Sensitivity Analysis for Linear Systems based on Reachability Sets

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***Abstract*— The problem of deciding which inputs in a model influence the most the state or output is often of practical importance, especially in the cases in which the system can be over-parameterized. In this context, a designer is required to perform sensitivity analyses so as to select which inputs are the most relevant to the problem at hand and remove those with smaller or no impact. In this paper, we tackle this issue by constructing the exact reachable set of a linear system that relates the inputs with the state of that system. By means of projections and solutions of linear optimization programs, we are able to assess which inputs drive the most the state or the output of a linear system. Illustrative examples are presented in order to provide insights on the proposed method.**

I. INTRODUCTION

Sensitivity analysis has been a long standing research topic addressed by multiple techniques. The problem relates to the identification of the inputs causing the largest variability of the state/output of a model. Different techniques have been proposed and lengthy discussions are presented in the books [1], [2], [3], while further information can be found in the review articles [4], [5], [6].

As defined in the aforementioned works, the sensitivity analysis is typically conducted by defining a model, assigning probability density functions to each input, generating inputs, and assessing the output of the model. The view in this paper is centered on a worst-case scenario where, instead of considering the probability information, the sensitivity analysis is driven by the most extreme impact each input can have on the state/output of the model.

The main motivation of considering the sensitivity of a model is to find which inputs are the most successful in achieving a control strategy or which contribute the most to the outputs of the system. We envisage as particular case of interest a formation of agents with their own dynamics. Which nodes contribute the most for a final decision depends on their individual dynamics but also on the topology. The techniques designed for sensitivity analysis in this paper aim to answer the question of which inputs should be used or, in the opposite direction, which elements are the ones that can cause the final state to drift the most in the worst-case scenario. In a smart grid environment, such techniques would be useful to determine what is the worst operating point if one of the inputs is compromised or faulty.

The first group of methods that has been used to address the problem can be categorized as *direct methods* or *differential sensitivity analysis* because the model equations are differentiated (similarly computed the difference equations) with respect to the inputs. The impact of each input can then be described by these derivatives. Many formulations exist for continuous-time models that are surveyed in [5] and that have been more recently developed in the works [7], [8], and also for the discrete-time case in [9], by exploiting the case of a Kalman filter.

The variance-based methods in [7], [8] determine the sensitivity of each input by computing the variance on the output caused by the inputs through an approximation model, by using Taylor series expansions. These methods have the advantage of considering a general nonlinear model of the type $Y = f(X_1, \cdots, X_k)$, whereas the focus of this paper is on the linear case and with the main difference that only the variance is being used to compute the sensitivity. Our proposal is to leverage on recent developments in reachability sets computational approaches as a measurement of the uncertainty or variability caused by a given input.

Additional methods exist that estimate the variance using for example FAST (Fourier Amplitude Sensitivity Test) [10], which resorts to the Fourier series to represent the ANOVA decomposition of the nonlinear model. Similarly, some also use the WASP (Walsh Amplitude Sensitivity Procedure) [11]. The main objective is to compute the ratio between the variance of Y given X for all possible values of X and the variance of Y as a measure of the sensitivity. Examples range from [12], in which a transformation is used to reduce the computational cost, to [13] for a general sensitivity analysis independent of the model and also [14] where different methods based on variance using FAST are compared with direct methods. All such techniques consider the sensitivity from a variance point-of-view, trying to identify which inputs cause the most variability. Another interesting question arises when focusing on the support of the distribution where finding which input generates the worst possible scenario among all the plant inputs.

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In [15], the authors show how the sensitivity can also be computed from the posterior probability given a prior on the inputs and thus describing it by means of a Bayesian approach. Using the previous method requires less runs than data-driven techniques such as Monte-Carlo tools. The work in [16] addresses the case of determining the sensitivity of medical parameters in a model by a Monte-Carlo approach, where the input space is sampled and propagated with the model, so as to determine its variance. The case of the over-parameterized hydrological models is also studied in [17] using a latin-hypercube sampling and a one-factor-at-a-time (OAT) sampling to produce a global sensitivity analysis. The same type of problem of water flow and quality is further discussed in [18].

Many applications benefit from a sensitivity analysis of their associated models. For example, in [19], an investigation is conducted on what inputs influence the most the spread of malaria through a sensitivity analysis to conclude the best course for prevention and containment of the disease. The authors in [20] propose the use of probabilistic sensitivity analysis to assess technology as motivated by the new requirements of the National Institute for Clinical Excellence. Also, the case of hybrid systems has been considered in [21] (the interested reader is referred to the recent survey in [22] for further information).

