Supervisory Control of Families of Stabilizing Controllers*

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Abstract

This paper deals with the problem of switching between several linear time invariant (LTI) controllers, all of them capable of stabilizing a single LTI process, in such a way that the stability of the closed loop system is guaranteed for any switching sequence. The motivation for this problem is the control of complex systems where conflicting requirements make a single LTI controller unsuitable.

1 Introduction

In any control system's design problem there are conflicting requirements: fast step response, small energy consumption, good disturbance/noise rejection, etc. Sometimes the designer is able to achieve a good trade-off between all the requirements, but in complex problems this is not likely. Consider for example the control of an autonomous underwater vehicle [1]. If the vehicle is moving in open water, reducing the energy consumption is probably more important than having a fast response, but if the vehicle is tracking an underwater pipeline to look for leaks, it is important to keep the pipeline steadily centered on the image of a CCD camera mounted on the vehicle. In this case a fast response is needed. Situations like this are not uncommon in control application.

A solution for this problem is to design several controllers, each one "tuned" for a specific task, and to build a high-level supervisor to decide which controller should be put in the feedback loop at every instant of time. The high-level supervisor could be, for example, a rule-based expert system or a finite-state automata. Using this approach, the overall system contains both continuous dynamics and discrete logic and it is often called hybrid. This paper deals with the continuous part of such a hybrid system. We assume that some discrete logic generates a sequence of symbols each one corresponding to a desired controller. This sequence is converted into a continuous time switching signal σ taking values on an index set $\mathcal{P}$. This paper addresses the problem of designing a continuous time multi-controller Σ_C which, at each time instant, generates the control signal corresponding to the controller indexed by the current value of σ. To make the analysis of the overall system tractable one wants to build Σ_C in such a way that for *any* switching sequence, the system is stable.

The remaining of this paper is organized as follows. The problem addressed is formulated in section 2. In section 3 it is shown how this problem relates to switching among a set of stable systems. In this section two alternative methods of switching that guarantee the stability of the switched system are proposed. In section 4 a structure for the multi-controller is described. Section 5 contains the main results of the paper. It is shown that the proposed structure for the multi-controller preserves basic properties of the original

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LTI controllers, namely the closed loop transfer matrix for frozen time, stability, and set-point regulation capability. Section 6 deals with the robustness of the overall system with respect to unmodeled dynamics. It is shown that if the nominal model used to design the multi-controller does not match the dynamics of the real process, the overall system is stable provided the "mismatch" is small in a $\mathcal{H}_\infty$ sense. Finally, section 7 contains a brief discussion of the results achieved so far and some directions for future research.

Notation: In the sequel $\mathbb{N}, \mathbb{R}, \mathbb{R}^+$, and $\mathbb{C}$ denote the sets of nonnegative integers, real numbers, nonnegative real numbers, and complex numbers, respectively. Given a complex number $s \in \mathbb{C}$, $\Re[s]$ denotes the real part of s. Given a nonnegative constant λ, $\mathrm{RH}_\infty^\lambda$ denotes the ring of transfer matrices whose entries are proper rational functions with real coefficients and poles in the open complex semiplane $\{s \in \mathbb{C} : \Re[s] < -\lambda\}$. Given a transfer matrix $\mathrm{H} : \mathbb{C} \to \mathbb{C}^{m\times n}$ and a piecewise constant signal $u : \mathbb{R}^+ \to \mathbb{R}^n$, $(\mathrm{H}u)$ denotes the signal $y : \mathbb{R}^+ \to \mathbb{R}^m$ defined by the convolution of the impulse response of H with u. In case $\mathrm{H} \in \mathrm{RH}_\infty^0$, we denote the $\mathcal{H}_\infty$-norm of H by $\|\mathrm{H}\|_\infty$, i.e.

$$\|\mathrm{H}\|_\infty \triangleq \sup_{\Re[s]\geq 0} \|\mathrm{H}(s)\|$$

where $\|\mathrm{H}(s)\|$ denotes the largest singular value of $\mathrm{H}(s)$. Given a matrix valued $\mathcal{L}_2$ signal $U : \mathbb{R}^+ \to \mathbb{R}^{n\times m}$, $\|U\|_2$ denotes the $\mathcal{L}_2$-norm of U, i.e.

$$\|U\|_2 \triangleq \left(\int_0^\infty \|U(t)\|^2\, dt\right)^{1/2}$$

where $\|U(t)\|$ denotes the largest singular value of $U(t)$.

2 Problem Formulation

Let Σ_P be a multivariable LTI process with strictly proper transfer matrix H_P and minimal realization

$$\begin{aligned} \dot{x}_P &= A_P x_P + B_P u \\ y &= C_P x_P \end{aligned} \tag{1}$$

where u denotes the control input and y the controlled output. The feedback configuration that is used in this paper is shown in Figure 1. In this figure $\mathbf{r}$ denotes a reference signal, $\mathbf{d}$ an unknown but bounded input disturbance, and $\mathbf{n}$ unknown but bounded measurement noise. A controller transfer matrix K is said to stabilize H_P if for any minimal realization of K the feedback connection shown in Figure 1 is asymptotically stable.

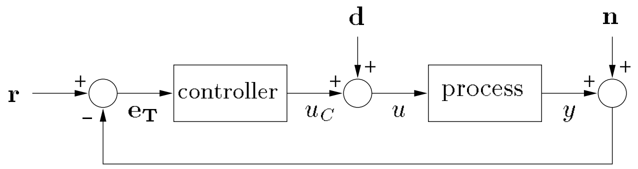

Figure 1: Feedback configuration

Consider an indexed set of controller transfer matrices $\mathcal{K} = \{\mathrm{K}_p : p \in \mathcal{P}\}$ such that K_p stabilizes H_P for every $p \in \mathcal{P}$. A signal $\sigma : \mathbb{R}^+ \to \mathcal{P}$ is said to be an (admissible) switching signal if it is piecewise constant and continuous from above at every time instant. The problem under consideration is the design a system Σ_C with inputs σ and $\mathbf{e_T}$ and output u_C such that the feedback connection shown in Figure 2 has the following properties:

Property 1 (Input/Output Matching). For every constant switching signal $\sigma(t) = p \in \mathcal{P}$, the dynamical system Σ_C is LTI from the input $\mathbf{e_T}$ to the output u_C and its transfer matrix is equal to K_p.

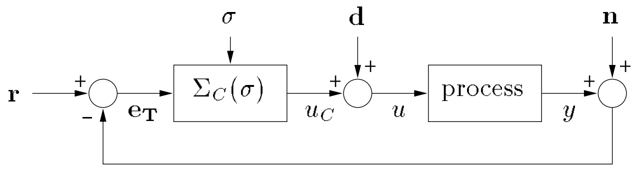

Figure 2: Feedback connection between H_P and Σ_C.

