Scale-Independent Hysteresis Switching* **

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Abstract. This paper introduces a new switching logic inspired by the hysteresis switching logic considered in [7, 11]. The new logic also uses hysteresis to prevent chatter, but unlike its predecessor in [7, 11], it is "scale-independent" as well. The logic is shown to have the requisite properties for adaptive control applications.

1 Introduction

"Scale-independence" is a property of certain switching algorithms used in an adaptive context which is key to proving an algorithm's correctness when operating in the face of noise and disturbance inputs [9]. The concept of *dwell-time switching* – exploited in [9] and elsewhere – has the advantage of being scale-independent. However, the existence of a prescribed dwell-time makes it impossible to rule out the possibility of finite escape in applications of dwell-time switching to the adaptive control of nonlinear systems [6]. On the other hand, the popular idea of *hysteresis switching* [7, 11] does not have this shortcoming. Unfortunately, hysteresis switching is not a scale-independent algorithm. The objective of this paper is to introduce a new form of chatter-free switching that does not employ a prescribed dwell-time and which is scale independent. We call this logic "scale-independent hysteresis switching" and we prove its correctness for applications to adaptive control [6].

Consider the hybrid dynamical system

$$\dot{x} = f_\sigma(x, t), \qquad x(0) = x_0 \tag{1}$$

where $\{f_p : p \in \mathcal{P}\}$ is an indexed family of locally Lipschitz functions taking values on a finite dimensional space $\mathcal{X}$ and defined on $\mathcal{X} \times [0, \infty)$, and σ is a

* This research was supported by the Air Force Office of Scientific Research, the Army Research Office, and the National Science Foundation.

** In F. W. Vaandrager and J. H. van Schuppen, editors, *Hybrid Systems: Computation and Control*, vol. 1569 of *Lecture Notes in Computer Science*, pp. 117–122. Springer-Verlag, Mar. 1999.

piecewise constant *switching signal* taking values in $\mathcal{P}$. The switching signal σ is chosen so as to cause the *performance signals*

$$\pi_p \stackrel{\triangle}{=} \Pi(p, x, t), \qquad\qquad p \in \mathcal{P} \tag{2}$$

to have certain desired properties. Here, Π is a *performance function* from $\mathcal{P} \times \mathcal{X} \times [0, \infty)$ to $\mathbb{R}$ that is continuous with respect to the second and third arguments for frozen values of the first.

The algorithm used to generate σ considered in this paper is called a *scale-independent hysteresis switching logic* and can be regarded as a hybrid dynamical system $\mathbb{S}_{\mathbb{H}}$ whose input is x and whose state and output are both σ. To specify $\mathbb{S}_{\mathbb{H}}$ it is necessary to first pick a positive number $h > 0$ called a *hysteresis constant.* $\mathbb{S}_{\mathbb{H}}$'s internal logic is then defined by the computer diagram shown in Figure 1 where the π_p are defined by (2) and, at each time t, $q \stackrel{\triangle}{=} \arg\min_{p \in \mathcal{P}} \Pi(p, x, t)$. In interpreting this diagram it is to be understood that σ's value at each of its

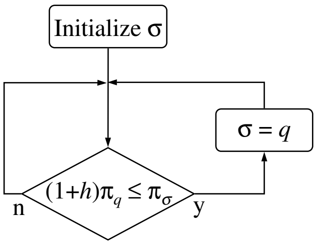

Fig. 1. Computer Diagram of $\mathbb{S}_{\mathbb{H}}$.

switching times $\bar{t}$ is equal to its limit from the right as $t \downarrow \bar{t}$. Thus if $\bar{t}_i$ and $\bar{t}_{i+1}$ are two consecutive switching times, then σ is constant on $[\bar{t}_i, \bar{t}_{i+1})$. The functioning of $\mathbb{S}_{\mathbb{H}}$ is roughly as follows. Suppose that at some time t_0, $\mathbb{S}_{\mathbb{H}}$ has just changed the value of σ to p. σ is then held fixed at this value unless and until there is a time $t_1 > t_0$ at which $(1 + h)\pi_q \leq \pi_p$ for some $q \in \mathcal{P}$. If this occurs, σ is set equal to q and so on. This type of logic has numerous applications in adaptive and supervisory control [7, 11, 8, 1, 6, 2, 3, 5].