In this paper, the focus is on constructing reachability sets for linear systems in order to quantify the impact of a given input on the state/output of the model. The intuition is that of measuring the variability of a state or an output with respect to the inputs by computing their respective interval through a reachable set. Following that motivation, the concept of Set-Valued Observers (SVOs) is going to be employed. In doing so, a polytope is generated that represents the restrictions on the state given the inputs and uncertainties on the initial state. The choice satisfies the need for an optimal representation for linear systems (i.e., no conservatism is added when there are no uncertainties in the dynamics). Motivated by the findings in [23] that OAT strategies might be justified for linear models, we sought to investigate that claim via a direct comparison between the two approaches, i.e., considering a single input versus considering multiple inputs simultaneously. The contributions of this paper can therefore be summarized as follows:

- The introduction of a formal method that makes use of reachability sets to compute the sensitivity of each input;
- A result proving, for linear models, the relationship between the sensitivity of OAT input with testing all the inputs at the same time.

The remainder of this paper is organized as follows. In Section II, we describe how to obtain the reachability sets of interest to linear models. Section III describes how given the sets one can use them both in a OAT and a global sensitivity analysis. The methodology is illustrated in a simple example where the impact of each input is tunned in Section IV. The technique is investigated in simulation in Section V. Concluding remarks and directions of future work are provided in Section VI.

Notation : The transpose of a matrix A is denoted by $A^\intercal$. For vectors a_i, $(a_1, \ldots, a_n) := [a_1^\intercal \;\; \ldots a_n^\intercal]^\intercal$. We let $1_n := [1 \ldots 1]^\intercal$ and $0_n := [0 \ldots 0]^\intercal$ indicate n-dimensional vector of ones and zeros, respectively, and I_n denotes the identity matrix of dimension n. Dimensions are omitted when clear from context. The vector e_i denotes the canonical vector whose components are equal to zero, except for the ith component. A diagonal matrix with its main diagonal equal to v is represented by $\mathrm{diag}(v)$. The symbol $\otimes$ denotes the kronecker product. The notation $||.||$ refers to $\|v\| := \sup_i |v_i|$ for a vector, and $\|A\| := \bar{\sigma}(A)$. The ith coordinate of a vector v is denoted by $[v]_i$.

II. Reachability for Linear Systems

In the context of sensitivity analysis, one is typically interested in determining what happens to the output if the inputs belong to some hypercube. Assuming a linear system, the optimal approach to model the smallest reachable set is using polytopes, as demonstrated in [24]. Therefore, we consider a definition that follows the same principles as those of Set-valued Observers (SVOs) that can be found in recent works [25], [26], [27], [28] and the references therein.

We consider a linear time-invariant model of the form:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Ed(k) \\ y(k) &= Cx(k) \end{aligned}, \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$, $y(k) \in \mathbb{R}^{n_y}$, $u(k) \in \mathbb{R}^{n_u}$ and $d(k) \in \mathbb{R}^{n_d}$, with matrices of appropriate size. In order to build a closed reachability set, we assume the bound $\forall_{1 \leq i < n_d}$: $|d_i(k)| \leq 1$ without loss of generality since matrix E can be appropriately scaled to enforce such bound on the vector of inputs $d(k)$. The main objective is to characterize the impact of each entry of input $d(\cdot)$ to the state $x(k)$ or the output $y(k)$.

To review the steps in the construction of reachable sets from the model in (1), we define $\mathrm{Set}(M, m) := \{q : Mq \leq m\}$, which represents a convex polytope, with the operator $\leq$ being a component-wise operation between the two vectors. The aim of an SVO (Set-Valued Observer) is to find the smallest set $X(k)$ containing all possible states of the system at time k, knowing that $\forall_{0 \leq i < H}, x(k-i) \in X(k-i)$ for all past H time steps and the dynamics of the system (1) for all possible values of inputs $d(k)$.

More precisely, the initial state satisfies $x(0) \in X(0)$, where $X(0) := \mathrm{Set}(M_0, m_0)$ and M_0 and m_0 are selected such that the corresponding polytope is guaranteed to contain the initial state. The notation $\bar{Z} := \begin{bmatrix} Z \\ -Z \end{bmatrix}$, for a matrix Z, and $\bar{v} := \begin{bmatrix} v \\ -v \end{bmatrix}$, for a vector v will be used to shorten the following equations. The information obtained by an additional output measurement $y(k+1)$, results in a set $X(k+1)$ that can be described as the set of points, $\mathbf{x}$, satisfying

$$\underbrace{\begin{bmatrix} M(k)A^{-1} & -M(k)A^{-1}E \\ \bar{C} & 0 \\ 0 & \bar{I} \end{bmatrix}}_{M(k+1)} \begin{bmatrix} \mathbf{x} \\ \mathbf{d(k-1)} \end{bmatrix} \leq \underbrace{\begin{bmatrix} m(k) + \tilde{u}(k,1) \\ \bar{y}(k+1) + \nu^\star \mathbf{1} \\ 1 \end{bmatrix}}_{m(k+1)}$$

for some $\mathbf{d(k-1)}$ where we used the notation $\tilde{u}(k,H) := \sum_{\tau=1}^{H} M(k)A^{-\tau}Bu(k-\tau+1)$.