Property 2 (Stability). For every switching signal σ and every bounded piecewise continuous exogenous signals $\mathbf{r}$, $\mathbf{n}$, and $\mathbf{d}$, the signals $\mathbf{e_T}$, u_C, u, and y remain bounded. Moreover, if $\mathbf{r} = \mathbf{d} = \mathbf{n} = 0$, these signals decay to zero exponentially fast with a rate of decay which is independent of σ.

Properties 1 and 2 are the main requirements for Σ_C, but there are a few more desirable properties:

Property 3 (Set-point). For a SISO process, if the process transfer matrix H_P has a pole at the origin, $\mathbf{d} = \mathbf{n} = 0$, and $\mathbf{r}$ is a constant signal, then for every switching signal σ, the signals $\mathbf{e_T}$, u_C, u, and y converge to constant values and, in particular, $y \to \mathbf{r}$ and $u \to 0$.

Property 4 (Robustness). In case the process transfer matrix H_P is not accurately known, but it is known to belong to a class of admissible processes $\mathcal{C}_P$ centered around a known nominal transfer matrix N_P, Property 2 still holds, for "small" $\mathcal{C}_P$.

Remark 1. In this paper we assume that both the process and the controllers have strictly proper transfer matrices, but all the results can be extended to the more general case of just proper transfer matrices.

3 Switching Between Stable Systems

Property 1 motivates the following approach to build Σ_C: for every $p \in \mathcal{P}$ compute some n_K dimensional stabilizable and detectable realization $(\bar{A}_p, \bar{B}_p, \bar{C}_p)$ of K_p and let Σ_C be the following dynamical system

$$\begin{aligned} \dot{x}_C &= \bar{A}_\sigma\, x_C + \bar{B}_\sigma\, \mathbf{e_T} \\ u_C &= \bar{C}_\sigma\, x_C \end{aligned}$$

In this case, defining $x \triangleq \begin{bmatrix} x_P' & x_C' \end{bmatrix}'$ he have

$$\dot{x} = A_\sigma\, x + B_\sigma \begin{bmatrix} \mathbf{d} & \mathbf{r} - \mathbf{n} \end{bmatrix} \tag{2}$$

where

$$A_p \triangleq \begin{bmatrix} A_P & B_P\, \bar{C}_p \\ -\bar{B}_p\, C_P & \bar{A}_p \end{bmatrix}, \quad B_p \triangleq \begin{bmatrix} B_P & 0 \\ 0 & \bar{B}_p \end{bmatrix}, \qquad p \in \mathcal{P}$$

Property 1 holds by construction and A_σ is a stability matrix for every fixed $t \in \mathbb{R}^+$. However, this is not enough for Property 2 to hold. To understand why this is so consider a family of stable realizations $\{(A_p, B_p, C_p) : p \in \mathcal{P}\}$ and the following dynamical system with input u and output y

$$\begin{aligned} \dot{x} &= A_\sigma\, x + B_\sigma\, u \\ y &= C_\sigma\, x \end{aligned} \tag{3}$$

Although A_σ is a stability matrix for every fixed $t \in \mathbb{R}^+$, x may become unbounded for some combinations of initial states, input signals and switching signals. In the following sections we explore two alternative ways to guarantee that the system (3) is stable for every switching signal σ. Before proceeding we make a few technical assumptions about the above family of stable realizations.

Definition 1. An indexed family of LTI state space realizations $\{(A_p, B_p, C_p) : p \in \mathcal{P}\}$ is *uniformly stable in the parameter* if the following conditions hold:

(i) There exists a positive constant λ such that $A_p + \lambda I$ is a stability matrix for every $p \in \mathcal{P}$.

(ii) There exists a positive constant c such that $\|C_p\| \le c$ for all $p \in \mathcal{P}$.

(iii) There exist positive constants g and ϵ such that

$$\|e^{(A_p+\lambda I)t} B_p\|_2 \le g, \quad \text{and} \quad \|C_p(sI - A_p - \lambda I)^{-1} B_p\|_\infty \le 1/\epsilon, \qquad \forall p \in \mathcal{P}$$

If $\mathcal{P}$ is a finite set, demanding a family of LTI asymptotically stable realizations to be uniformly stable in the parameter introduces no loss of generality. Indeed, in this case one can take

$$\lambda \triangleq \min_{p\in\mathcal{P}} \{\lambda : A_p + \lambda I \text{ is asymptotically stable}\}, \quad c \triangleq \max_{p\in\mathcal{P}} \|C_p\|,$$
$$g \triangleq \max_{p\in\mathcal{P}} \|e^{(A_p+\lambda I)t} B_p\|_2, \quad \text{and} \quad \epsilon \triangleq \min_{p\in\mathcal{P}} \left(\|C_p(sI - A_p)^{-1} B_p\|_\infty\right)^{-1}$$

When $\mathcal{P}$ is an infinite set, further assumptions must be made in order to guarantee uniform stability in the parameter. For example, one may require $\mathcal{P}$ to be compact and appropriate continuity conditions to hold.

3.1 State Reseting

One possibility to guarantee that the state of (3) is bounded for every initialization of $x(0)$, bounded input signal u, and switching signal σ is to reset x to zero whenever there is a discontinuity in σ. The following lemma is an expanded version of this simple observation.

Lemma 1. *Given a family of state space realizations $\{(A_p, B_p, C_p) : p \in \mathcal{P}\}$ which is uniformly stable in the parameter, consider the dynamical system whose state x is reset to zero at every time instant for which σ is discontinuous and such that between discontinuities of σ equation* (3) *holds. For every switching signal σ, and piecewise continuous input u, the following inequalities hold:*

(i) For every $t \in \mathbb{R}^+$

$$\|x(t)\|^2 \le g^2 \int_0^t \|e^{-\lambda(t-\tau)}\, u(\tau)\|^2\, d\tau \tag{4}$$

$$\|y(t)\|^2 \le c^2 g^2 \int_0^t \|e^{-\lambda(t-\tau)}\, u(\tau)\|^2\, d\tau \tag{5}$$

$$\int_0^t \|e^{\lambda\tau}\, y(\tau)\|^2\, d\tau \le \frac{1}{\epsilon^2} \int_0^t \|e^{\lambda\tau}\, u(\tau)\|^2\, d\tau \tag{6}$$

(ii) If there exist nonnegative constants c_u and λ_u such that $\|u(t)\| \le c_u e^{-\lambda_u t}$ for every $t \in \mathbb{R}^+$, then

$$\|x(t)\| \le \frac{g\, c_u\, e^{-\lambda_m t}}{\sqrt{2(\lambda - \lambda_m)}}, \quad \textit{and} \quad \|y(t)\| \le \frac{c\, g\, c_u\, e^{-\lambda_m t}}{\sqrt{2(\lambda - \lambda_m)}}, \qquad \forall t \in \mathbb{R}^+ \tag{7}$$

where λ_m is any positive constant strictly smaller than both λ and λ_u.