The main result of this paper is the scale-independent hysteresis switching theorem. This theorem states that under appropriate "open-loop" assumptions, switching will stop at some finite time. Being able to establish that switching stops in finite time is crucial to the stability analysis of adaptive and supervisory control algorithms using hysteresis switching.

The switching logic described above is new. Its main advantage over the hysteresis switching logic considered in [7, 11] is that it is "scale-independent" in that its output σ remains unchanged if its performance function/input signal pair $\{\Pi, x\}$ is replaced by another pair $\{\bar{\Pi}, \bar{x}\}$ satisfying

$$\bar{\Pi}(p, \bar{x}, t) = \vartheta \Pi(p, x, t), \qquad \forall p \in \mathcal{P},\ t \geq 0$$

where ϑ is a positive time function. This is because, for any fixed time t, (i) the value of p that minimizes $\Pi(p, x, t)$ is the same value of p that minimizes $\bar{\Pi}(p, \bar{x}, t)$ and (ii) $(1 + h)\Pi(q, x, t) \leq \Pi(p, x, t)$ is exactly equivalent to $(1 + h)\bar{\Pi}(q, \bar{x}, t) \leq \bar{\Pi}(p, \bar{x}, t)$ for every $p, q \in \mathcal{P}$. Scale-independence often simplifies considerably the analysis of estimator-based supervisory control algorithms [9, 10, 6, 2, 3, 5].

This paper is organized as follows. Section 2 contains the statement and proof of the scale-independent hysteresis switching theorem. This theorem states that under appropriate "open-loop" assumptions, switching will stop at some finite time. In Sect. 3 it is illustrated how the scale-independence property can be used to apply the scale-independent hysteresis switching theorem to some systems for which the "open-loop" assumptions stated in Sect. 2 might be violated. Section 4 contains some concluding remarks.

2 Scale-Independent Hysteresis Switching Theorem

Let $\mathcal{X}_0$ denote a given subset of $\mathcal{X}$, and $\mathcal{S}$ the class of all piecewise-constant functions $s : [0, \infty) \to \mathcal{P}$. In what follows, for each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$, $T_{\{x_0,s\}}$ is the length of the maximal interval of existence of solution to the equations

$$\dot{x} = f_{s(t)}(x, t), \qquad x(0) = x_0$$

and $x_{\{x_0,s\}}$ is the corresponding solution. The following "open-loop" assumptions are made:

Assumption 1 (Open-Loop). *For each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$ the following is true:*

1. *There exists a positive constant ϵ such that for each $p \in \mathcal{P}$, the performance signal $\pi_p \stackrel{\triangle}{=} \Pi(p, x_{\{x_0,s\}}, t)$ is bounded below on $[0, T_{\{x_0,s\}})$ by ϵ.*
2. *For each $p \in \mathcal{P}$, the performance signal $\pi_p \stackrel{\triangle}{=} \Pi(p, x_{\{x_0,s\}}, t)$ has a limit*[1] *{which may be infinite} as $t \to T_{\{x_0,s\}}$.*
3. *There exists at least one $p^* \in \mathcal{P}$ such that the performance signal $\pi_{p^*} \stackrel{\triangle}{=} \Pi(p^*, x_{\{x_0,s\}}, t)$ is bounded on $[0, T_{\{x_0,s\}})$.*

First note that, because of the definition of $\mathbb{S}_{\mathbb{H}}$, $\sigma(0)$ must be such that $\pi_{\sigma(0)}(0)$ is strictly smaller than $(1 + h)\pi_p(0)$ for every $p \in \mathcal{P}$. Thus, because of the Lipschitz continuity of the f_p and the continuity of Π, there must exist an interval $[0, t_1)$ of maximal length on which $\pi_{\sigma(0)}$ remains strictly smaller than $(1 + h)\pi_p$ for every $p \in \mathcal{P}$. This means that σ is constant on $[0, t_1)$. Either

[1] This assumption holds, for example, if the performance signals are monotone functions of time.

this interval is the maximal interval of existence for x or it is not, in which case x is bounded on $[0, t_1)$ {cf. [4]}. If the latter is true, a switch must occur at t_1 and therefore $\pi_{\sigma(t_1)}(t_1) \leq \pi_p(t_1)$ for every $p \in \mathcal{P}$. Again, because of the Lipschitz continuity of the f_p, the continuity of Π, and the assumed boundedness below of each $h\pi_p$ by a positive constant, there must exist an interval $[t_1, t_2)$ of maximal length on which $\pi_{\sigma(t_1)}$ remains strictly smaller than $(1+h)\pi_p$ for every $p \in \mathcal{P}$. σ will then be constant on such an interval. Continuing this reasoning one concludes that there must be an interval $[0, T)$ of maximal length on which there is a unique pair $\{x, \sigma\}$ with x continuous and σ piecewise constant, which satisfies (1) with σ generated by $\mathbb{S}_{\mathbb{H}}$. Moreover, on each strictly proper subinterval $[0, \tau) \subset [0, T)$, σ can switch at most a finite number of times.