We will be assuming an invertible matrix of the dynamics A to enable the previous strategy, although it is possible to construct the same set otherwise by resorting to the strategy in [24].

The above computations assume a horizon value $H = 1$, i.e., only the measurements from time $k+1$ and the input signal from time k are used to compute the set-valued estimate of the state at time $k+1$. One can leverage on this idea to construct the set corresponding to the constraints that respect the model from time zero to some time k, since this will be suitable for the purpose of sensitivity analysis. The computations can be extended to a general horizon H by defining the set as:

$$M(H) \begin{bmatrix} \mathbf{x} \\ \mathbf{d(0)} \\ \vdots \\ \mathbf{d(H-1)} \end{bmatrix} \leq m(H) \tag{2}$$

where

$$M(H) := \left[\begin{array}{c|ccc} M_0A^{-H} & -M_0A^{-1}E & \cdots & -M_0A^{-H}E \\ 0_{2Hn_d \times n_x} & & I_H \otimes \bar{I} & \end{array} \right],$$

$$m(H) := \begin{bmatrix} m_0 + \hat{u}(H) \\ 1_{2Hn_d} \end{bmatrix},$$

with $\hat{u}(H) := \sum_{\tau=1}^{H} M_0 A^{-\tau} Bu(\tau-1)$.

The inequality in (2) describes an optimal polytope that contains all constraints given by the model (1) and the hypercube for inputs $d(0), d(1), \cdots d(H)$ for which we would like to analyze the associated contributions to the state. The relationship between the state $x(H)$ and the values of $d(\cdot)$ are captured in the polytope.

III. Sensitivity Analysis using Polytopes

The general sensitivity analysis considers the individual contributions as well as the possible interactions among the inputs as opposed to the OAT framework where each input is taken independently. In this section, the two cases will be addressed.

A. General Sensitivity

The set $X(H)$ defined by (2) comprises all points of the type $(x(H), d(0), \cdots, d(H-1))$ that satisfy the dynamics (1) and can be reached for at least a point on the set of values being tested for the inputs. Notice that the definition of the reachable set $X(H)$ satisfies this property even if it is not convex. The idea behind the general sensitivity approach is that one can project this set into a lower dimension space and obtain the set defining how $x(H)$ varies with respect to a subset of the inputs and setting the remaining to zero. Then, it is possible to check the sensitivity of a state or output by computing the maximum and the minimum value attained with a worst-case selection. This formulation allows for considering more general linear models and to add constraints to the set of admissible states. For instance, one can answer questions like "what is the sensitivity of state j at time H with respect to inputs 2 and 3, given some unknown initial condition, and in the case where the measurement at time $H-5$ is zero?".

Returning to the assumptions in this paper, since $X(H)$ is a polytope, one could resort to the so-called Fourier-Motzkin elimination method [29] to remove the dependence on some of the inputs and obtain the description of $x(H)$ on a smaller subset. However, such tools have a heavy computational complexity and, since the objective is solely to compute the amplitude of the state or output with respect to the admissible values of the inputs, a different technique is proposed in this paper.

Since the computation of the sensitivity is going to be written as an optimization problem, we will constrain the state to belong to the space with a single non-zero ith input. In order to do so, we introduce the linear map $\Pi_i : \mathbb{R}^{n_x+1} \mapsto \mathbb{R}^{n_x+Hn_d}$ defined as

$$\Pi_i(x) = \begin{bmatrix} I_{n_x} & 0 \\ 0 & \mathsf{e}_i \end{bmatrix} x$$

where e_i is the ith vector of the canonical basis of $\mathbb{R}^{Hn_d}$, and is helpful in formulating the problem of finding the sensitivity amplitude for input i, where $1 \leq i \leq Hn_d$. However, it can be generalized to select more than one input by replacing e_i with a matrix $\begin{bmatrix} \mathsf{e}_{i_1} & \mathsf{e}_{i_2} & \cdots \end{bmatrix}$ to select inputs $i_1, i_2, \cdots$. Thus, the selection can take inputs at different time instants or entries in the input vector. In Section IV, an example is presented when comparing inputs of different times whereas Section V tests the impact of inputs on the same time instant.

Let the notation $X_i(H) := \{x : \Pi_i(x) \in X(H)\}$ represent the projected set, which will not be computed explicitly. Remark that input i is an entry of the input vector at a given time and therefore there are Hn_d inputs. Then, the sensitivity function can be given as in the following definition.