The proof of this lemma can be found in the appendix.

3.2 Common Lyapunov Function

If one is only interested in the input/output maps that correspond to the given realizations $\{(A_p, B_p, C_p) : p \in \mathcal{P}\}$, an alternative way to guarantee stability of (3) for arbitrary switching signals is to apply similarity transformations to the given realizations in such a way that there exists a Lyapunov function for the autonomous part of (3) which is independent of the switching signal. The following lemma guarantees the existence of similarity transformations that make $V(x) \stackrel{\triangle}{=} \|x\|^2$ a Lyapunov function for the autonomous part of (3) and uses it to give bounds on the state and output in the general case when u is nonzero.

Lemma 2. *Given a family of state space realizations $\{(\tilde{A}_p, \tilde{B}_p, \tilde{C}_p) : p \in \mathcal{P}\}$ which is uniformly stable in the parameter, there exists a family of nonsingular real matrices $\{T_p : p \in \mathcal{P}\}$ such that defining*

$$A_p = T_p^{-1} \tilde{A}_p T_p\,, \quad B_p = T_p^{-1} \tilde{B}_p, \quad \textit{and} \quad C_p = \tilde{C}_p T_p, \qquad p \in \mathcal{P}$$

for every switching signal σ and piecewise continuous input u, the system defined by (3) with zero initial state is such that

(i) For every $t \in \mathbb{R}^+$

$$\|x(t)\|^2 \leq \frac{1}{\epsilon^2} \int_0^t \|e^{-\lambda(t-\tau)}\, u(\tau)\|^2\, d\tau \tag{8}$$

$$\|y(t)\|^2 \leq \frac{c^2}{\epsilon^2} \int_0^t \|e^{-\lambda(t-\tau)}\, u(\tau)\|^2\, d\tau \tag{9}$$

$$\int_0^t \|e^{\lambda\tau}\, y(\tau)\|^2\, d\tau \leq \frac{1}{\epsilon^2} \int_0^t \|e^{\lambda\tau}\, u(\tau)\|^2\, d\tau \tag{10}$$

(ii) If there exist nonnegative constants c_u and λ_u such that $\|u(t)\| \leq c_u e^{-\lambda_u t}$ for every $t \in \mathbb{R}^+$, then

$$\|x(t)\| \leq \frac{c_u\, e^{-\lambda_m t}}{\epsilon\sqrt{2(\lambda-\lambda_m)}}, \quad \textit{and} \quad \|y(t)\| \leq \frac{c\, c_u\, e^{-\lambda_m t}}{\epsilon\sqrt{2(\lambda-\lambda_m)}}, \qquad \forall t \in \mathbb{R}^+ \tag{11}$$

where λ_m is any positive constant strictly smaller than both λ and λ_u.

The proof of this lemma can be found in the appendix. It is important to emphasize that this proof is constructive and it gives an efficient algorithm to compute the coordinate transformation matrices $\{T_p : p \in \mathcal{P}\}$. In fact, computing this matrices amounts to solving a quadratic matrix inequality that can be easily converted into a LMI [2].

If one could either reset to zero the state of (2) or apply a different similarity transformation at every switching time, the problem addressed in this paper would be solved. However, this is not the case: it is possible to apply a different coordinate transformation to the state of the controller x_C at every switching time or even reset it to zero instantaneously but one cannot do the same to the state of the process x_P. Nevertheless, it will be seen that one can always choose realizations for the multi-controller that make Properties 2, 3 and 4 hold.

4 Multi-controller

Throughout this and the next section it is assumed that the process transfer matrix H_P is equal to a known nominal transfer matrix function N_P. In order to build the multi-controller Σ_C one must determine matrices A_E, B_E, C_E, D_E, F_E, and $\{(\tilde{A}_p, \tilde{B}_p, \tilde{C}_p) : p \in \mathcal{P}\}$ such that for some $\lambda > 0$ and $p_0 \in \mathcal{P}$

(i) $A_E + \lambda I$ is a stability matrix;

(ii) $(A_E + D_E\, C_E, B_E, C_E)$ and $(A_E - B_E\, F_E, D_E, F_E)$ are stabilizable and detectable realizations of N_P and K_{p_0}, respectively, with stability margin[1] λ, in particular

$$\mathrm{N}_P(s) = C_E(sI - A_E - D_E\, C_E)^{-1} B_E$$
$$\mathrm{K}_{p_0}(s) = F_E(sI - A_E + B_E\, F_E)^{-1} D_E\,;$$

(iii) $\tilde{A}_p + \lambda I$ is a stability matrix and, defining $\mathrm{S}_p(s) \triangleq \tilde{C}_p(sI - \tilde{A}_p)^{-1}\tilde{B}_p$,

$$\mathrm{K}_p = (\mathrm{X}_C - \mathrm{S}_p\, \mathrm{Y}_P)^{-1}(\mathrm{Y}_C + \mathrm{S}_p\, \mathrm{X}_P)\,, \quad p \in \mathcal{P} \tag{12}$$

where

$$\begin{bmatrix} \mathrm{X}_C & -\mathrm{Y}_C \\ \mathrm{Y}_P & \mathrm{X}_P \end{bmatrix} \triangleq \begin{bmatrix} F_E \\ C_E \end{bmatrix} (sI - A_E)^{-1} \begin{bmatrix} B_E & -D_E \end{bmatrix} + I \tag{13}$$

Lemma 4 in the appendix {a reformulation of results that can be found in [3], [4], and [5]} guarantees that matrices A_E, B_E, C_E, D_E, and F_E satisfying Properties (i) and (ii) always exist. Lemma 5 also in the appendix, guarantees that the set of stable transfer matrices $\mathcal{S} = \{\mathrm{S}_p \in \mathrm{RH}_\infty^\lambda : p \in \mathcal{P}\}$ such that (12) holds always exists.

