dot{\omega}_{us}$ with (29)–(30) and using (44) together with the previous results for the ENDI case, it is straightforward to conclude that if $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ are selected as in (28), then (42)–(43) (with input disturbances $v \in M_{\mathcal{V}}$) in closed-loop with the seesaw controller $u = \alpha_\sigma(q+n)$, $n \in M_{\mathcal{N}}$, $\mathcal{N} := \{n \in \mathbb{R}^6 : |n| \le \bar{n}\}$ is a switched seesaw system on $\mathcal{X} \subset \{q \in \mathbb{R}^6 : |x_1| \ge \delta,\ |(x_1 - \kappa, \dot{x}_1)| \le \mu\}$. The existence of such a set $\mathcal{X}$ can be proved using the same arguments as in Proposition 9. These results are summarized in the following theorem.

Theorem 13 *Consider the system* (42)–(43) *subject to input disturbances and measurement noise, and select* σ *such that the assumptions of Theorem 5 hold. Then,*

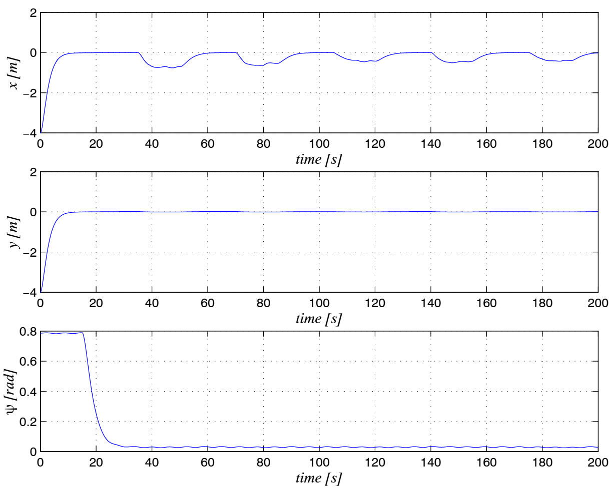

Fig. 3. Time evolution of the position $\mathbf{x}$, $\mathbf{y}$ and orientation ψ.

there exists $\mu_0 > 0$ such that for every initial condition in $S_0 := \{q \in \mathbb{R}^6 : |(x_1 - \kappa, \dot{x}_1)| \leq \mu_0\}$ the closed-loop system with the seesaw controller $u = \alpha(q+n)$ is ISpS on $\mathcal{X}$ w.r.t. $\omega(q) = |q|$.

Remark 14 *As in the ENDI case (see Remark 11), it is always possible to make sure that q starts in S_0 by applying during a finite amount of time the control law $u = \alpha_2(q)$ before the normal switching takes over.* $\square$

3.2.4 Simulation results

Simulations were done using a dynamic model of the *Sirene* AUV (Aguiar, 2002). Figure 3 shows the simulation results for a sample initial condition given by $(\mathbf{x}, \mathbf{y}, \psi, \mathbf{u}, \mathbf{v}, \mathbf{r}) = (-4\,m, -4\,m, \pi/4, 0, 0, 0)$. Zero mean uniform random noise was introduced in every sensed signal: the $\mathbf{x}$ and $\mathbf{y}$ positions, the orientation angle ψ, the linear velocities $\mathbf{u}, \mathbf{v}$, and the angular velocity $\mathbf{r}$. The amplitudes of the noise signals were set to $(0.5\,m, 0.5\,m, 5\pi/180, 0.1, 0.1, 0.1)$. There is also a small input disturbance: $v_1 = 10\sin(t)$, $v_2 = 10\sin(t+\pi/2)$. The dwell-time constants were set to $\tau_1 = \tau_1^{min} = \tau_1^{max} = 15\,s$ and $\tau_2 = \tau_2^{min} = \tau_2^{max} = 20\,s$. Clearly, the vehicle converges to a small neighborhood of the target position while the heading angle is attracted to a neighborhood around zero.

4 Conclusions

A new class of switched systems was introduced and mathematical tools were developed to analyze their stability and disturbance/noise attenuation properties. A so-called seesaw control design methodology was also proposed that yields input-to-state stability of these systems using switching. Applications were made to the stabilization of the extended nonholonomic double integrator and to the dynamic model of an underactuated autonomous underwater vehicle in the presence of input disturbances and measurement noise.

Seesaw controllers explore switching between two modes, each one driving a different sub-component of the closed-loop state to the origin. Recent work in (Hespanha *et al.*, 2005) suggests that instead of switching between different modes of operation, one could use the continuous flow to drive a subset of the state to the origin and instantaneous jumps to drive a complementary subset of the state to the origin. These ideas were suggested by one of the anonymous reviewers and provide a promising direction for future work.

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