Definition 1: Given a set $X(H)$ built for a given horizon H, the general sensitivity of state j to the input i can be defined as the function:

$$\mathcal{S}(X_i(H), j) := x_j^{\max}(H) - x_j^{\min}(H),$$

$$x_j^{\max}(H) = \max_{\begin{bmatrix} x(H) \\ d_i \end{bmatrix} \in X_i(H)} x_j(H)$$

$$x_j^{\min}(H) = \min_{\begin{bmatrix} x(H) \\ d_i \end{bmatrix} \in X_i(H)} x_j(H).$$

In Definition 1, d_i is an input where $1 \leq i \leq Hn_d$. Notice that this function measures the sensitivity of the state (similarly the definition can represent an output if the polytope is constructed for the output) to the ith input. The problem can be solved independently by computing two linear optimization programs:

$$\begin{array}{ll} \underset{z}{\text{minimize}} & \begin{bmatrix} \mathsf{e}_j \\ 0_{n_d} \end{bmatrix}^\mathsf{T} z \\ \text{subject to} & z = \Pi_i(x), \\ & M(H)z \leq m(H). \end{array} \tag{3}$$

and

$$\begin{array}{ll} \underset{z}{\text{minimize}} & -\begin{bmatrix} \mathsf{e}_j \\ 0_{n_d} \end{bmatrix}^\mathsf{T} z \\ \text{subject to} & z = \Pi_i(x), \\ & M(H)z \leq m(H). \end{array} \tag{4}$$

The optimization programs (3) (minimum) and (4) (maximum) directly compute the two terms in function $\mathcal{S}_i(X(H), j)$. Remark that the solution of both problems is a linear objective function in a projected space from the original reachability set, which can be extended to other ways of computing the sets and more general models than that in (1). The steps are summarized in Algorithm 1.

Algorithm 1 General Sensitivity Analysis

Require: Linear model of the form (1), time horizon H.
Ensure: Computation of the influence of each input in the final state.

/* *Compute the full reachability set* $X(H)$ */
$X(H)$ from (2)
for each input i **do**
    /* Compute minimum and maximum of $x(H)$ */
    $\min x_j(H)$ from (3)
    $\max x_j(H)$ from (4)
    /* *Compute sensitivity* $\mathcal{S}(X_i(H), j)$ */
    $\mathcal{S}(X_i(H), j) = \max x_j(H) - \min x_j(H)$
    return *Sensitivity values*
end for

The next lemma proves the intuition stated in [23], clarifying the relationship to the dynamics applied to the initial state.

Lemma 1 (General sensitivity for Linear Systems): Consider an initial state x_0 satisfying $\|x_0\|_\infty \leq 1$ for the linear system in (1) and a vector of inputs d. Also consider the definition of $\mathcal{S}(X_i(H, \mathcal{J}), j)$ as the sensitivity amplitude of the state $x_j(H)$ to input i assuming all inputs in $\mathcal{J}$ are zero.

Then, the following holds:

$$\mathcal{S}(X(H, \emptyset), j) = \sum_{\ell \in \mathcal{D}} \mathcal{S}(X_\ell(H, \mathcal{D} \backslash \{\ell\}), j) - (|\mathcal{D}| - 1)\mathcal{S}_{x_0}.$$

where $\mathcal{D} = \{\kappa \leq Hn_d\}$ with integer κ, and $\mathcal{S}_{x_0}$ is the worst-case contribution of the initial state on state x_j at time H, i.e.:

$$\mathcal{S}_{x_0} := \mathcal{S}(X(H, \mathcal{D}), j).$$

Proof: Let us write the solution to the state equation in (1):

$$x(H) = A^H x_0 + \sum_{\tau=0}^{H-1} A^{H-1-\tau}\big(Ed(\tau) + Bu(\tau)\big). \tag{5}$$

To compute $\mathcal{S}(X(H, \emptyset), j)$, given the bounds for the inputs and the initial state $\|x_0\|_\infty \leq 1$, one needs to compute

$$\begin{array}{ll} \underset{z}{\text{argmin}} & \mathsf{e}_j^\mathsf{T} \mathcal{L}(z) \\ \text{subject to} & \|z\|_\infty \leq 1. \end{array} \tag{6}$$

and

$$\begin{array}{ll} \underset{z}{\text{argmax}} & \mathsf{e}_j^\mathsf{T} \mathcal{L}(z) \\ \text{subject to} & \|z\|_\infty \leq 1. \end{array} \tag{7}$$

using different linear functions $\mathcal{L}(z)$. In particular, when $\mathcal{L}(z) = A^H z$ the optimization programs in (6) and (7) give us $x_{\min}$ and $x_{\max}$ as the arguments that minimize and maximize that linear function subject to the constraints. Therefore,

$$\mathcal{S}_{x_0} = \mathsf{e}_j^\mathsf{T} A^H (x_{\max} - x_{\min})$$

as the amplitude of the sensitivity caused by the dynamics on the (unknown) initial state. Given that the objective function in the maximization of the jth entry of the state vector in (5) is a separable function (i.e., linear), it holds:

$$\mathcal{S}(X(H, \emptyset), j) = \mathcal{S}_{x_0} + \mathsf{e}_j^\mathsf{T}\Big(\sum_{\tau=0}^{H-1} A^{H-1-\tau} E(d^\tau_{\max} - d^\tau_{\min})\Big)$$