To establish existence of solution to (1) with σ generated by $\mathbb{S}_{\mathbb{H}}$, only the first Open-Loop Assumption was used. The remaining assumptions enable us to draw conclusions regarding the limiting behavior of σ as $t \to T$. The following is the main result of this chapter.

Theorem 1 (Scale-Independent Hysteresis Switching). *Let $\mathcal{P}$ be a finite set, assume that the Open-Loop Assumptions 1 hold and, for fixed initial state $\{x_0, \sigma_0\} \in \mathcal{X}_0 \times \mathcal{P}$, let $\{x, \sigma\}$ denote the unique solution to* (1) *with σ generated by $\mathbb{S}_{\mathbb{H}}$ with input x. If $[0, T)$ is the largest interval on which this solution is defined, there is a time $T^* < T$ beyond which σ is constant and $\pi_{\sigma(T^*)} \triangleq \Pi(\sigma(T^*), x, t)$ is bounded on $[0, T)$.*

Proof. Let $\{x, \sigma\}$ denote the unique solution to (1) with σ generated by $\mathbb{S}_{\mathbb{H}}$ and suppose that $[0, T)$ is the largest interval on which this solution is defined. Also, let ϵ be a positive constant that bounds below each performance signal π_p, $p \in \mathcal{P}$. For each $t \in [0, T)$, the switching logic guarantees that

$$\pi_{\sigma(t)}(t) < (1 + h)\pi_q(t) \tag{3}$$

with $\pi_q(t) \leq \pi_p(t)$ for every $p \in \mathcal{P}$. This is because if at some time t (3) were violated, σ would have to switch {at that precise instant of time} to that value q in $\mathcal{P}$ that minimizes $\pi_q(t)$ and therefore (3) would still be valid.

Because of the third Open-Loop Assumption 1, there is some $p^* \in \mathcal{P}$ such that $\pi_{p^*}(t)$ is bounded on $[0, T)$ by some positive constant K. From this and (3) one must have $\pi_{\sigma(t)}(t) \leq (1+h)K$ on $[0, T)$. Because of this and the monotonicity of the π_p, if a performance signal π_p becomes larger than $(1 + h)K$, σ will not switch to p ever again that. Thus, after some finite time $\bar{T} < T$, $\sigma(t)$ must remain inside the following nonempty set

$$\mathcal{P}^* \triangleq \Big\{ p \in \mathcal{P} : |\pi_p(\tau)| \leq (1 + h)K,\ \tau \in [0, T) \Big\}$$

To conclude the proof it is then enough to shown that there is a time $T^* \in [\bar{T}, T)$ beyond which σ actually becomes constant. Because of the second Open-Loop Assumption 1, for each $p \in \mathcal{P}^*$, $\pi_p(t)$ converges to a finite limit as $t \to \infty$. Thus, for each $p \in \mathcal{P}^*$, there must exist a time $T_p \in [\bar{T}, T)$ such that

$$|\pi_p(t) - \pi_p(\tau)| < \frac{h\epsilon}{2}, \qquad \forall t, \tau \in [T_p, T) \tag{4}$$

Because $\mathcal{P}$ is a finite set, $\mathcal{P}^*$ is also a finite set and therefore the set $\{T_p : p \in \mathcal{P}^*\}$ has a maximum element. Let $T_1 \in [\bar{T}, T)$ be such element. If there is no switching in $[T_1, T)$ then one can take $T^* \triangleq T_1$ and the proof is finished. Otherwise, let $T_2 \in (T_1, T)$ be the time instant at which the next switching occurs. For a given $t \in [T_2, T)$, let q be that element in $\mathcal{P}$ that minimizes $\pi_q(t)$. Since $\sigma(T_2), q \in \mathcal{P}^*$, $T_1 \geq T_{\sigma(T_2)}$, and $T_1 \geq T_q$, from (4) one concludes that

$$\pi_{\sigma(T_2)}(T_2) > \pi_{\sigma(T_2)}(t) - \frac{h\epsilon}{2} \qquad \text{and} \qquad \pi_q(T_2) < \pi_q(t) + \frac{h\epsilon}{2} \tag{5}$$

