Self-Triggered Set-Valued Observers

Daniel Silvestre, Paulo Rosa, João P. Hespanha, Carlos Silvestre

Abstract—This paper addresses the problem of high computational requirements in the implementation of Set-Valued Observers, which places stringent constraints in terms of their use in applications where low computational power is available or the plant is sensitive to delay. It is firstly shown how to determine an overbound for the set-valued estimates, which reduces the overhead by limiting the number of inequalities defining those set-valued state estimates. In the particular setting of distributed gossip problems, the proposed algorithm is shown to have constant complexity. This algorithm is of prime importance to reduce the computational load and enable the use of such estimates for real-time applications. Results are also provided regarding the frequency of the triggers both in the deterministic and stochastic cases. The performance of the proposed method is evaluated through simulation.

I. INTRODUCTION

The problem of distinguishability for general linear systems relates to that of determining, for any given set of inputs, noise and disturbances, if two systems produce different outputs. In particular, fault detection can be cast into such a framework by considering the distinguishability of the faulty system and the nominal fault-free one. An interesting instance of the fault detection problem is detecting faults in an asynchronous distributed environment, which refers to determining if any of the nodes enters an incoherent state given the observed history of measurements.

In the context of fault detection, a special case of interest rises in the domain of randomized distributed algorithms, both due to their relevance in certain problems but also because of their unstructured nature, i.e., all nodes play the same role in the algorithm, while the messages need not satisfy any particular type of time sequence, since any two messages are regarded as having the same function. This class of algorithms is used for iterative solutions, because they offer robustness characteristics against packet drops and node failure. Applications of randomized algorithms range from selection and sorting [1] to consensus [2] and solving problems for which the solution requires a heavy computational burden.

The area of distributed systems and, in particular, randomized algorithms, poses challenges to the fault detection scheme since, due to the random behavior of gossip algorithms, for each possible transmission, the state can belong to a set of possible state realizations originated by that transmission and the previous state. To consider the worst-case scenario, one needs to perform the union of all possible state sets, as this, in general, represents a disconnected set. Thus, at each transmission time, a state estimator must compute the set of possible states generated by each transmission and compute the union of all of them. By definition, the number of sets grows exponentially with the number of past time instants that we are considering, i.e., the horizon $N$.

The study of Fault Detection and Isolation (FDI) problems has been a long standing research topic, since the early 70’s (see [3]), but still poses remarkable challenges to both the scientific community and the industry (see, for example, the survey in [4] and references therein). Classical fault detection methods such as the ones proposed in [3], [5], [6], [7], [8], [9] and [10], rely on designing filters that generate residuals that should be large under faulty environments. These strategies aim to derive bounds (or thresholds) on these residuals that can be used to decide whether a fault has occurred or not. However, calculation of these thresholds is typically cumbersome or poses stringent assumptions on the exogenous disturbances and measurement noise acting upon the system.

An alternative approach, based on Set-Valued Observers (SVOs) was described in [11]. The concept of SVOs was first introduced in [12] and [13] and further information can be found in [14] and [15] and the references therein. The SVO-based solution alleviates the design complexity, while posing mild assumptions on the system. However, it also requires increased computational power when compared to classical FDI methodologies.

The choice for representing the set of possible states depends on a mathematical formulation that enables fast and non-conservative intersections and unions of sets, as those are major and normally time-consuming operations when implemented in a computer. One alternative is to use the concept of zonotopes, described in [16] and further developed in [17] and [18]. However, it is normally the case that each proposal represents a compromise between the speed of the unions and intersections. Alternatively, the idea of interval analysis [19] may also be adopted, although it introduces conservatism by not considering higher horizon values in their formulation, unlike the SVOs [11]. An alternative approach is adopted in this article, as described in the sequel, in order to attain the desired convergence guarantees, while keeping the computational requirements to a tractable level.

In this paper, we address the main issue regarding the use of SVOs for fault detection described before by resorting to
overbounding with ellipsoids the polytope where the state can take values in. These are propagated using the possible dynamics matrices, which is computationally lighter than running the standard SVO propagation procedure. In doing so, the hyper-volume of the set-valued estimates increases since we added conservatism to the solution. Whenever such measure is greater than the initial size of the polytope, the algorithm self-triggers the standard mechanism to reduce the size of the set-valued estimates.