where $d^\tau_{\max}$ and $d^\tau_{\min}$ are the respective solutions to (7) and (6) with $\mathcal{L}(z) = A^{H-1-\tau}Ez$. If we redo the calculations for $\mathcal{S}(X_\ell(H, \mathcal{D} \setminus \{\ell\}), j)$, and since all inputs are set to zero except ℓ, one gets that:

$$\mathcal{S}(X_\ell(H, \mathcal{D} \backslash \{\ell\}), j) = \mathcal{S}_{x_0} + \mathsf{e}_j^\mathsf{T} A^{H-1-\tau} E(d^\ell_{\max} - d^\ell_{\min})\Big)$$

i.e., it is the same state solution as in (5) but with all inputs except ℓ set to zero. Thus, the conclusion follows by noticing two facts. The first one is that $\mathcal{S}(X(H, \emptyset), j)$ is the sum of $\mathcal{S}_{x_0}$ and one term for each input ℓ. Second fact is that summing all $\mathcal{S}(X_\ell(H, \mathcal{D} \backslash \{\ell\}), j)$ equals to adding $|\mathcal{D}|$ times the term $\mathcal{S}_{x_0}$ and the same terms related with the inputs ℓ. ■

The previous result asserted a relationship between the general sensitivity and the OAT strategy where the key step was a result of the linearity of (1). The next corollary reaches the same intuitive conclusion provided in [23].

Corollary 1: Assume that the initial condition x_0 is known, then the general sensitivity is the sum of the OAT sensitivities of each input.

Proof: The result follows from the fact that if x_0 is known then $x_{\max} = x_{\min}$ and $\mathcal{S}_{x_0} = 0$. ■

Given the relationship between the general sensitivity and the OAT strategy for linear systems, in the next section further details are provided on how to compute this sensitivity for linear systems in an efficient manner.

B. One-Factor-At-A-Time (OAT)

The discussion about the general sensitivity pointed towards the adoption of an OAT strategy to the linear case. As a consequence, the computational complexity of the method is largely reduced because the number of variables is much smaller. Intuitively, OAT aims at fixing all inputs except one and computing the amplitude of change on the state caused by the analyzed input.

For simplicity of notation, let us say that the parameter i corresponds to the entry ℓ at time t, i.e., that the labeling $d_i = d_\ell(t)$. Fixing all but the ith input to zero enables rewriting the polytope definition in (2) as:

$$M_i(H)\begin{bmatrix}\mathbf{x}\\ \mathbf{d_i}\end{bmatrix} \leq m_i(H) \tag{8}$$

where

$$\begin{aligned} M_i(H) &:= \left[\begin{array}{c|c} M_0 A^{-H} & -M_0 A^{t-1} E \mathsf{e}_\ell \\ \mathsf{0}_{1\times n_x} & 1 \\ \mathsf{0}_{1\times n_x} & -1 \end{array}\right], \\ m_i(H) &:= \begin{bmatrix} m(k) + \hat{u}(H) \\ \mathsf{1}_2 \end{bmatrix}. \end{aligned}$$

In turn, the optimizations required for computing the sensitivity also simplify to:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \begin{bmatrix}\mathsf{e}_j \\ \mathsf{0}_{n_d}\end{bmatrix}^{\mathsf{T}} x \\ \text{subject to} \quad & M_i(H)x \leq m_i(H). \end{aligned} \tag{9}$$

and

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & -\begin{bmatrix}\mathsf{e}_j \\ \mathsf{0}_{n_d}\end{bmatrix}^{\mathsf{T}} x \\ \text{subject to} \quad & M_i(H)x \leq m_i(H). \end{aligned} \tag{10}$$

Notice that both (9) and (10) do not involve any projection and work directly on the much smaller space of $\mathbb{R}^{n_x+1}$ instead of $\mathbb{R}^{n_x+Hn_d}$. For comparison with Algorithm 1, it is summarized the steps of the OAT approach in Algorithm 2.

Algorithm 2 OAT Sensitivity Analysis

Require: Linear model of the form (1), time horizon H.
Ensure: Computation of the influence of each input in the final state.

for each input i **do**
  /* *Compute the ith reachability set $X_i(H)$* */
  $X_i(H)$ given by (8)
  /* Compute minimum and maximum of $x_j(H)$ */
  $\min x_j(H)$ from (9)
  $\max x_j(H)$ from (10)
  /* *Compute sensitivity $\mathcal{S}(X_i(H), j)$* */
  $\mathcal{S}(X_i(H), j) = \max x_j(H) - \min x_j(H)$
  return $\mathcal{S}(X_i(H), j)$
end for

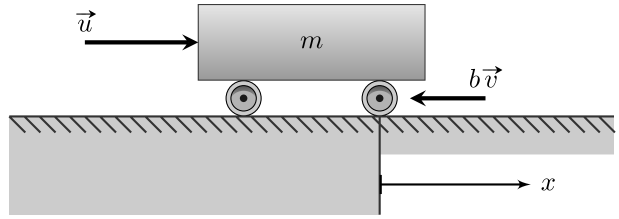

Fig. 1. Schematic of the moving cart.

