Stochastic and Deterministic State-Dependent Social Networks

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***Abstract*—This paper investigates a political party or an association social network where members share a common set of beliefs. In modeling it as a distributed iterative algorithm with network dynamics mimicking the interactions between people, the problem of interest becomes that of determining i) the conditions when convergence happens in finite-time and ii) the corresponding steady-state opinion. For a traditional model, it is shown that finite-time convergence requires a complete topology and that by removing neighbors with duplicate opinions reduces in half the number of links. Finite-time convergence is proved for two novel models even when nodes contact two other nodes of close opinion. In a deterministic setting, the network connectivity influences the final consensus and changes the relative weight of each node on the final value. In the case of mobile robots, a similar communication constraint is present which makes the analysis of the social network so relevant in domain of control systems as a guideline to save resources and obtain finite-time consensus. Through simulations, the main results regarding convergence are illustrated paying special attention to the rates at which consensus is achieved.**

***Index Terms*—Diffusion Processes; Control Systems; Computer Networks; Distributed Algorithms; Social Factors.**

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I. Introduction

A. Motivation

Understanding the mechanisms of a social networks means to investigate how a group of agents decides a given issue. In particular, focus is given to determine the key agents that contribute the most to driving the general opinion of the network to the final state. In another direction, importance is given to identifying the general properties of the social network that ensure convergence of opinion given a model with iterative dynamics, representing the interaction between agents along time. This paper tackles the problem of, given a state-dependent social network, showing the conditions for convergence and how they influence the settling time. In practice, understanding such factors can guide systems engineers to make options that favor a fast information dissemination, reducing the convergence time. In [1], preliminary results about convergence are given for the deterministic case. In this article, those results are extended to the stochastic case by showing the converse results in terms of expected value, when the nodes communicate in a random fashion. In addition, steady-state solutions are characterized in terms of the contribution of each agent and also in the presence of leaders.

In this paper, we deal with social networks where agents contact similar minded people in terms of their opinions on a subject, i.e., we adopt the multi-agent view of the diffusion process (see [2] for a broader discussion). It is assumed that these beliefs describe objective arguments for rational people that take them into consideration regardless of the person who sent them. A similar terminology of *rational* innovations is used in [3] where the opinion of an agent towards an innovation is *rational* if it depends only on the quality of the innovation, as opposed to *controversial* innovations. The work in [4] also points to the same characterization that social networks evolve in a rational manner. Examples range from scientific discussions, focus groups with undisclosed brand names, and others where the sentiment of the agents does not influence their opinion. The fact that nodes belong to a community is perceived as the bounded confidence model meaning that nodes with closer opinion influence each other.

An example motivation in the field of control for mobile networks is that one might want to replicate a social behavior in a distributed system by enforcing the same rules for neighbor selection. In these scenarios, it is useful to study deterministic networks where communication is not a key aspect and acknowledgment of messages can be performed. However, for other cases of interest, one might relax this assumption and consider an asynchronous model better represented by stochastic networks. A group of mobile robots agreeing on the location to rendezvous, equipped with communication devices of variable transmitting power, would have crucial features such as: saving resources, as nodes limit the number of interconnections; having finite-time convergence as opposed to asymptotic when using a consensus algorithm; working both synchronously and asynchronously; and the generated network topology is regular and robust to link failures. These illustrative scenarios motivate the current problem.

In the literature, it is often considered a deterministic and synchronized model to account when people update their opinions. Direct consequences of these assumptions are: i) all people update their opinion in a round-fashion manner and ii) the model cannot account for irregular patterns of updates. Moreover, the decision of when each person updates its opinion is better modeled by a stochastic process given the non-deterministic features of human behavior. In this paper, the adopted model allows to consider stochastic behavior in the opinion update even though each person can still follow a rule on establishing its influence connections. A major issue is that the analysis of deterministic networks, when selection of discussion partners is still based on his/her own opinion, can benefit from a cluster-based analysis or techniques from dynamical systems whereas proofs for stochastic networks rely on computing expected values and variances and all theoretical results are now in probabilistic sense.

B. Contribution

The main contributions of this paper can be summarized as:

- A social network is modeled as an iterative distributed algorithm with a state-dependent dynamics for the topology that uses a fixed parameter of connectivity;
- Finite-time convergence is shown for the base network and discarding similar opinions reduces in half the number of required links to achieve the same speed;
- Results relating the interaction dynamics with the relative weight of each node in the final opinion are provided;
- Two proposed strategies that have finite-time convergence with each node having two neighbors;
- Stochastic versions of the network dynamics are studied and converse results are obtained with convergence in mean square and almost surely. The case of a pure random selection is also visited with exact convergence rate in terms of expected value.