The main contributions of this paper are as follows:

- given a specific structure for the matrix defining the polytope (i.e., the set-valued state estimate), it is shown how to compute an overbounding ellipsoid or ball;
- based on the concept of singular vectors, we show how a rotation can be found to prevent the approximation error of using boxes from going to infinity when the matrix defining the polytope is ill-conditioned;
- an algorithm is introduced that uses approximations to the optimal SVO estimates based on the previous methods, which is less computationally demanding, and self-triggers the computation of the aforementioned estimates only when necessary to ensure convergence;
- results are provided regarding the worst-case frequency of the triggers for a general Linear Parameter-Varying (LPV) system;
- finally, for the special case of a distributed linear algorithm with a gossip property, it is shown that the overbounds are efficient to compute and propagate, since its complexity is constant, and a stochastic analogous to the smallest sampling frequency is computed.

The remainder of this paper is organized as follows. In Section II, we describe the problem of fault detection using SVOs estimates when in the presence of Byzantine faults. The proposed solution to reduce the computational cost and limit the conservativeness is given in Section III. Section IV presents the algorithm for Self-Triggered SVOs, while the main properties of the obtained results are stated in Section V and illustrated in simulation in Section VI. Concluding remarks are provided in Section VII.

**Notation:** The transpose of a matrix $A$ is denoted by $A^T$. For vectors $a_i$, $(a_1, \ldots, a_n) := [a_1^T \ldots a_n^T]^T$. We let $1_n := [1 \ldots 1]^T$ and $0_n := [0 \ldots 0]^T$ indicate $n$-dimensional vector of ones and zeros, respectively, and $I_n$ denotes the identity matrix of dimension $n$. Dimensions are omitted when clear from context. The vector $e_i$ denotes the canonical vector whose components are equal to zero, except for the $i$th component. The notation $||v||$ refers to $\|v\| := \sup_i |v_i|$ for a vector, and $||A|| := \sigma(A)$ and $||A||_2 := \sqrt{\text{tr}(A^TA)}$. The Orthogonal group of dimension $n$ is represented by $O(n)$.

## II. PROBLEM STATEMENT

We consider the dynamics of a Linear Parameter-Varying (LPV) system $S$ of the form:

$$
S: \begin{cases}
x(k+1) = A(\Delta(k))x(k) + B(\Delta(k))u(k) \\
y(k) = C(\Delta(k))x(k)
\end{cases}
$$

where $x \in \mathbb{R}^n$, matrices $A(\Delta(k))$ represent the dynamics of the system and are taken from $\{Q_i, i = 1, \ldots, m\}$ with each $Q_i$ being the possible dynamics in each time instant. The signal $u$ is designed so as to ensure the state remains bounded. Explicitly considering the uncertainty in the time-varying matrix $A(\Delta(k))$, we rewrite it as a central matrix and a sum of uncertainties resulting in the state equation in (1) being given by

$$
x(k+1) = (A_0 + \sum_{\ell=1}^{n_{\Delta}} \Delta_{\ell}(k)A_{\ell})x(k) + B(k)u(k) \tag{2}
$$

where $n_{\Delta}$ is the number of required uncertainties and each $\Delta_{\ell}(k)$ is a scalar uncertainty with $|\Delta_{\ell}(k)| \leq 1$. Whenever clear from context that matrices depend on the parameter $\Delta(k)$ we will drop the $\Delta$ and refer to the matrices as $A(k)$, $B(k)$ and $C(k)$.

It is necessary to maintain a set of all possible state realizations at each time instant to determine when sampling the output of the system is necessary. We use the Set-Valued Observers (SVOs) framework from [20] and [21] and define, at transmission time $k$,

$$
X(k) := \text{Set}(M(k), m(k))
$$

where $\text{Set}(M, m) := \{q : Mq + m \leq 0\}$ represents a convex polytope, where the operator $\leq$ is understood component-wise. The aim of an SVO is to find an approximation of the smallest set containing all possible states of the system at time $k$, $\hat{X}(k)$, with the knowledge that $\forall_{0 \leq t < N}, x(k) \in \hat{X}(k-i)$ and that the dynamics of the system are as in (2). In other words, at each time $k$, $\hat{X}(k)$ is an approximation of the set containing all possible states, $X(k)$, such that $X(k) \subseteq \hat{X}(k)$.