The multi-controller Σ_C is a dynamical system such that between discontinuities of σ,

$$\begin{aligned} \dot{x}_C &= \bar{A}_\sigma\, x_C + \bar{B}_\sigma\, \mathbf{e_T} \\ u_C &= \bar{C}_\sigma\, x_C \end{aligned} \tag{14}$$

where

$$\bar{A}_p \triangleq \begin{bmatrix} A_E - B_E\, F_E & B_E\, \tilde{C}_p \\ \tilde{B}_p\, C_E & \tilde{A}_p \end{bmatrix}, \qquad \bar{B}_p \triangleq \begin{bmatrix} -D_E \\ \tilde{B}_p \end{bmatrix}, \qquad \bar{C}_p \triangleq \begin{bmatrix} F_E & \tilde{C}_p \end{bmatrix}, \qquad p \in \mathcal{P} \tag{15}$$

The state x_C is partitioned into two components x_E and x_V according to the partition of the dynamic matrix $\bar{A}_p$ shown in equation (15). At switching times, x_E is maintained constant and x_V is reset to zero. Σ_C can be regarded as the interconnection of two subsystems (cf. Figure 3): a subsystem Σ_E defined by

$$\begin{aligned} \dot{x}_E &= (A_E - B_E\, F_E)\, x_E + B_E\, v_u - D_E\, \mathbf{e_T} \\ u_C &= -F_E\, x_E + v_u \\ e_y &= C_E\, x_E + \mathbf{e_T} \end{aligned} \tag{16}$$

and a subsystem Σ_V whose state x_V is reset to zero at switching times and such that between switching times

$$\begin{aligned} \dot{x}_V &= \tilde{A}_\sigma\, x_V + \tilde{B}_\sigma\, e_y \\ v_u &= \tilde{C}_\sigma\, x_V \end{aligned} \tag{17}$$

In the definition given above for the multi-controller, state reseting is used to guarantee that Σ_V is "well-behaved" in the sense of Lemma 1. Another option would be to choose the realizations $\{(\tilde{A}_p, \tilde{B}_p, \tilde{C}_p) : p \in \mathcal{P}\}$ for the transfer matrices in $\mathcal{S}$ in such a way that there exists a Lyapunov function for the autonomous part of (17) which is independent of the switching signal. Lemma 2 guarantees that such realizations always exist. In the sequel we work with the definition for Σ_V in which state reseting is done, but this could be avoided.

Remark 2. Denoting by n_P the McMillan degree of N_P and by n_K an upper bound on the McMillan degree of the transfer matrices in $\mathcal{K}$, one can show that the order of the state of Σ_C is no larger than $2\, n_P + 3\, n_K$ *no matter what the number of controllers in* $\mathcal{K}$ *is.* For the particular case in which K_{p_0} has the structure of an observer with state feedback, i.e. $\mathrm{N}_P(s) = C(sI - A)^{-1}B$ and $\mathrm{K}(s) = F(sI - A - KC + BF)^{-1}K$, one can show that the dimension of the state of Σ_C can be reduced to $2\, n_P + n_K$. The dimension of Σ_C can often be further reduced by applying a model reduction algorithm to the family of realizations $\{(\tilde{A}_p, \tilde{B}_p, \tilde{C}_p) : p \in \mathcal{P}\}$.

[1] A realization (A, B, C) is said to be stabilizable (detectable) with stability margin λ if and only if $(A + \lambda I, B, C)$ is stabilizable (detectable).

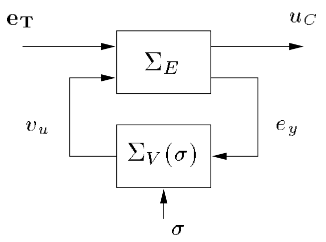

Figure 3: Proposed structure for Σ_C.

5 Main Result

Theorem 1. *If the family of state space realizations* $\{(\tilde{A}_p, \tilde{B}_p, \tilde{C}_p) : p \in \mathcal{P}\}$ *is uniformly stable in the parameter then Properties 1 and 2 hold.*

Proof of Theorem 1. Equation (16) can be rewritten as

$$\begin{aligned}\dot{x}_E &= A_E\, x_E + B_E\, u_C - D_E\, \mathbf{e_T}\\ v_u &= F_E\, x_E + u_C\\ e_y &= C_E\, x_E + \mathbf{e_T}\end{aligned}$$

and therefore

$$\begin{bmatrix} v_u \\ e_y \end{bmatrix} = \begin{bmatrix} \mathrm{X}_C & -\mathrm{Y}_C \\ \mathrm{Y}_P & \mathrm{X}_P \end{bmatrix} \begin{bmatrix} u_C \\ \mathbf{e_T} \end{bmatrix} + \epsilon \tag{18}$$

where the signal ϵ steams from nonzero initializations of x_E and therefore decays to zero no slower than $e^{-\lambda t}$ (this term is omitted in the sequel). For constant $\sigma(t) = p \in \mathcal{P}$, Σ_V is a LTI system with transfer matrix equal to S_p and therefore

$$v_u = \mathrm{S}_p\, e_y \tag{19}$$

Combining equations (18) and (19) yields

$$\mathrm{S}_p\, e_y = \mathrm{X}_C\, u_C - \mathrm{Y}_C\, \mathbf{e_T}, \qquad \text{and} \qquad e_y = \mathrm{Y}_P\, u_C + \mathrm{X}_P\, \mathbf{e_T}$$

which implies that

$$u_C = (\mathrm{X}_C - \mathrm{S}_p \mathrm{Y}_P)^{-1} (\mathrm{Y}_C + \mathrm{S}_p \mathrm{X}_P)\, \mathbf{e_T}$$

By the way the matrices S_p were selected {cf. equation (12)} one concludes that Property 1 holds. As for Property 2, using equation (18) and the fact that $\mathbf{e_T} = \mathbf{r} - y - \mathbf{n}$, $u_C = u - \mathbf{d}$, and $y = \mathrm{H}_P\, u$, one concludes that

$$e_y = (\mathrm{Y}_P - \mathrm{X}_P\, \mathrm{H}_P)\, u + \mathrm{X}_P\, (\mathbf{r} - \mathbf{n}) - \mathrm{Y}_P\, \mathbf{d}.$$

But $\mathrm{H}_P = \mathrm{N}_P = \mathrm{X}_P^{-1}\, \mathrm{Y}_P$ (cf. Lemma 5) and therefore $\mathrm{Y}_P - \mathrm{X}_P\, \mathrm{H}_P = 0$ which means that

$$e_y = \mathrm{X}_P\, (\mathbf{r} - \mathbf{n}) - \mathrm{Y}_P\, \mathbf{d}.$$

Since $\mathbf{r}$, $\mathbf{n}$, and $\mathbf{d}$ are bounded, e_y is also bounded and, by virtue of Lemma 1, so are x_V and v_u. Since the inverse of $\left[\begin{smallmatrix} \mathrm{X}_C & -\mathrm{Y}_C \\ \mathrm{Y}_P & \mathrm{X}_P \end{smallmatrix}\right]$ belongs to $\mathrm{RH}_\infty^\lambda$ (cf. Lemma 5), one concludes from equation (18) that u_C and $\mathbf{e_T}$ are also bounded.