C. Related Work

In [5], a classical model of influence and opinion formation processes found in sociology is studied where the relative weight of each agent is based on their influence in the outcome of past discussions. The model has two equations: one governing the evolution of the current opinion and another where the power of each agent is updated at the end of each previous discussion based on the final preponderance of that agent. The analysis focuses on the convergence properties of the Friedkin-Degroot model [6] (see also [7] for the related Friedkin and Johnsen model), which models social interactions by means of a linear system, where each agent updates its opinion as a weighted average of their previous opinion and that of their neighbors. Generalizations of these models include [8] where agents communicate belief functions.

The main view in this paper concerns how people belonging to the same political party, sports association, or other organizations, are inherently contacting with agents sharing similar opinions. The work of [9], [10], [11] and [12] support the same argument. In particular, [9] studies various models of interaction to analyze when nodes converge to the same opinion or fragment into various opinion clusters that do not communicate. In [11], a model is investigated where, as in a gossip fashion, random pairs of nodes with close opinions evolve their belief to their average. Conditions for single or multi-cluster convergence are provided. Both works share a common view that the connectivity graphs depend on the state. A more recent work [13] studies the community cleavage problem as the result of stubborn leaders. A more comprehensive discussion of this topic can be found in [14].

Randomized algorithms for information aggregation have attracted attention due to its decentralization and accurate modeling of people interactions. In particular, [15] generalizes the concept for a set of agents with a state that reflects many opinions on different topics. This can be seen as a generalization of the randomized gossip algorithm proposed in [16], which encompasses other interesting particular cases such as for political voting, as mentioned in [15]. The proposed model differs from [15] in the sense that the evolution of the network is deterministic, having an environment with a set of rules and where people are rewarded for their cooperation. The present work differs from these models by assuming a different update rule and focusing on having network dynamics that mimic social interactions. Expressing the interconnections as a stochastic graph model has also been investigated in [17, 18, 19] where algorithms are given to estimate the probability distributions of such links existing in the topology.

Since social networks can be treated as first-order models, their analysis is similar to linear consensus (see, e.g., [20], [21], [22], [23], [24] and [25]). Stability analysis in both fields share similar tools [26]. In [27], the authors assume randomized directional communication in a consensus system. Some of these concepts have counterparts in the analysis of social networks. The work of [28] tackles the state-dependent consensus problem with the proofs for the some of the results being similar to those in this paper.

When addressing convergence, a meaningful characterization will describe the rate at which the process reaches the final value. For the average consensus problem, [29] analyzes the examples of complete and Cayley graphs with tools based on computing the expected value of the difference between the state and the average. These results follow a similar reasoning to what is presented in this paper for the stochastic social network (for the deterministic case, we follow another line-of-proof, as the objective is to get a finite number of steps, instead of an asymptotic convergence rate). The main difference between the approach provided in this paper and that of [29] is the focus on a different Lyapunov function, since the final consensus value is not known *a priori*.

II. Problem Statement

A social network comprises a set of n agents, interchangeably called nodes, that interact and influence the personal belief or opinion of others about a subject or a topic. In this paper, the opinion of node i is denoted by the scalar state $x_i(k), 1 \leq i \leq n$, for the discrete time domain variable k, which is incremented whenever a communication occurs. The terms *opinion* and *state* are used interchangeably given that both refer to the same concept, except that the latter is from the dynamical system's point-of-view of the model for the interaction. The objective is to study under which conditions the limit $x_\infty := \lim_{k\to\infty} x(k)$ exists, i.e., the discussion finishes, and to understand how each agent impacts the value x_∞.

The network topology modeling how each agent affects the opinion of a neighbor is given as a time-varying directed graph $G(k) = (\mathcal{V}, E(k))$, where $\mathcal{V}$ represents the set of n nodes, and $E(k) \subseteq \mathcal{V} \times \mathcal{V}$ is the set of communication links that change over time. Node i contacts j, at time k, if $(i, j) \in E(k)$.

$N_i(k)$ denotes the set of neighbors of agent i, i.e., $N_i(k) = \{j : (j, i) \in E(k)\}$.