More precisely, the initial state $x(0) \in X(0)$ where $X(0) := \text{Set}(M_0, m_0)$ and we can select $M_0$ and $m_0$ such that the corresponding polytope is guaranteed to contain the initial state. If $\Delta^*$ is known, then the set $X(k+1) := \text{Set}(M_{\Delta^*}(k+1), m_{\Delta^*}(k+1))$, which contains all the possible states of the system at time $k+1$, can be described by the set of points, $x$, satisfying

$$
\begin{pmatrix}
M(k)(A_0 + A_{\Delta^*})^{-1} & -m(k) \\
C(k + 1) & -y(k + 1)
\end{pmatrix}
\begin{pmatrix}
x \\
y(k + 1)
\end{pmatrix}
\leq
\begin{pmatrix}
-m(k) \\
y(k + 1) \\
-M(k)
\end{pmatrix}
\tag{3}
$$

where

$$
A_{\Delta^*} = \sum_{\ell=1}^{n_{\Delta^*}} \Delta^*_\ell A_{\ell}
$$

and $\Delta^*_\ell$ is the realization of the uncertainty for the current time instant. This procedure assumes an invertible transmission matrix. When this is not the case, we can adopt the strategy in [22] and solve the inequality

$$
\begin{pmatrix}
I & - A_0 - A_{\Delta^*} & -M(k) \\
-I & C(k + 1) & 0 \\
0 & -C(k + 1) & M(k)
\end{pmatrix}
\begin{pmatrix}
x \\
x^-
\end{pmatrix}
\leq
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\tag{4}
$$
by applying the Fourier-Motzkin elimination method [23] to remove the dependence on $x^-$ and obtain the set described by $M(k+1)x \leq -m(k+1)$.

Let the coordinates of each vertex of the hypercube $H := \{ \delta \in \mathbb{R}^{N_\Delta} : |\delta| \leq 1 \}$ be denoted by $\theta_i, i = 1, \ldots, 2^{N_\Delta}$. Using (4), we compute $X_\theta(k)$ with $\Delta^* = \theta_i$. Thus, the set of all possible states at time $k+1$ can be obtained by

$$X(k+1) = \bigcup_{\theta_i \in H} \text{Set}(M_{\theta_i}(k+1), m_{\theta_i}(k+1))$$

where we make the union for all the vertices $\theta_i$ and where $M_{\theta_i}$ and $m_{\theta_i}$ are obtained using (4). The convex hull, $X(k+1)$, of set $X(k)$ is then obtained by using the methods described in [24], since, in general, the set $X(k+1)$ is non-convex even if $X(k)$ is convex. For additional properties of the set $X(k)$, the interested reader is referred to [25] and references therein.

An important issue regarding the SVOs is its computational burden, since the complexity grows exponentially both on the number of matrices $Q_i$ and also on the horizon $N$. This problem can render the SVOs inapplicable for some systems, especially those with stringent time constraints such as control in real-time applications.

The main goal of this paper can therefore be stated as: reduce the necessary number of calculations by only performing the SVO update routine when necessary, according to some user-defined criterion, while lowering the conservatism of the solution.

### III. ELLIPSOID OVERBOUNDING

The main issue when applying SVO-based techniques to a real-time or time-sensitive application is its computational burden. In each iteration, generating the set containing all possible state realizations amounts to the union of the sets obtained by propagating all possible combinations of the system dynamics and intersecting it with the set of states compatible with the current measurements. This process is time-consuming due to the number of possible combinations - see [25].

In this section, we assume the particular structure $X(k) = \{ q : M(k)q \leq 1 \}$, where matrix $M(k)$ takes the form

$$M(k) = \begin{bmatrix} m_1(k) \\ -m_1(k) \\ \vdots \\ m_{\ell}(k) \\ -m_{\ell}(k) \end{bmatrix}$$

and where the polytope $X(k)$ is defined as the intersection of $2\ell$ half-planes.

**Theorem 1:** Consider a convex set

$$S = \{ x : Mx \leq 1 \}$$

such that $M$ can be written in the form (5), we have the implication $\forall x : Mx \leq 1_n \Rightarrow x^T M^T Mx \leq 1_{n \times n}$. Performing a singular-valued decomposition on $M$ we get $\frac{1}{n} x^T V S^T U S V^T \leq 1$ and, since $U$ is a unitary matrix, we get the conclusion. ■

A corollary of the previous result can be derived in order to provide an overbound in terms of the maximum norm of any point belonging to the set.

**Corollary 2:** Consider a convex set $S$ such that matrix $M$ can be written as in (5). Then, we can overbound $S$ by a ball described by

$$x^T x \leq \frac{n}{\sigma_{\min}(M)}$$

which means that $||x||_2 \leq \sqrt{\frac{n}{\sigma_{\min}(M)}}$.