For the particular case $\mathbf{r} = \mathbf{n} = \mathbf{d} = 0$, e_y converges to zero no slower that $e^{-\lambda t}$, and therefore, by virtue of Lemma 1, x_V and v_u also converge to zero no slower $e^{-\lambda_m t}$, where λ_m is any positive constant smaller that λ. This is enough to conclude that all the signals converge to zero no slower than $e^{-\lambda_m t}$. ∎

Remark 3. Realizing the process as

$$\dot{x}_P = (A_E + D_E\, C_E)\, x_P + B_E\, u$$
$$y = C_E\, x_P$$

and defining $x \triangleq \begin{bmatrix} x_P' & x_V' & x_E' - x_P' \end{bmatrix}'$, between switchings the closed loop system is given by

$$\dot{x} = A_\sigma\, x + B_\sigma \begin{bmatrix} \mathbf{d} & \mathbf{r} - \mathbf{n} \end{bmatrix} \tag{20}$$

where

$$A_p \triangleq \begin{bmatrix} A_E + D_E\, C_E - B_E\, F_E & B_E\, \tilde{C}_p & -B_E\, F_E \\ 0 & \tilde{A}_p & \tilde{B}_p\, C_E \\ 0 & 0 & A_E \end{bmatrix}, \quad B_p \triangleq \begin{bmatrix} B_E & 0 \\ 0 & \tilde{B}_p \\ -B_E & -D_E \end{bmatrix}, \qquad p \in \mathcal{P}$$

Since K_{p_0} stabilizes N_P, $A_E + D_E\, C_E - B_E\, F_E$ is a stability matrix and therefore there exist symmetric positive definite matrices P_{11}, P_{33}, and $\{\tilde{P}_p : p \in \mathcal{P}\}$ such that defining

$$V(t,x) \triangleq x' \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & \tilde{P}_{\sigma(t)} & 0 \\ 0 & 0 & P_{33} \end{bmatrix} x$$

$\dot{V}(t, x(t)) < 0$ between switchings and, at switching times, $V(t, x(t))$ is discontinuous but nonincreasing. In case the reseting to zero of x_V is avoided by using the similarity transformations given by Lemma 2 (cf. Section 4), one can take $\tilde{P}_p = I$ $(p \in \mathcal{P})$ and $V(t,x)$ is a time-invariant Lyapunov function for the homogeneous part of (20).

Set-point Regulation

Corollary 1. *If $\mathbf{d} = \mathbf{n} = 0$, $\mathbf{r}$ is a constant signal, and the transfer matrix X_P defined by (13) has zero steady state gain, i.e. $\mathrm{X}_P(0) = 0$, then for any switching signal σ, all the remaining signals converge to constant values and in particular $y \to \mathbf{r}$ and $u \to 0$.*

Remark 4. In case K is a SISO system and has a pole at the origin, one can show that $\mathrm{X}_P(0) = 0$ (cf. Lemma 5) and therefore Property 3 holds.

Proof of Corollary 1. The proof of this corollary is just a small modification of the argument used in the proof of Theorem 1. Under the hypothesis of this theorem $\mathrm{X}_P\,(\mathbf{n} - \mathbf{r}) - \mathrm{Y}_P\,\mathbf{d} = 0$, therefore e_y converges to zero exponentially fast and, by virtue of Lemma 1, so does v_u. Once again, using the fact that the inverse of $\left[\begin{smallmatrix} \mathrm{X}_C & -\mathrm{Y}_C \\ \mathrm{Y}_P & \mathrm{X}_P \end{smallmatrix}\right]$ belongs to $\mathrm{RH}_\infty^\lambda$ (cf. Lemma 5), one concludes from equation (18) that u_C and $\mathbf{e_T}$ also converge to zero exponentially fast. ■

6 Robustness

In this section we discuss the effect of modeling errors in the properties of the feedback system. In order to do this it is assumed that the process transfer matrix H_P does not exactly match the nominal transfer matrix N_P used to build Σ_C. In this setup we require the set of controllers $\mathcal{K} = \{\mathrm{K}_p : p \in \mathcal{P}\}$ to be such that for any $p \in \mathcal{P}$, K_p stabilizes both H_P and N_P. Since H_P is not accurately known this requirement must be guaranteed by a robust stability argument.

Theorem 2. *If the family of state space realizations $\left\{(\tilde{A}_p, \tilde{B}_p, \tilde{C}_p) : p \in \mathcal{P}\right\}$ is uniformly stable in the parameter then, for any process transfer matrix H_P in*

$$\mathcal{C}_P = \left\{ (\mathrm{X}_P + \delta\, \mathrm{Y}_C)^{-1} (\mathrm{Y}_P - \delta\, \mathrm{X}_C) : \delta \in \mathrm{RH}_\infty^\lambda,\ \|\delta(s - \lambda)\|_\infty < \epsilon \right\},$$

Properties 1 and 2 hold.

Remark 5. One advantage of the parameterization for $\mathcal{C}_P$ given in Theorem 2 over the more usual multiplicative or additive parameterizations, like for example

$$\mathcal{C}_m \triangleq \left\{ \mathrm{X}_P^{-1}\mathrm{Y}_P(I+\delta) : \delta \in \mathrm{RH}_\infty^\lambda,\ \|\delta(s-\lambda)\|_\infty \leq \epsilon_m \right\}$$

is that in case the nominal transfer function $\mathrm{X}_P^{-1}\mathrm{Y}_P$ has an unstable pole then all the transfer functions in $\mathcal{C}_m$ must have a pole in the same location, whether in $\mathcal{C}_P$, every pole (and zero) can be perturbed.

In order to proof Theorem 2 one needs the following technical lemma whose proof can be found in the appendix.

Lemma 3. *Given a process transfer matrix*

$$\mathrm{H}_P = (\mathrm{X}_P + \delta\,\mathrm{Y}_C)^{-1}(\mathrm{Y}_P - \delta\,\mathrm{X}_C)\,, \quad \delta \in \mathrm{RH}_\infty^\lambda \tag{21}$$

Then

$$e_y = (\mathrm{X}_P + \delta\,\mathrm{Y}_C)\,(\mathbf{r} - \mathbf{n} - \mathrm{H}_P\,\mathbf{d}) + \delta\,v_u + \epsilon_1 \tag{22}$$

$$\begin{bmatrix} u_C \\ \mathbf{e_T} \end{bmatrix} = \mathrm{S}\begin{bmatrix} v_u \\ e_y \end{bmatrix} + \epsilon_2 \tag{23}$$

where S *is a transfer matrix that belongs to* $\mathrm{RH}_\infty^\lambda$ *and* ϵ_1, ϵ_2 *are signals that decays to zero no slower than* $e^{-\lambda t}$.