The edge set $E(k)$ evolves according to a "nearest" policy which is motivated by agents searching for a diverse set of opinions. In real-life, when people want to make a decision, they search for positive and negative feedback within other nodes with opinions similar to the node state [9], [11], with a constraint on the amount of feedback they can read or consult. In the next section, four definitions are formalized for the neighbor sets $N_i(k)$.

In this paper, we will not consider consensus-like updating rule (see, for instance, [6], [5] for the deterministic consensus-like dynamics and [30], for the stochastic counterpart) translated as a linear function. These works allow for an approximation to the complex decision-making process of humans. Instead, the opinion is seen as translating a set of arguments in the social network. In [31], a comprehensive discussion on how a decision opinion is based on the positive arguments compensating the negative ones, is presented, which motivates one to consider the average between the worst and the best sets of arguments. Agents are objective, i.e., *rational* in the nomenclature of [3], meaning that there is no own sentiment towards a final choice. If each agent was presented with all facts composing the individual beliefs he/she would reach the same conclusion. Nevertheless, that evaluation can change over time which will be modeled by a pessimistic/optimistic parameter α_k. Combining the previous description, an agent i has the following dynamics:

$$x_i(k+1) = \alpha_k \min_{j \in N_i(k)} x_j(k) + (1 - \alpha_k) \max_{j \in N_i(k)} x_j(k), \tag{1}$$

where parameter $\alpha_k \in [0, 1]$ accounts for how objectively agents balance their opinion between their neighbors' extremes (minimum and maximum) beliefs. Note that everything is well-defined in (1) since $N_i(k) \neq \emptyset, \forall k$, because at least the node itself is in the neighbor set. We will refer to a *deterministic social network* when, at each time instant $k + 1$, all nodes update their opinion based on (1) and by *stochastic social network* when the people updating their opinions, i.e. node i in (1), are randomly selected according to some stochastic variable. The difference is that in a deterministic social network, all agents update synchronously their opinion.

Parameter α_k represents the level of optimism/pessimism of the agents. Associating a positive stance to high values of the belief, then $\alpha_k = 0$ would correspond to optimistic agents that only take into account beliefs more positive than their own, whereas $\alpha_k = 1$ would correspond to pessimistic agents. When considering a single value α_k for all the nodes, focus is being given to a specific type of decision-making. However, it is also interesting to study the case where each node might have a different value. Apart from the asymptotic convergence in the deterministic and stochastic cases, the proofs of the theorems would no longer be valid. In future work, it is of relevance to consider different values for α_k. In particular, extending the results in this paper would characterize under what circumstances there is still finite-time convergence.

In summary, we are interested in characterizing the number of required neighbors to guarantee convergence and whether it is possible to find a finite number of discrete updates (k_f) after which convergence is achieved, i.e.,

$$\exists k_f : \forall k \geq k_f, i, j \in \mathcal{V}, |x_i(k) - x_j(k)| = 0.$$

In addition, we would like to find the smallest k_f for each choice of number of neighbors η in a n-node network. On the other hand, *asymptotic convergence* is obtained if

$$\forall i, j \in \mathcal{V}, \lim_{k \to \infty} |x_i(k) - x_j(k)| = 0.$$

We are also interested in comparing different definitions for the graph dynamics to determine key features influencing the rate of convergence and final opinion shared by the nodes. We start by introducing the deterministic version of the network dynamics and then progress to analyze the stochastic setting which reflects more accurately other real-life examples where nodes are not forced to a synchronous update.

III. Neighbor Selection Rules

In order to get a simple definition, we introduce the notation for permutation $\{(i) : i \in \mathcal{I}\}$ of the indices in the index set $\mathcal{I}$ such that $x_{(i)}(k) \leq x_{(i+1)}(k)$ and $x_{(i)}(k) = x_{(i+1)}(k) \implies (i) < (i+1)$ (i.e., the permutation (i) is such that all the opinions become sorted and when two opinions are equal the sorting is resolved by the indices of the nodes). Based on this permutation of an index set, we have the following definition.

Definition 1 (order of): Take a node i and a set $\mathcal{S}$ of indices for which we have a permutation (j) as before. We define that j is the *order of* i in the set $\mathcal{S}$ if $(j) = i$.