**Proof:** Consider the previous result where we have upperbounded the set $S$ by an ellipse. By upperbounding the singular values, we get $P = \frac{\sigma_{\min}}{n} V IV^T = \frac{\sigma_{\min}}{n} I$ which leads to the result. ■

**Remark 3:** It is stressed that alternative methods have been developed in the literature to obtain ellipsoidal set-valued state estimates, such as the seminal work by [13], as described, for instance, in [14]. However, the algorithm proposed in this work has some relevant properties, as discussed in the sequel, including the low-computational power required, as well as guaranteeing that the state of the system is indeed contained within the ellipsoid.

### IV. SELF-TRIGGERED SET-VALUED OBServers

Self-Triggered Set-Valued Observers aim to reduce the computational cost associated with the classical SVOs by only computing the set-valued state estimate (using the previously described tools) when the set grows in size above a certain threshold. As an alternative, we compute and propagate overbounds, which are less computationally demanding, and perform the intersection with the measurement set for the worst-case in terms of system dynamics.

The main advantage of this method is its real-time application due to diminished computational costs associated with three main factors:

- The matrix defining the polytope generally belongs to $\mathbb{R}^{\ell \times n}$, where $\ell \gg n$ and represents the number of restrictions associated with the edges of the polytope whereas the proposed overbound matrix belongs to $\mathbb{R}^{n \times n}$.
- Running the SVO computations at some time instants allows to use the idle moments to pre-compute the necessary combinations of matrices products;
- In some special cases of interest, such as in distributed systems, it is possible to discard dynamics matrices based on the observation set and compute the worst estimate with minimal processing effort.

The case of systems with invertible dynamics is analyzed in the sequel.

Start by noticing that the first rows of (3) are equivalent to

$$M(k)(A_0 + A_{\Delta^*})^{-1}x \leq -m(k)$$

**Proof:** We have that $Mx \leq 1_n$ and, since matrix $M$ can be written in the format (5), we have the implication $\forall x : Mx \leq 1_n \Rightarrow x^T M^T Mx \leq 1_{n \times n}$. Performing a singular-valued decomposition on $M$ we get $\frac{1}{n} x^T V S^T U S V^T \leq 1$ and, since $U$ is a unitary matrix, we get the conclusion. ■
where (6) defines the set of points that originate from propagating the previous points of the set defined by Set(\(M(k), m(k)\)). If the sets are always defined to have \(m(k) = -1\), then we are in the conditions to bound the state at time \(k\) with Theorem 1 and propagate using the dynamics of the system, considering each given instantiation of the uncertainties in \(\Delta^*\). The assumption only forces the set to include the origin, although it is not very stringent as we can always take a translation. Thus, \(X(k + 1) = \{(A_0 + A_\Delta)^{-1}x \land x \in X(k)\}\).

From the previous discussion, the set \(X(k + 1)\) can be described as

\[
X(k + 1) = \bigcup_{\Delta \in \mathcal{H}} \text{Set}(M(k)(A_0 + A_\Delta)^{-1}, -1) \quad (7)
\]

If the original set defined by the matrix \(M(k)\) is over-bounding using the procedure found in the previous section, the resulting overbounding set \(X(k + 1)\) is a union of ellipsoids with the singular vectors of each matrix of possible dynamics resulting in an expansion or contraction along the singular vectors basis.

The ellipsoids describing the set of possible state realizations are subject to a translation as to reflect the control input. Notice that the control input has no influence in the hyper-volume of the ellipsoids obtained using equation (7). However, the result of the intersection phase (see Figure 1) with the measurement set depends on the control input as it might render the intersection to be the empty set.

Due to sensor noise or the inability to measure the full state of the system, the observations can be defined as a set \(Y(k)\) which is a polytope posing constraints on the current state. The singular values and the associated singular vectors of the matrix defining such a polytope indicate the directions where the uncertainty is greater. Therefore, to test whether to execute the operation of computing the actual set, one can resort to intersecting the observation set \(Y(k + 1)\) with the ellipsoids and evaluate if it the norm of the state increases over the current iteration to prevent it from becoming non-convergent. In other words, this allows us to derive conditions to decide whether a full iteration of the SVO described in Section II is necessary.