Proof of Theorem 2. The proof that Property 1 holds is the same as in Theorem 1. As for Property 2, by virtue of Lemma 3, $e_y = \delta\,v_u + w$ where $w \triangleq (\mathrm{X}_P + \delta\,\mathrm{Y}_C)\,(\mathbf{r} - \mathbf{n} - \mathrm{H}_P\,\mathbf{d}) + \epsilon_1$. Defining $\epsilon_P \triangleq \|\delta(s-\lambda)\|_\infty$ ($< \epsilon$) one can show [6] that

$$\left(\int_0^t \|e^{\lambda\tau}e_y(\tau)\|^2\,d\tau\right)^{1/2} \leq \epsilon_P\left(\int_0^t \|e^{\lambda\tau}v_u(\tau)\|^2\,d\tau\right)^{1/2} + \left(\int_0^t \|e^{\lambda\tau}w(\tau)\|^2\,d\tau\right)^{1/2} \qquad \forall t \in \mathbb{R}^+ \tag{24}$$

On the other hand, by virtue of Lemma 1,

$$\left(\int_0^t \|e^{\lambda\tau}v_u(\tau)\|^2\,d\tau\right)^{1/2} \leq \frac{1}{\epsilon}\left(\int_0^t \|e^{\lambda\tau}e_y(\tau)\|^2\,d\tau\right)^{1/2} \qquad \forall t \in \mathbb{R}^+ \tag{25}$$

Combining inequalities (24) and (25) yields

$$\left(\int_0^t \|e^{\lambda\tau}e_y(\tau)\|^2\,d\tau\right)^{1/2} \leq \frac{\epsilon}{\epsilon - \epsilon_P}\left(\int_0^t \|e^{\lambda\tau}w(\tau)\|^2\,d\tau\right)^{1/2} \qquad \forall t \in \mathbb{R}^+$$

and therefore

$$\bar{e}_y(t) \triangleq \left(\int_0^t \|e^{-\lambda(t-\tau)}e_y(\tau)\|^2\,d\tau\right)^{1/2} \leq \frac{\epsilon}{\epsilon - \epsilon_P}\left(\int_0^t \|e^{-\lambda(t-\tau)}w(\tau)\|^2\,d\tau\right)^{1/2} \qquad \forall t \in \mathbb{R}^+$$

Since w is bounded, $\bar{e}_y$ is also bounded and, by virtue of Lemma 1, so are x_V and v_u. Equation (23) further implies that u_C and $\mathbf{e_T}$ are also bounded.

For the particular case $\mathbf{r} = \mathbf{n} = \mathbf{d} = 0$, $\bar{e}_y$ converges to zero no slower that $e^{-\lambda t}$, and therefore x_V, v_u and the remaining signals converge to zero no slower $e^{-\lambda_m t}$, where λ_m is any positive constant smaller that λ. ■

Remark 6. Theorem 2 shows that if the constant ϵ in the definition of uniform stability in the parameter is increased, the class of admissible plants for which stability is guaranteed becomes larger. This means that small $\mathcal{H}_\infty$ norms for the transfer matrices in $\mathcal{S}$ produce more robust systems. Since the set $\mathcal{S}$ whose existence is guaranteed by Lemma 5 is not unique one should choose it in such a way that robustness is maximized. Furthermore, one can even design the original controllers in such a way that ϵ is minimized. This would be possible, for example, in the setup of [7].

7 Concluding Remarks

When using a hybrid system to control a continuous process it is often possible to design the continuous dynamics of the hybrid controller in such a way that the continuous signals (namely the input and output of the process) remain bounded *independently* of the behavior of the discrete logic. In this sense the "stability" of the continuous dynamics becomes *decoupled*. In the context of switching between several LTI controllers, all of them capable of stabilizing a single LTI process, it was shown that stability decoupling can be achieved by "carefully" choosing the realizations for the multi-controller.

Current research focus in more complex versions of this problem which arise in the presence of "large" unmodeled dynamics. In particular, situations in which some of the controllers may not stabilize some of the admissible process models. Preliminary results in this area can be found in [8].

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A Appendix

Proof of Lemma 1. Let $\{t_k : t_0 = 0,\ t_k < t_{k+1},\ k \in \mathbb{N}\}$ be the sequence of times for which σ is discontinuous, $t \in \mathbb{R}^+$ be an arbitrary time instant, and n be the largest integer such that $t_n \leq t$.

(i) As far as inequality (6) is concerned it does not matter what happens to σ after time t, therefore one can assume without loss of generality, that $t_{n+1} = t$. Consider the following set of $\mathcal{L}^2$ signals

$$u_k(\tau) = \begin{cases} u(\tau) & \text{if } \tau \in [t_k, t_{k+1}) \\ 0 & \text{otherwise} \end{cases} \qquad k \in \{0, \ldots, n\}$$

$$y_k(\tau) = \begin{cases} y(\tau) & \text{if } \tau \in [t_k, t_{k+1}) \\ 0 & \text{otherwise} \end{cases} \qquad k \in \{0, \ldots, n\}$$

In every interval $[t_k, t_{k+1})$ $(k \in \{1, \ldots, n\})$ the system defined by (3) is LTI with $\mathcal{H}_\infty$-norm shifted by λ no larger then $1/\epsilon$ and zero initial condition, therefore [6]

$$\int_{t_k}^{t_{k+1}} \|e^{\lambda\tau} y_k(\tau)\|^2 \, d\tau \leq \frac{1}{\epsilon^2} \int_{t_k}^{t_{k+1}} \|e^{\lambda\tau} u_k(\tau)\|^2 \, d\tau$$

Integrating over the whole interval $[0, t)$,

$$\int_0^t \|e^{\lambda\tau} y(\tau)\|^2 \, d\tau = \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \|e^{\lambda\tau} y_k(\tau)\|^2 \, d\tau$$

$$\leq \frac{1}{\epsilon^2} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \|e^{\lambda\tau} u_k(\tau)\|^2 \, d\tau = \frac{1}{\epsilon^2} \int_0^t \|e^{\lambda\tau} u(\tau)\|^2 \, d\tau$$

Which concludes the proof of inequality (6). As for inequalities (4) and (5), since $x(t_n) = 0$

$$x(t) = \int_{t_n}^{t} e^{A_{\sigma(t_n)}(t-\tau)} B_{\sigma(t_n)} u(\tau) \, d\tau$$

taking norms, squaring and using the Schwartz inequality yields

$$\begin{aligned} \|x(t)\|^2 &\leq \left(\int_{t_n}^{t} \|e^{(A_{\sigma(t_n)}+\lambda I)(t-\tau)} B_{\sigma(t_n)}\|^2 \, d\tau \right) \left(\int_{t_n}^{t} \|e^{-\lambda(t-\tau)} u(\tau)\|^2 \, d\tau \right) \\ &\leq g^2 \left(\int_0^t \|e^{-\lambda(t-\tau)} u(\tau)\|^2 \, d\tau \right) \end{aligned}$$

Since $\|y(t)\| \leq c\|x(t)\|$, this concludes the first part of this proof.