We can now present four definitions for neighbor selection (depicted in Fig. 1) that aim at capturing different behaviors. With a slight abuse of notation, we will use $N_i(k)$ and redefine it. The reader can recognize $N_i(k)$ as the set of in-neighbors of i and, in each result, the appropriate definition is referred. The following definition uses the set $\mathcal{V}_i(k) := \{\ell : x_\ell(k) \neq x_i(k)\} \cup \{i\}$. For $\eta = 1$ this definition matches a consensus dynamics.

Definition 2 (base network): For each node $i \in \mathcal{V}$ of order j in the set $\mathcal{V}_i(k)$, we define the set of at most η neighbors with opinion smaller than that of i as $N_i^-(k)$, i.e.,

$$N_i^-(k) = \begin{cases} \{(j-\eta), (j-\eta+1), \cdots, (j)\}, & \text{if } j-\eta \geq 1 \\ \{(1), (2), \cdots, (j)\}, & \text{otherwise.} \end{cases}$$

and the set of at most η neighbors with higher opinion $N_i^+(k)$ defined as

$$N_i^+(k) = \begin{cases} \{(j), (j+1), \cdots, (j+\eta)\}, & \text{if } j+\eta \leq n \\ \{(j), (j+1), \cdots, (n)\}, & \text{otherwise.} \end{cases}$$

and the set of all neighbors as $N_i(k) := N_i^-(k) \cup N_i^+(k)$, where $\eta \in \mathbb{Z}^+$.
Notice that $0 < |N_i(k)| \leq 2\eta + 1$, thus nothing is being assumed about node degree.

The previous definition drafts a topology dynamics that may be slow due to nodes close to the minimum or the maximum having fewer links, because either $|N_i^-(k)| < \eta$ or $|N_i^+(k)| < \eta$. Even though, in realistic scenarios, extremist people may indeed have fewer interactions because of their extreme views, it is desirable to investigate how small deviation from the definition can speed up convergence.

In real-life, the next policy is observed when people disregard the opinions of some of their acquaintances because they know that two individuals share the same positive or negative points towards the subject being discussed. In a different direction, one can resort to this definition in distributed systems or virtual social networks (such as Facebook) to reduce resource allocation by removing connections to neighbors that share the same opinion. Before introducing the proposed network dynamics, it is useful to consider the set of neighbors with distinct values. In particular, we denote by $\mathcal{D}_i(k)$ the set of distinct possible neighbors of node i at time k, i.e., obtained by going through all the elements of $\mathcal{V}_i(k)$ and adding them to $\mathcal{D}_i(k)$ if there does not exist an element in $\mathcal{D}_i(k)$ already with equal state. In doing so, for all the nodes with duplicate state, there exists only one in $\mathcal{D}_i(k)$.

Definition 3 (distinct value): For each node $i \in \mathcal{V}$ of order j in the set $\mathcal{D}_i(k)$, we define the set of at most η neighbors with opinion smaller than that of node i as $N_i^-(k)$, i.e.,

$$N_i^-(k) = \begin{cases} \{(j-\eta), (j-\eta+1), \cdots, (j)\}, & \text{if } j-\eta \geq 1 \\ \{(1), (2), \cdots, (j)\}, & \text{otherwise.} \end{cases}$$

and

$$N_i^+(k) = \begin{cases} \{(j), (j+1), \cdots, (j+\eta)\}, & \text{if } j+\eta \leq n \\ \{(j), (j+1), \cdots, (n)\}, & \text{otherwise.} \end{cases}$$

and define the set of all neighbors $N_i(k) := N_i^-(k) \cup N_i^+(k)$.
The network in Definition 3 is depicted in Fig. 2b.

The previous definition did not take into account the behavior of some people that want to assure an informed decision and therefore get exactly 2η neighbors. One of their possibilities is to look for other closer nodes which motivates a second network structure (or policy), referred to as *nearest distinct neighbors*, which satisfies the following definition:

Definition 4 (distinct neighbors): For each node $i \in \mathcal{V}$ of order j in the set $\mathcal{D}_i(k)$, we define the set of at most η neighbors with opinion smaller than that of node i as $N_i^-(k)$, i.e.,

$$N_i^-(k) = \begin{cases} \{(j-\eta), \cdots, (j)\}, & \text{if } j-\eta \geq 1 \wedge j+\eta \leq n \\ \{(1), (2), \cdots, (j)\}, & \text{if } j-\eta < 1 \wedge j+\eta \leq n \\ \{(\max\{1, n-2\eta\}), \cdots, (j)\}, & \text{otherwise.} \end{cases}$$

and

$$N_i^+(k) = \begin{cases} \{(j),(j+1),\cdots,(j+\eta)\}, & \text{if } j+\eta \le n \wedge j-\eta \ge 1 \\ \{(j),(j+1),\cdots,(n)\}, & \text{if } j+\eta > n \wedge j-\eta \ge 1 \\ \{(j),\cdots,(\min\{n,2\eta+1\})\}, & \text{otherwise.} \end{cases}$$

and define the set of all neighbors $N_i(k) := N_i^-(k) \cup N_i^+(k)$.