IV. Illustrative Example

In this section, the model for a moving cart is used for its simplicity to show all the steps of the algorithm using the OAT strategy. The schematic for this dynamical system is depicted in Fig. 1. All expressions are presented to provide the reader with a concrete example of what computations are required and a small discussion on the correctness of the results.

Summing the forces in the x-direction and applying the second Newton's law, we get the equation corresponding to the cart in Fig. 1:

$$m\dot{v} + bv = u$$

which can be rewritten in continuous-time state space format as:

$$\dot{v} = \frac{-b}{m}v + \frac{1}{m}u$$

for mass $m = 100$ kg, damping coefficient $b = 50$ Ns/m, and force u in N. After discretization using a zero-order hold and a sampling time of 0.1 s the system can be written in the format of (1), as follows:

$$x(k+1) = 0.9512x(k) + 0.9754u + 0.9754d(k). \tag{11}$$

To include all the elements in the model of (1), it is assumed the cart is driven by a constant force of 500 N (i.e., $u = 0.5$ since we multiplied matrix B by a one thousand factor) and the question is: which value of $d(k)$ is some future state $x(H)$ most sensitive to? In order to allow a simplified representation, we will consider $H = 2$ and depict the reachability sets when $d(0)$ and $d(1)$ are taken from the interval $[-1, 1]$.

Resorting to the OAT technique, the first step is to construct the polytopes $X_i(H)$ setting all the remaining variables to zero. In addition, it was also assumed that the initial state space satisfies $\|x(0)\|_\infty \leq 1$. From this assumption, $M_0 = \begin{bmatrix}1\\-1\end{bmatrix}$ and $m_0 = \mathsf{1}_2$. According to (8), it follows that:

$$M_1(H) = \begin{bmatrix} 1.105 & -1.025 \\ -1.105 & 1.025 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, M_2(H) = \begin{bmatrix} 1.105 & -1.078 \\ -1.105 & 1.078 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \tag{12}$$

whereas for both cases $m_i(H) = \begin{bmatrix}2.0517 & -0.0517 & 1 & 1\end{bmatrix}^{\mathsf{T}}, i \in \{1, 2\}$.

Given the matrices in (12), one can plot the reachability sets produced in OAT fashion for each of the inputs $d(0)$

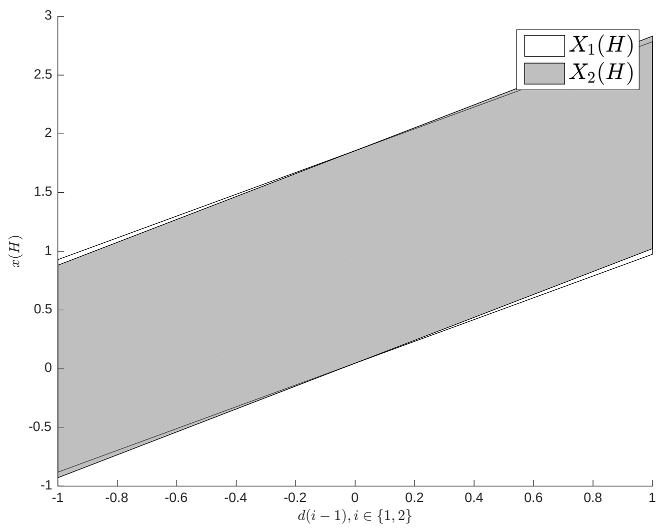

Fig. 2. Reachability sets for the illustrative case of the moving cart.

TABLE I
SENSITIVITY FOR THE MOVING CART.

	min (9)	max (10)	$\mathcal{S}(X(H))$
$X_1(H)$	-0.8811	2.7843	3.6654
$X_2(H)$	-0.9286	2.8319	3.7605

and $d(1)$. Figure 2 depicts the two sets relating the impact of $d(0)$ and $d(1)$ on state $x(H)$.

After the computation of $X_1(H)$ and $X_2(H)$, by solving the four optimization problems (one for the minimum and another for the maximum for both sets) given in (9) and (10), the solution for this example returns the values in table I.