An easy-to-compute new estimate for the norm of the state at time \(k + 1\) is simply obtained by solving

\[
\begin{align*}
\text{maximize} & \quad ||p|| \\
\text{subject to} & \quad p^TM(k + 1)^TM(k + 1)p \leq 1 \\
& \quad p \in Y(k + 1). \\
\end{align*}
\]

The previous problem translates into finding an intersection of ellipsoids and then computing the point in that set with the greatest norm. Matlab’s Ellipsoidal Toolbox [26] provides computationally efficient methods to tackle this problem. The complexity associated with computing the intersection of two sets is constant in terms of required iterations and each is cubic in the dimension of the state as it amounts to solve a Second-Order Cone Programming (SOCP). We now summarize the algorithm to self-trigger the SVO calculations (see Section II) only when the hyper-volume of the overbounding set grows.

\begin{algorithm}
\textbf{Algorithm 1 Self-trigger SVO}

\begin{enumerate}
\item \textbf{for each} \(A_0 + A_\Delta\) \textbf{do}
\item \textbf{Compute the ellipsoids overbounding} \(M(k)(A_0 + A_\Delta)^{-1}\)
\item \textbf{Determine upper bound of} \(||x(k + 1)||\) \textbf{using} (8)
\item \textbf{if} \(||x(k + 1)|| > \text{threshold}\) \textbf{then}
\item \textbf{return} \textbf{Run the SVO update}
\item \textbf{end if}
\item \textbf{end for}
\end{enumerate}
\end{algorithm}

Regarding the previous algorithm, see that the constant \(c\) here translates the hyper-volume computed using the over-approximation ellipsoid introduced before and, therefore, the main idea is to trigger the standard SVO update when the current estimation yields a hyper-volume greater than the previous value at triggering time. An interesting question is to determine under what conditions the set-estimate overbound converges to zero and no observations are needed. This is particularly important in fault detection, as it reduces the problem to checking whether the norm of the state estimate is convergent.

\begin{proposition}
Consider an LPV system

\[
S: \begin{cases}
x(k + 1) = A(\Delta(k))x(k) + B(\Delta(k))u(k) + w(k) \\
y(k) = C(\Delta(k))x(k)
\end{cases}
\]

If

\[
||w(k)|| \leq (1 - \sigma_{\text{max}}(A_0 + A_\Delta))||x||
\]

then, the hyper-volume of the overbounding set is a decreasing function.

\begin{proof}
We need to show that \(||x(k + 1)|| \leq ||x(k)||\), regardless of the disturbance, \(w(k)\), which amounts to

\[
||(A_0 + A_\Delta)x(k) + w|| \leq ||x(k)||, \forall \Delta
\]

Using the definition of the 2-norm we get

\[
\frac{\sigma_{\text{max}}(A_0 + A_\Delta)||x(k)|| + ||w(k)||}{||x(k)||} \leq 1
\]

\end{proof}

Fig. 1: Original set and ellipsoid overbound with the set resulting from the intersection with the measurement set to form the new set-valued estimate.
By rearranging the terms we reach the conclusion. 

Remark 5: Proposition 4 addressed the deterministic case where we are looking at the worst possible case. The result states a condition to ensure that there is no need for triggering the standard SVOs. To satisfy the condition, the system needs to be stable since we can take a higher horizon and change the discrete time by “grouping” sequences of time steps. This ensures that the condition is satisfied at the expenses of considering the worst product of all combinations of dynamics matrices within those sequences of time steps.

Proposition 4 provides the result that for the self-triggered mechanism to never be activated, one possible solution is to ensure that the magnitude of the disturbance is canceled by a distributed system by considering that each node is represented as a state and that the sequence of actions of each node must satisfy is that the dynamics are stable.

Let us define the parameter $\Delta_j$ as an example of a network for a distributed system. A possible instantiation for each edge in the graph as to select the dynamics matrices being permutations of the same matrix to ensure that the magnitude of the disturbance is canceled by a distributed system by considering that each node is represented as a state and that the sequence of actions of each node must satisfy is that the dynamics are stable.

A. Distributed Systems

The case of distributed systems is particularly relevant when considering fault detection schemes. In particular, given the formulation in (1), it is possible to accommodate a distributed system by considering that each node is represented as a state and that the sequence of actions of each node defines the $\Delta$ parameters that selects a given overall system dynamics at any time instant. In Figure 2, we depict an example of a network for a distributed system. A possible definition for the parameter $\Delta$ in a gossip algorithm is to have each dynamics matrix being written as $A_0 + A_{\Delta_j} = P(A_0 + A_{\Delta_j})P^\top$, where $P$ is a permutation matrix. From (9) and the fact that the permutation matrix is orthogonal (i.e., $P^\top P = I$), we get