In the former, nodes seek to have 2η neighbors by establishing contact with other nearest nodes (see Fig. 1b). The next definition is somehow counterintuitive as nodes contact with others with opposite opinions to correct their lower degree.

Definition 5 (circular value): For each node $i \in \mathcal{V}$ of order j in the set $\mathcal{D}_i(k)$, we define the set of at most η neighbors considered as $N_i^-(k)$ as

$$N_i^-(k) = \begin{cases} \{(j-\eta),\cdots,(j)\}, & \text{if } j-\eta \ge 1 \wedge j+\eta \le n \\ \{(1),(2),\cdots,(j)\}, & \text{if } j-\eta < 1 \wedge j+\eta \le n \\ \{(1),\cdots,(j+\eta-n)\} \\ \quad \cup\{(j-\eta),\cdots,(j)\}, & \text{otherwise.} \end{cases}$$

and

$$N_i^+(k) = \begin{cases} \{(j),(j+1),\cdots,(j+\eta)\}, & \text{if } j+\eta \le n \wedge j-\eta \ge 1 \\ \{(j),(j+1),\cdots,(n)\}, & \text{if } j+\eta > n \wedge j-\eta \ge 1 \\ \{(n+j-\eta),\cdots,(n)\} \\ \quad \cup\{(j),\cdots,(j+\eta)\}, & \text{otherwise.} \end{cases}$$

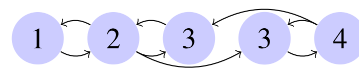
and define the set of all neighbors $N_i(k) := N_i^-(k) \cup N_i^+(k)$.

The nearest circular value guarantees that all nodes have 2η links, as shown in Fig. 1c.

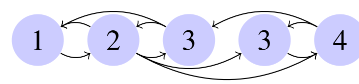
IV. Stochastic State-Dependent Social Network

In this section, we introduce a randomized version of the social network presented in Section III. Intuitively, at each discrete time instant, one agent is selected randomly according to the probabilities in the diagonal matrix $\mathrm{diag}(p_1, p_2, \cdots, p_n)$, where each $p_\ell \in (0,1)$ represents the probability that agent ℓ is selected, with $\sum_\ell p_\ell = 1$. We denote by i_k the random variable accounting for the selection of the node updating its state at communication time k. All random variables i_k are independent and identically distributed (i.i.d.), following the distribution given by matrix P, i.e., $i_k = \ell$ with probability p_ℓ. If a given agent ℓ is selected at time k, $i_k = \ell$, then its state is updated according to the update law in (1) and the remaining states stay unchanged.

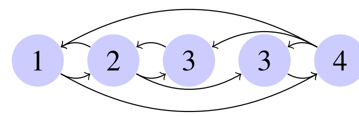
Parameter α_k is assumed to be randomly selected at each time instant k from a probability distribution with $\bar{\alpha} := \mathbb{E}[\alpha_k], \forall k \ge 0$ and support $[0,1]$. This definition is assuming implicitly that the distribution for the choice of α is the same at every time instant, independent across time, and is common to all the nodes in the network. From the definition of the α parameter, we also have that $0 \le \bar{\alpha} \le 1$. All the random variables are measurable on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.



(a) base network / distinct value

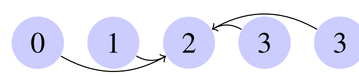


(b) distinct neighbor

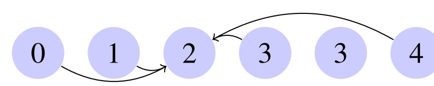


(c) circular value

Fig. 1: Network generated for each definition using $\eta = 1$ and $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 3$ and $x_5 = 4$.



(a) base network



(b) distinct value

Fig. 2: Detail of the links from node x_3 when using $\eta = 2$ and $x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3, x_5 = 3$ and $x_6 = 4$ for the base and distinct value networks.