Analyzing the values of $\mathcal{S}(X_i(H))$, it can be concluded that $x(H)$ is more sensitive to $d(1)$ than $d(0)$, which was expected given that it is a single state and $d(0)$ is multiplied by a constant smaller than one. For such a simple example, the sensitivity could be computed by means of interval analysis, by propagating using the model in (11) for the largest value, for all the signals, to get the maximum and the converse for the minimum. For $d(0) = 1$ and $d(1) = 0$ we get respectively:

$$\min x(H) = -A^2 + ABu + Bu - AB = -0.8811$$
$$\max x(H) = A^2 + ABu + Bu + AB = 2.7843$$

and

$$\min x(H) = -A^2 + ABu + Bu - B = -0.9286$$
$$\max x(H) = A^2 + ABu + Bu + B = 2.8319$$

thus obtaining the same results. However, for more complicated examples, the interval analysis introduces conservatism in the computed sets, thus losing optimality of the sensitivity calculation. By resorting to the SVO formulation, the obtained sets are optimal in the sense that no conservatism was added provided there are no uncertainties in the dynamics equations.

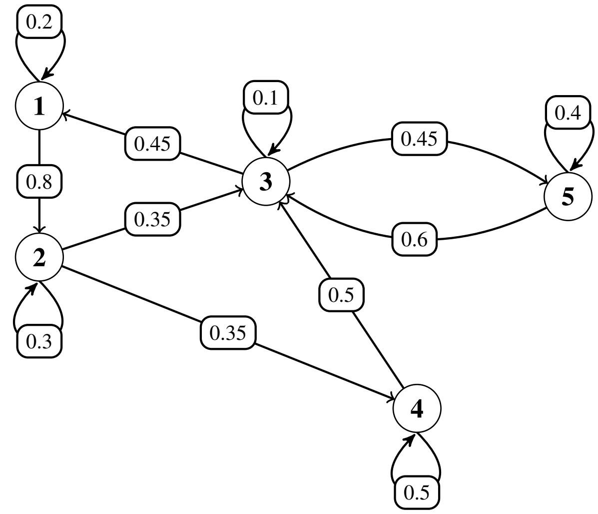

Fig. 3. Communication graph between the different vehicles.

V. SIMULATION RESULTS

In the previous section, the sensitivity was computed for a very simple example which could be equivalently calculated using interval analysis. The sensitivity values were expected since we were comparing inputs from different time instants. In a sense, if a system is stable older inputs will have less impact whereas newer values contribute the most. In this section, we present simulations for a network comprised of $N = 5$ vehicles with unicycle dynamics described in [30]. The formation follows the graph in Figure 3 where it is annotated the weights that each vehicle uses in a directional consensus algorithm to decide on a group velocity and direction values.

Each of the vehicles numbered from 1 to 5 in Figure 3 corresponds to the schematic presented in Figure 4. As described in [30], the discrete-time model for the ith vehicle can be written as:

$$\begin{bmatrix} p_i \\ q_i \end{bmatrix}(k+1) = \begin{bmatrix} p_i \\ q_i \end{bmatrix}(k) + TA_i(\theta_i)\begin{bmatrix} v_i \\ w_i \end{bmatrix}(k)$$

where the state (p_i, q_i) identify the position of the front of the vehicle and the inputs (v_i, w_i) account for the linear velocity and rotation. Moreover, T stands for the sampling time, θ_i for the orientation and matrix $A_i(\theta_i)$ is given as:

$$A_i(\theta_i) = \begin{bmatrix} \cos\theta_i & -l\sin\theta_i \\ \sin\theta_i & l\cos\theta_i \end{bmatrix}.$$

In this scenario, the vehicles initial orientation is known to be $\theta(0) = \begin{bmatrix} \pi/4 & \pi/6 & \pi/3 & \pi/5 & -\pi/3 \end{bmatrix}$ but their initial position is unknown with the constraint that $\forall_{1 \leq i \leq 5} : \| \begin{bmatrix} p_i(0) & q_i(0) \end{bmatrix}^\intercal \|_\infty \leq 1$. At each time step, after injecting their input, the nodes follow a consensus algorithm to decide the common position based on their state. Given the network topology and weights in Figure 3, one can define the iteration of a linear consensus algorithm of the type $x(k+1) = Px(k)$

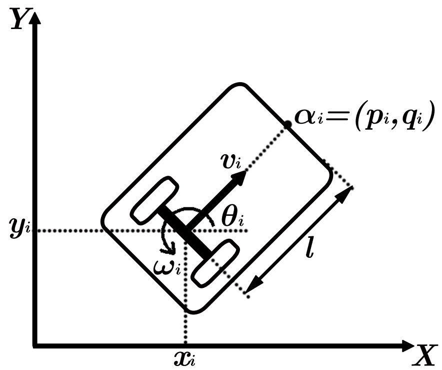

Fig. 4. Schematic of the unicycle model for the vehicles.

to be given by the matrix:

$$P = \text{Adj} \otimes I$$

and

$$\text{Adj} = \begin{bmatrix} 0.20 & 0 & 0.45 & 0 & 0 \\ 0.80 & 0.30 & 0 & 0 & 0 \\ 0 & 0.35 & 0.10 & 0.50 & 0.60 \\ 0 & 0.35 & 0 & 0.50 & 0 \\ 0 & 0 & 0.45 & 0 & 0.40 \end{bmatrix}.$$