$$\sigma(A_0 + A_{\Delta_j}) = \sigma(A_0 + A_{\Delta_1}).$$

Let us define $M_j$ as the matrix generating the ellipsoid containing the set-valued state estimate, which is the set of points satisfying $\{q : q = (A_0 + A_{\Delta_j})x, x^\top x \leq 1\}$. Then, the ellipsoid for any $\Delta_j$ can be defined using just $\Delta_1$ as

$$x^\top M_j x \leq 1 \quad \iff \quad x^\top PM_j P^\top x \leq 1 \quad \iff \quad (P^\top x)^\top M_1 P^\top x \leq 1$$

which defines the ellipsoid for $\Delta_j$ at the expenses of a rotation matrix $P$ and the ellipsoid matrix $M_1$. Thus, conclusion arises that for the case of distributed systems sharing that gossip property, the set estimates overbounds will be equal ellipsoids apart from a rotation.

Remark 6: The previous argument focused on the case of distributed gossip algorithms, and found the property of the dynamics matrices being permutations of the same matrix to suffice in demonstrating that all overbounds will be the same ellipsoid up to a rotation. However, the key condition is that the dynamics matrices can be written up to an orthogonal change of basis (i.e., $\forall j, A_0 + A_{\Delta_j} = P(A_0 + A_{\Delta_j})P^\top$ with $P \in O(n)$).

Theorem 7: Consider a distributed gossip algorithm with each dynamics matrix being written as $A_0 + A_{\Delta_j} = U_j S V_j^\top$, where $v_{\text{max}}^j$ is the singular vector associated with the largest singular vector and, conversely, an observation set $Y = U_j S_j V_j^\top$ and $v_{\text{max}}^j$. Then, the worst-case set-valued state estimate overbound is given by the intersection with the ellipsoid defined by $\Delta_j$ such that

$$\max_j (v_{\text{max}}^j)^\top V_j^\text{max}$$

Proof: We start by noticing that we are solving the following program

\[
\begin{align*}
\text{maximize} & \quad ||p|| \\
\text{subject to} & \quad p^\top M_j p \leq 1 \\
& \quad p \in Y.
\end{align*}
\]

where the $M_j = U_j S_j^{-2} V_j^\top$ which amounts to the conversion between ellipsoid representations. Since all matrices $M_j$ are the same apart from a rotation, we are solving the equivalent problem

\[
\begin{align*}
\text{maximize} & \quad ||p|| \\
\text{subject to} & \quad (R^\top p)^\top M_1 (R^\top p) \leq 1 \\
& \quad p \in Y \\
& \quad R \in \{R_1, \ldots, R_j\}.
\end{align*}
\]

which, when $R$ is unconstrained, has a closed-form solution given by $R V_0 = V_1$ and the cost function evaluates to $\min(\sigma_{\text{max}}(Y), \sigma_{\text{max}}(A_0 + A_{\Delta_1}))$ (i.e., the maximum singular vectors align), since it is monotonically increasing with the inner product of the maximum singular vectors. Thus,
the ellipsoid with the maximum inner product between its singular vectors and the singular vector of \( Y \) is the worst case producing the largest intersection and the conclusion follows.

Theorem 7 establishes that the self-triggered technique presented in this paper is particularly effective when dealing with fault detection in the context of distributed system with a gossip feature. In such systems the worst case scenario can be found by checking the inner product between the maximum singular vector of each possible dynamics matrix and the singular vectors of the observation set. This fact is illustrated in Figure 3 where two ellipsoid overapproximation sets are shown with the correspondent intersection with the set \( Y \). As a consequence, at each time instant, the mechanism selects the overbounding ellipsoid producing the largest intersection. By doing so, the computational cost associated with the combinatorial behavior of the SVOs becomes constant given that only one ellipsoid needs to be computed along with one intersection and no unions are needed. This is irrespective of the number of dynamics matrices (i.e., the number of agents and states in the network).

V. MAIN PROPERTIES

The method introduced in this paper targets a specific issue related to the use of Set-Valued Observers, which is also one of its major drawbacks: the exponential growth of complexity with the number of states of the system. In this section, we aim to present results regarding the rate at which such costly computations are going to be triggered, when using the ellipsoid overbounding described previously.

We first present a result which allows the node to compute the worst-case time instant for the next trigger given that we must guarantee that, in between triggers, the size of the set-valued estimates is bounded. The following theorem presents the value \( \tau \) of time instant to the next trigger given that no measurement set is collected (i.e., transmitted to the observer). We recall that by triggering, it is understood the request for the computation of a “full” iteration of the standard SVO.