But $\sigma(T_2)$ is the element $p \in \mathcal{P}$ that minimizes $\pi_p(T_2)$, thus $\pi_{\sigma(T_2)}(T_2) \leq \pi_q(T_2)$. From this and (5) one concludes that

$$\pi_{\sigma(T_2)}(t) - \frac{h\epsilon}{2} < \pi_{\sigma(T_2)}(T_2) \leq \pi_q(T_2) < \pi_q(t) + \frac{h\epsilon}{2}$$

and therefore that

$$\pi_{\sigma(T_2)}(t) < \pi_q(t) + h\epsilon \tag{6}$$

Moreover, because of the first Open-Loop Assumption 1, $\pi_q(t) > \epsilon$. From this and (6) one concludes that

$$\pi_{\sigma(T_2)}(t) < (1+h)\pi_q(t)$$

Thus there can be no more switching at any time $t \in [T_2, T)$ and one can take $T^* \triangleq T_2$. □

3 Relaxing the Open-Loop Assumptions

The scale-independence property can be use to somewhat relax the "open-loop" assumptions stated in the previous section. Suppose that the following assumptions hold.

Assumption 2 (Relaxed Open-Loop). *There exists a positive time-function ϑ such that for each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$ the following is true:*

1. *There exists a positive constant ϵ such that for each $p \in \mathcal{P}$, the scaled performance signal $\bar{\pi}_p \triangleq \vartheta \Pi(p, x_{\{x_0,s\}}, t)$ is bounded below on $[0, T_{\{x_0,s\}})$ by ϵ.*
2. *For each $p \in \mathcal{P}$, the scaled performance signal $\bar{\pi}_p \triangleq \vartheta \Pi(p, x_{\{x_0,s\}}, t)$ has a limit $\{$which may be infinite$\}$ as $t \to T_{\{x_0,s\}}$.*
3. *There exists at least one $p^* \in \mathcal{P}$ such that the scaled performance signal $\bar{\pi}_{p^*} \triangleq \vartheta \Pi(p^*, x_{\{x_0,s\}}, t)$ is bounded on $[0, T_{\{x_0,s\}})$.*

In light of the scale independence property, $\mathbb{S}_{\mathbb{H}}$'s output σ remains unchanged if its performance function/input signal pair $\{\Pi, x\}$ is replaced by another pair $\{\bar{\Pi}, \bar{x}\}$ satisfying

$$\bar{\Pi}(p, \bar{x}, t) = \vartheta \Pi(p, x, t), \qquad \forall p \in \mathcal{P}$$

Since Assumptions 2 guarantee that the Open-Loop Assumptions 1 hold for the pair $\{\bar{\Pi}, \bar{x}\}$, we obtain the following corollary of Theorem 1.

Corollary 1 (Scale-Independent Hysteresis Switching). *Let $\mathcal{P}$ be a finite set, assume that the Relaxed Open-Loop Assumptions 2 hold and, for fixed initial state $\{x_0, \sigma_0\} \in \mathcal{X}_0 \times \mathcal{P}$, let $\{x, \sigma\}$ denote the unique solution to* (1) *with σ generated by $\mathbb{S}_{\mathbb{H}}$ with input x. If $[0, T)$ is the largest interval on which this solution is defined, there is a time $T^* < T$ beyond which σ is constant and $\bar{\pi}_{\sigma(T^*)} \triangleq \vartheta \Pi(\sigma(T^*), x, t)$ is bounded on $[0, T)$.*

It should be emphasized that in supervisory control it is often the case that Assumptions 1 might be violated, whereas Assumptions 2 can be shown to hold [6, 2, 3, 5].

4 Conclusions

A new type of switching logic called scale-independent hysteresis switching was introduced. This logic is inspired on the one in [7, 11], but has the advantage that it is scale-independent in that its output remains unchanged if the performance signals are scaled by a positive time function. The reader is referred to [7, 11, 8, 1, 6, 2, 3, 5] for the use of hysteresis switching logics in adaptive and supervisory control.

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