The matrix Adj can be selected that guarantees (all that is needed is to have the columns sum to one) convergence to a weighted average of the inputs. In this example, the weights associated with each node contribution to the final value are $\begin{bmatrix}0.0826 & 0.0944 & 0.1468 & 0.0661 & 0.1101\end{bmatrix} \otimes \begin{bmatrix}1 & 1\end{bmatrix}$. Therefore, the sensitivity analysis by traditional methods would have to take into consideration two different aspects. First, the inputs of each vehicle, even though equal, will have a different impact on their position depending on each initial orientation. On the other hand, the fact that those positions are subject to a consensus algorithm with different weights can change the relative impact of a node position to the final agreed one. Using the proposed strategy in this paper, the SVOs produce a set for the final output (as opposed to the example where the sensitivity was with respect to the state) and the inputs are the different signals to each of the vehicles (instead of being the same input in different time instants in the example). The aim is to show the versatility of this approach and confirm the theoretical result of possible separation of the sensitivities for each input.

The reported sensitivities are presented in Table V when the initial position of the nodes is unknown but their orientation is given by the initial state. These values translate that the largest amplitude in the final output is obtained when changing the vehicle 3 input.

TABLE II

SENSITIVITY VALUES FOR DIFFERENT VEHICLES IN THE NETWORK WHEN THEIR INITIAL POSITION IS UNKNOWN.

# vehicle	1	2	3	4	5
$\mathcal{S}(X_i(H), 1)$	2.0467	2.0654	2.1017	2.0428	2.0763

TABLE III

SENSITIVITY VALUES FOR DIFFERENT VEHICLES IN THE NETWORK WHEN THEIR INITIAL POSITION IS KNOWN AND BELONGING TO A UNIFORM DISTRIBUTION.

# vehicle	1	2	3	4	5
$\mathcal{S}(X_i(H), 1)$	0.0467	0.0654	0.1017	0.0428	0.0763

In order to verify in an example the result in Lemma 1, we computed the sensitivity of the model to zero inputs and obtained $\mathcal{S}(X(H), 1) = 2$, which was expected given that with no input the vehicles will not move and their final positions are going to be within the same initial state irrespective of how long it has passed since the initial time. The simulation was repeated with a known $(p_i(0), q_i(0))$ taken from a uniform distribution by setting the initial polytope to be a singleton. The new sensitivities are reported in Table V, which satisfies the result in Corollary 1 that states the sensitivity of considering all inputs to be the sum of taking one-factor-at-time. In order to confirm the result in Lemma 1, the general sensitivity for the unknown model was computed to be 2.3329 which is the sum of the sensitivities with known initial state and $\sum_{1\leq \ell \leq 5} \mathcal{S}(X_\ell(H, \mathcal{D} \setminus \{\ell\}), 1) - 4\mathcal{S}_{x_0}$.

From the above simulation, it is clear that the use of SVOs for sensitivity analysis allows to compute different variations of interest of how a model reacts to the initial state uncertainty and amplitude in the state or output of a given input. Moreover, the above simulations require only one SVO computation for the unknown case and another for the known initial state and two linear optimization programs for each sensitivity computations. Given that there are no uncertainties in the model, both operations are computationally efficient.

VI. CONCLUSIONS AND FURTHER RESEARCH

This paper addressed the problem of computing the sensitivity of the state of a linear model to some inputs. By building on results from reachability analysis, a novel solution is proposed that computes the impact of an input on the state by checking the maximum interval of realizations of the state. Finding the minimum and the maximum is written as two linear optimization programs, thus avoiding the high complexity projections. Two possibilities are presented: a general version where the full reachable set is computed and the optimizations involve a projection; and, the One-Factor-At-A-Time (OAT) where, for each input, a reachable set is calculated by taking all the remaining to zero and proceeding in a similar fashion from that point. The latter method enables considerable computational savings, since the number of variables involved in the optimization is smaller.

Simulations are presented for a case of a network of dynamic systems (vehicles modeled as unicycles) where different weights to the links measure various contributions to the overall output. The simulations allow to illustrate the main result of this paper where it is shown that for linear systems the general sensitivity is related with the OAT approach. The precise relationship is quantified and the intuition that taking all inputs is the sum of taking one by one is only valid when the initial state is known. More interestingly, the approach herein proposed resorts to reachability sets that can also be computed for more general models. As future work, the idea is going to be extended for Linear Time-Varying (LTV) models, inheriting the same nice features. The results found in this paper motivate future investigations, as it provides an alternative solution for the sensitivity problem. Two main directions are envisioned: addressing the same issue on more complicated linear models (such as Linear Parameter-Varying) and resorting to other reachability tools; and, performing the same analysis for nonlinear and hybrid models using the general sensitivity based on their correspondent reachable sets.

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