*Theorem 8:* A Self-Triggered SVO, with maximum state norm at the last trigger time \( T \) given by \( ||x(T)|| \leq C \), will trigger after \( \tau \) time instants, where

\[
\tau = \left[ \log_{\sigma_{\max}(A_0 + A_\Delta)} \frac{\sigma_{\min}(M(T))C}{\sqrt{n}} \right]
\]

*Proof:* Any point \( x_1 \in X(k+1) \) will satisfy \( x_1 = (A_0 + A_\Delta)x_0 \) for a point \( x_0 \in X(k) \) and an instantiation of

the uncertainties \( \Delta \). Then,

\[
||x(T + \tau)|| = \|[(A_0 + A_\Delta)\cdots(A_0 + A_\Delta_T)]x(T)\|
\]

\[
\leq \sigma_{\max}(A_0 + A_\Delta)^T ||x(T)||
\]

\[
\leq \sigma_{\max}(A_0 + A_\Delta)^T \frac{\sqrt{n}}{\sigma_{\min}(M(T))}
\]

To maintain the norm bounded \( ||x(T + \tau)|| \leq C \) it thus required that

\[
\sigma_{\max}(A_0 + A_\Delta)^T \frac{\sqrt{n}}{\sigma_{\min}(M(T))} \leq C
\]

\[
\iff \sigma_{\max}(A_0 + A_\Delta)^T \leq \frac{\sigma_{\min}(M(T))C}{\sqrt{n}}
\]

\[
\tau \leq \log_{\sigma_{\max}(A_0 + A_\Delta)} \frac{\sigma_{\min}(M(T))C}{\sqrt{n}}
\]

Thus, leading to the conclusion.

The previous result is of particular interest for the case where the module running the observer and the plant itself are connected using a network and where communication is expensive. Under these circumstances, the worst-case triggering decision must be made based on the model dynamics and not on the observation set. Another interesting setup is to determine what is the expected time between trigger events if a probability distribution is known for the parameters associated with the systems. Practical examples range over distributed stochastic system where some decision is random or the nodes acting at a given time instant are stochastically chosen. For that purpose, we introduce the following result.

*Theorem 9:* Take a distributed system where the difference between the volume of the overbounding estimate set from the last trigger to the current time instant is a stochastic variable defined as \( V_0 = V_i \) and \( V_{n+1} = \sum_{i=1}^{n+1} Z_i, n \geq 0 \), where \( Z_i \) represents the change in the difference at time \( i \) with \( \mathbb{E}[Z_i] = \mu \). Then,

i) If \( \mu < 0 \), the volume of the overbounding converges almost surely to 0;

ii) If \( \mu = 0 \), the expected triggering time is given by \( \mathbb{E}[\tau] = V_i(V_N - V_i) \), where \( V_i \) is the starting volume variation and \( V_N \) is the maximum allowed volume variation.

*Proof:* i) Take the martingale \( V_{n+1} = V_n + Z_{n+1} \) and the correspondent filtration \( \mathcal{F}_n = \{V_0, V_1, \ldots, V_n\} \) (for additional information on martingale theory, see [27]). Computing the conditioned expectation, we get

\[
E[V_{n+1} | \mathcal{F}_n] = E[V_n + Z_{n+1} | \mathcal{F}_n]
\]

\[
= E[V_n] + E[Z_{n+1} | \mathcal{F}_n]
\]

\[
= V_n + \mu
\]

which implies that \( E[V_{n+1} | \mathcal{F}_n] \leq V_n \) and therefore it is a supermartingale and by the concept of Markov Chains, it converges almost surely to 0 as \( n \to \infty \) [27].

ii) We start by writing the stochastic variable \( W_n = V_n^2 - n \) and showing that it can indeed be made a martingale. Take
Due to the Monotone Convergence theorem, as $E$ Using both (10) and (11), we get $E[W_{\tau \land n}] = V_i^2 - n$