(ii) Inequality (4) implies that

$$\|e^{\lambda_m t} x(t)\|^2 \leq g^2 c_u^2 \int_0^t \left(e^{2\lambda_m t - 2\lambda(t-\tau) - 2\lambda_u \tau} \right) \, d\tau$$

If $\lambda = \lambda_u$, this implies that

$$\|e^{\lambda_m t} x(t)\|^2 \leq g^2 c_u^2 \, t \, e^{-2(\lambda-\lambda_m)t} \leq \frac{g^2 c_u^2}{2e(\lambda - \lambda_m)}$$

If $\lambda > \lambda_u$ ($> \lambda_m$) then[2]

$$\|e^{\lambda_m t} x(t)\|^2 \leq \frac{g^2 c_u^2}{2(\lambda - \lambda_u)} \left(e^{-2(\lambda_u - \lambda_m)t} - e^{-2(\lambda-\lambda_m)t} \right) \leq \frac{g^2 c_u^2}{2(\lambda - \lambda_m)}$$

and finally if $\lambda_u > \lambda$ ($> \lambda_m$) then

$$\|e^{\lambda_m t} x(t)\|^2 \leq \frac{g^2 c_u^2}{2(\lambda_u - \lambda)} \left(e^{-2(\lambda - \lambda_m)t} - e^{-2(\lambda_u-\lambda_m)t} \right) \leq \frac{g^2 c_u^2}{2(\lambda_u - \lambda_m)}$$

In any case

$$\|e^{\lambda_m t} x(t)\|^2 \leq \frac{g^2 c_u^2}{2(\lambda - \lambda_m)}$$

which guarantees that inequalities (7) hold. ∎

Notation Given a symmetric positive definite matrix Q, let $Q^{1/2}$ denote any square matrix (not necessarily symmetric) such that $Q = Q^{1/2'} Q^{1/2}$. The symmetry and positive definiteness of Q guarantees that at least one such matrix always exists. Since $Q^{1/2}$ must necessarily be a nonsingular matrix it has an inverse which will be denoted by $Q^{-1/2}$.

[2] Given two positive constants a and b such that $b \geq a$, $e^{-ax} - e^{-bx} \leq (b-a)/b$ for any $x \geq 0$. Indeed, the maximum of $e^{-ax} - e^{-bx}$ is attained for a point x^* such that

$$-ae^{-ax^*} + be^{-bx^*} = 0 \quad \Leftrightarrow \quad e^{-(b-a)x^*} = a/b$$

But then $e^{-ax^*} - e^{-bx^*} = e^{-ax^*}(b-a)/b \leq (b-a)/b$.

Proof of Lemma 2. By virtue of the bounded real lemma, the uniform stability in the parameter of $\{(\tilde{A}_p, \tilde{B}_p, \tilde{C}_p) : p \in \mathcal{P}\}$ guarantees that there exists a family of symmetric positive definite matrices $\{P_p : p \in \mathcal{P}\}$ such that for all $p \in \mathcal{P}$

$$\left(\tilde{A}_p + \lambda I\right)' P_p + P_p \left(\tilde{A}_p + \lambda I\right) + \tilde{C}_p' \tilde{C}_p + \epsilon^2 P_p \tilde{B}_p \tilde{B}_p' P_p \leq 0$$

Defining $T_p \triangleq P_p^{-1/2}$, the above matrix inequality becomes

$$P_p^{1/2'} \left(A_p' + A_p + 2\lambda I + C_p' C_p + \epsilon^2 B_p B_p'\right) P_p^{1/2} \leq 0$$

and since $P_p^{1/2}$ is nonsingular for every $p \in \mathcal{P}$, one concludes that

$$A_p' + A_p + 2\lambda I + C_p' C_p + \epsilon^2 B_p B_p' \leq 0 \quad \forall p \in \mathcal{P} \tag{26}$$

We will see next that all the inequalities than need to be proven follow from this set of matrix inequalities.

(i) Let $V(\tau, x) \triangleq \|e^{\lambda\tau} x\|^2$ $(\tau \in \mathbb{R}^+)$, along any trajectory of (3)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau} V(\tau, x(\tau)) &= e^{2\lambda\tau} x' \left(A_\sigma' + A_\sigma + 2\lambda I\right) x + e^{2\lambda\tau} (x' B_\sigma u + u' B_\sigma' x) \\
&\leq -e^{2\lambda\tau} x' \left(C_\sigma' C_\sigma + \epsilon^2 B_\sigma B_\sigma'\right) x + e^{2\lambda\tau} (x' B_\sigma u + u' B_\sigma' x) \\
&= -e^{2\lambda\tau} \left(y' y - \epsilon^{-2} u' u\right) - \epsilon^2 e^{2\lambda\tau} \left(x' B_\sigma - \epsilon^{-2} u'\right) \left(B_\sigma' x - \epsilon^{-2} u\right) \\
&\leq -\|e^{\lambda\tau} y\|^2 + \epsilon^{-2} \|e^{\lambda\tau} u\|^2
\end{aligned}$$

Integrating over $[0, t)$ one gets

$$\|e^{\lambda t} x(t)\|^2 + \int_0^t \|e^{\lambda\tau} y(\tau)\|^2 \, d\tau \leq \frac{1}{\epsilon^2} \int_0^t \|e^{\lambda\tau} u(\tau)\|^2 \, d\tau \tag{27}$$

Inequalities (10), (8) and (9) are directly implied by (27).

(ii) Similar to the second part of the proof of Lemma 1 with g replaced by $1/\epsilon$. ■

Remark 7. The matrix inequality (26) guarantees that $A_p' + A_p < 0$ and therefore $V(x) \triangleq x'x$ is a Lyapunov function for the autonomous part of (3).

Lemma 4. *Given two transfer matrices* H *and* K *such that* K *stabilizes* H *with stability margin* λ*, there exist matrices* A_E*,* B_E*,* C_E*,* D_E*, and* F_E*, such that*

1. $A_E + \lambda I$ *is a stability matrix;*
2. $(A_E + D_E\, C_E, B_E, C_E)$ *and* $(A_E - B_E\, F_E, D_E, F_E)$ *are stabilizable and detectable realizations of* H *and* K*, respectively, with stability margin* λ*, in particular*

$$\begin{aligned}
\mathrm{H}(s) &= C_E (sI - A_E - D_E\, C_E)^{-1} B_E \\
\mathrm{K}(s) &= F_E (sI - A_E + B_E\, F_E)^{-1} D_E
\end{aligned}$$

Proof of Lemma 4. Let $(A_\mathrm{H}, B_\mathrm{H}, C_\mathrm{H})$ and $(A_\mathrm{K}, B_\mathrm{K}, C_\mathrm{K})$ be minimal realizations of H and K respectively, and K_H and K_K matrices such that $A_\mathrm{H} + K_\mathrm{H} C_\mathrm{H} + \lambda I$ and $A_\mathrm{K} + K_\mathrm{K} C_\mathrm{K} + \lambda I$ are asymptotically stable. Defining

$$A_E \triangleq \begin{bmatrix} A_\mathrm{K} + K_\mathrm{K} C_\mathrm{K} & 0 \\ 0 & A_\mathrm{H} + K_\mathrm{H} C_\mathrm{H} \end{bmatrix} \qquad B_E \triangleq \begin{bmatrix} K_\mathrm{K} \\ B_\mathrm{H} \end{bmatrix}$$

$$D_E \triangleq \begin{bmatrix} B_\mathrm{K} \\ -K_\mathrm{H} \end{bmatrix} \qquad C_E \triangleq \begin{bmatrix} 0 & C_\mathrm{H} \end{bmatrix} \qquad F_E \triangleq \begin{bmatrix} C_\mathrm{K} & 0 \end{bmatrix}$$

simple algebra shows the two properties stated in the lemma hold. □

Remark 8. For the particular case in which K has the structure of an observer with state feedback, i.e. $\mathrm{H}(s) = C(sI - A)^{-1}B$ and $\mathrm{K}(s) = F(sI - A - KC + BF)^{-1}K$ one can pick $A_E = A + KC$, $B_E = B$, $C_E = C$, $D_E = -K$, and $F_E = F$.