Let us introduce a stopping time

$$\tau = \inf \{ n \geq 0 : V_n \in \{ 0, V_N \} \}$$

Due to the martingale properties, $V_{\tau \land n}$ is a martingale which implies that

$$E[V_{\tau \land n}] = E[V_{\tau \land 0}] = E[V_0] = V_i.$$  

We can also compute the probability of hitting the maximum volume variation $V_N$ by computing

$$E[V_\tau] = 0P[V_\tau = 0] + V_nP[V_\tau = V_n]$$

$$\Leftrightarrow$$

$$E[V_0] = V_nP[V_\tau = V_n]$$

$$\Leftrightarrow$$

$$P[V_\tau = V_n] = \frac{V_i}{V_n}.$$  

Using the new martingale

$$E[W_{\tau \land n}] = E[W_{\tau \land 0}] = E[W_0] = V_i^2,$$

and also

$$E[W_{\tau \land n}] = E[V_{\tau \land n}^2 - \tau \land n]$$

Using both (10) and (11), we get

$$E[V_{\tau \land n}^2] = V_i^2 + E[\tau \land n].$$

Due to the Monotone Convergence theorem, as $\tau \land n \to \tau$, $E[\tau \land n] \to E[\tau]$, which combined with (12) leads to

$$E[V_{\tau}^2] = V_i^2 + E[\tau]$$

but by definition

$$E[V_{\tau}^2] = V_N^2P[V_\tau = V_N] = V_N^2 \frac{V_i}{V_N}$$

Using (13) and (14), we reach the conclusion that $E[\tau] = V_i(V_N - V_i)$.

VI. SIMULATION RESULTS

In this section, we show some preliminary simulation results regarding the use of self-triggered SVOs. In Section III, we assumed a particular structure on the matrix defining the polytopes, however, the aim of this paper is not concerning to finding such an algorithm and, during simulation, we performed a simple method that duplicates the constraints and removes the unnecessary ones. Such procedure introduces conservatism and is a future research topic. Focus is given on how the triggering time reduces the computational cost associated with calculating the set-valued estimates and also on the effect of the problem on the performance of the method.

We consider a wind turbine problem where the continuous system has dynamics

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\chi \omega \end{bmatrix}$$

and it is discretized using a sampling period of 0.01s and the whole simulation lasts 100 iterations. We assume noise in the measurements and disturbances in the state that are overbounded. The control input is set to be a constant signal and no faults are introduced. The objective is to simulate how the set-valued estimates evolve in comparison with the self-trigger approach in a disadvantageous scenario as the generated polytope is not centered at the origin, which introduces conservatism in the solution.

Figure 4 depicts a typical run of the algorithm with the triggering times reducing the volume of the estimate set, which in this context is a ellipse. The volume increment depends on the singular values of the dynamics matrix as well as the overbounds for the disturbance and input signal. The triggering times become more frequent as the polytope resulting from the standard SVO framework becomes less centered at the origin and, by the end of the simulation, each 2 iterations it happens a trigger.

In order to show the main part of the proposed method that still required additional research effort, we depict in Figure 5 the volume of the hyper-parallelepiped approximation of the polytope defining the set-valued estimates. If we compare Figure 4 against Figure 5, the volume of the approximation when the computation is triggered goes from around 40 to 70. Part of that conservatism makes the volume grows faster as we are integrating the error in each iteration. Thus, future research attention will focus on how to enforce the symmetry with less conservatism.

![Fig. 4: Volume of the set-valued estimates when using the self-triggering approach.](image-url)
Figure 6 depicts the execution time of each iteration for both cases. The key observation is that when we get a trigger, the execution time is very similar between the two cases but, we gain from refraining the computation in each time instant. The peak at the beginning are not very meaningful in the sense that correspond to the first iteration where all the data structures need to be allocated and do not correspond solely to the computation of the new estimates.

VII. CONCLUSIONS

In this paper, the problem of high computational demand from the use of SVOs in the fault detection domain is addressed using an overapproximation technique, which only triggers the standard SVO computations when the volume of the set-valued state estimation is not decreasing. The method uses ellipsoids instead of the original polytopes, which inherits two main advantages, namely i) the number of rows in the matrices defining ellipsoids is in general much smaller; and ii) to compute linear transformations we can use the maximum singular value of the matrix defining the ellipsoid.

The proposed self-trigger method allows the observer to use the approximations whenever the hyper-volume of the set is converging, and compute the actual (or an improved approximation of the) polytope whenever such measure is above the previous triggering time. By doing so, the computations can be performed only at triggering times, avoiding the complex and heavy computations of the standard SVOs. For the particular case of distributed gossip algorithms, it is shown that the singular vectors are identical up to a rotation matrix, which enables the estimate update to be performed using only the worst-case singular vector of the measurement and prediction sets. The problem of computing the trigger rate and the time to the next trigger is also investigated, both in the deterministic setup and the stochastic case.

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