Lemma 5. *Given two transfer matrices* H *and* K *such that* K *stabilizes* H *with stability margin* λ *and matrices* A_E, B_E, C_E, D_E, *and* F_E, *such that*

1. $A_E + \lambda I$ *is a stability matrix;*
2. $(A_E + D_E\, C_E, B_E, C_E)$ *and* $(A_E - B_E\, F_E, D_E, F_E)$ *are stabilizable and detectable realizations of* H *and* K, *respectively, with stability margin* λ, *in particular*

$$
\begin{aligned}
\mathrm{H}(s) &= C_E(sI - A_E - D_E\, C_E)^{-1} B_E \\
\mathrm{K}(s) &= F_E(sI - A_E + B_E\, F_E)^{-1} D_E
\end{aligned}
$$

Defining

$$
\begin{bmatrix} \mathrm{X}_C & -\mathrm{Y}_C \\ \mathrm{Y}_P & \mathrm{X}_P \end{bmatrix} \triangleq \begin{bmatrix} F_E \\ C_E \end{bmatrix} (sI - A_E)^{-1} \begin{bmatrix} B_E & -D_E \end{bmatrix} + I
$$

1. $\mathrm{X}_P, \mathrm{Y}_P, \mathrm{Y}_C, \mathrm{X}_C \in \mathrm{RH}_\infty^\lambda$ *are a simultaneous right coprime factorization of* H *and* K, *i.e. (i)* X_P *and* X_C *have causal inverse, (ii)* $\left[\begin{smallmatrix} \mathrm{X}_C & -\mathrm{Y}_C \\ \mathrm{Y}_P & \mathrm{X}_P \end{smallmatrix}\right]$ *is a unit in* $\mathrm{RH}_\infty^\lambda$, *and (iii)* $\mathrm{H} = \mathrm{X}_P^{-1}\mathrm{Y}_P$ *and* $\mathrm{K} = \mathrm{X}_C^{-1}\mathrm{Y}_C$.
2. *Given any transfer matrix* K_1 *such that* K_1 *stabilizes* H *with stability margin* λ, *there exists a transfer matrix* $\mathrm{S} \in \mathrm{RH}_\infty^\lambda$ *such that*

$$
\mathrm{K}_1 = (\mathrm{X}_C - \mathrm{S}\,\mathrm{Y}_P)^{-1}(\mathrm{Y}_C + \mathrm{S}\,\mathrm{X}_P) \tag{28}
$$

Proof of Lemma 5. Property 1. can be proved by direct computation. As for Property 2., pick

$$
\mathrm{S} = (-\mathrm{Y}_C + \mathrm{X}_C\mathrm{K}_1)(\mathrm{X}_P + \mathrm{Y}_P\mathrm{K}_1)^{-1}
$$

The stability of $(\mathrm{X}_P + \mathrm{Y}_P\mathrm{K}_1)^{-1}$ and $\mathrm{K}_1(\mathrm{X}_P + \mathrm{Y}_P\mathrm{K}_1)^{-1}$ is guaranteed by the fact that K_1 stabilizes $\mathrm{H} = \mathrm{X}_P^{-1}\mathrm{Y}_P$. □

Proof of Lemma 3. Equation (16) can be rewritten as

$$
\begin{aligned}
\dot{x}_E &= A_E\, x_E + B_E\, u_C - D_E\, \mathbf{e_T} \\
v_u &= F_E\, x_E + u_C \\
e_y &= C_E\, x_E + \mathbf{e_T}
\end{aligned}
$$

and therefore

$$
\begin{bmatrix} v_u \\ e_y \end{bmatrix} = \begin{bmatrix} \mathrm{X}_C & -\mathrm{Y}_C \\ \mathrm{Y}_P & \mathrm{X}_P \end{bmatrix} \begin{bmatrix} u_C \\ \mathbf{e_T} \end{bmatrix} + \epsilon \tag{29}
$$

where the signal ϵ steams from nonzero initializations of x_E and therefore decays to zero no slower than $e^{-\lambda t}$ (this term is omitted in the sequel). Using the fact that $\mathbf{e_T} = \mathbf{r} - y - \mathbf{n}$, $u_C = u - \mathbf{d}$, and $y = \mathrm{H}_P\, u$, one further concludes that

$$
\begin{aligned}
v_u &= (\mathrm{X}_C + \mathrm{Y}_C\, \mathrm{H}_P)\, u - \mathrm{Y}_C\, (\mathbf{r} - \mathbf{n}) - \mathrm{X}_C\, \mathbf{d} \\
e_y &= (\mathrm{Y}_P - \mathrm{X}_P\, \mathrm{H}_P)\, u + \mathrm{X}_P\, (\mathbf{r} - \mathbf{n}) - \mathrm{Y}_P\, \mathbf{d}
\end{aligned}
$$

and therefore

$$u = (\mathrm{X}_C + \mathrm{Y}_C\,\mathrm{H}_P)^{-1}(\mathrm{Y}_C\,(\mathbf{r}-\mathbf{n}) + \mathrm{X}_C\,\mathbf{d} + v_u)$$

On the other hand (21) is equivalent to

$$\delta = (\mathrm{Y}_P - \mathrm{X}_P\,\mathrm{H}_P)(\mathrm{X}_C + \mathrm{Y}_C\,\mathrm{H}_P)^{-1}$$

and therefore

$$e_y = (\mathrm{X}_P + \delta\,\mathrm{Y}_C)\,(\mathbf{r}-\mathbf{n}) - (\mathrm{Y}_P - \delta\,\mathrm{X}_C)\,\mathbf{d} + \delta\,v_u$$

Equation (23) is a direct consequence of (29) with $\mathrm{S} \triangleq \left[\begin{smallmatrix} \mathrm{X}_C & -\mathrm{Y}_C \\ \mathrm{Y}_P & \mathrm{X}_P \end{smallmatrix}\right]^{-1}$. Lemma 5 guarantees that $\mathrm{S} \in \mathrm{RH}_\infty^\lambda$. ∎

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