For stochastic social networks, we consider the following convergence definition to a final opinion $x_\infty(\omega) := c(\omega)1_n$, for some constant c that depends on the outcome $\omega \in \Omega$ encompassing the outcomes of the random variables α_k and i_k.

Definition 6: We say that the social network with graph dynamics as in Section III and stochastic selection of agents converges, in the mean square sense, to a final opinion, if there exists a random variable, given the outcome ω, of the form $x_\infty(\omega) := c(\omega)1_n$ such that

$$\lim_{k\to\infty} \mathbb{E}[\|x(k,\omega) - x_\infty(\omega)\|^2] \to 0.$$

An alternative to the dynamics considered above is also studied which we refer in the sequel to as "random neighbors social network". The selection of updating node is maintained, using the random variables i_k to represent the node i selected at time k, and α_k as the random choice for the parameter to use in (1). However, at time k, the selection of neighbors ignores the previous definitions for the connectivity graph. Instead, a set of neighbors is selected at random with equal probability from the all possible non-empty subsets constructed using the nodes in $\mathcal{V}$. In addition, it is made the union of the selected set with the node i_k itself, as to reflect that it is always possible for node i_k to use its own opinion. As a consequence, (1) is still well-defined. Let us also define the random variables j_k, as the node with minimum opinion from the selected set of neighbors at time k, and, conversely, ℓ_k as the node with the maximum opinion at time k.

The random neighbors social network mimics the behavior of interaction where nodes just randomly encounter others and the stochastic updates follow the asynchronous setting of the real world. As an example of how nodes interact, consider a 6-node network with initial state $\begin{bmatrix}1 & 3 & 20 & -4 & 7 & 0\end{bmatrix}^\intercal$, where $i_k = 1$ and node 1 selects nodes 2, 3 and 6 to update its opinion. This would mean that $x_1(k+1) = \alpha_k x_6(k) + (1-\alpha_k)x_3(k)$.

V. Convergence Results

A. Auxiliary results

The next Lemmas are presented to lighten subsequent proofs for the results regarding convergence using the proposed dynamics.

Lemma 1 (order preservation): Take any two nodes $i, j \in \mathcal{V}$ with the update rule (1) and graph dynamics described either by Definition 2, Definition 3, or Definition 4. If $x_i(k) \leq x_j(k)$ for some k, then $x_i(k+1) \leq x_j(k+1)$.

Proof. The lemma results from the relationship that if $x_i(k) \leq x_j(k)$, then

$$\min_{\ell\in N_i(k)} x_\ell(k) \leq \min_{m\in N_j(k)} x_m(k)$$

and also

$$\max_{\ell\in N_i(k)} x_\ell(k) \leq \max_{m\in N_j(k)} x_m(k)$$

and since the update (1) performs a weighted average between minimum and maximum opinions, the conclusion follows. ■

Notice that Lemma 1 is not valid for the case of the nearest circular value of Definition 5, as nodes interact with neighbors that are the "farthest". The result can be interpreted as each agent knowledge of advantages and disadvantages remain ordered as nodes contact with closer-in-opinion neighbors who in turn interact with other nodes with knowledge of more extreme facts about the topic in discussion. Lemma 1 is going to be helpful since given the relative ordering is maintained we can use the initial sorting as labeling.

Lemma 2 (convergence for higher connectivity): Take any of the network dynamics in Definition 2, Definition 3, or Definition 4, and two integers $1 \leq \eta_1 \leq \eta_2$. Define

$$V^\eta(k) := \max_{i\in\mathcal{V}} x_i^\eta(k) - \min_{i\in\mathcal{V}} x_i^\eta(k),$$

where $x_i^\eta(k)$ represents the state at time instant k evolving according to (1) when the maximum number of larger or smaller neighbors is η. Then, for any initial conditions $x(0)$, $V^{\eta_1}(k) \geq V^{\eta_2}(k)$.

Proof. Regardless of the value of η and given the iteration in (1), any element of $x(k+1)$ is going to be a weighted average of the elements in $x(k)$ with weights α_k and $1-\alpha_k$. Applying (1) recursively yields that any opinion is going to be a weighted average of the initial state with weights being all the combinations from $\alpha_0 \cdots \alpha_k$ to $(1-\alpha_0)\cdots(1-\alpha_k)$. If we use a binary vector b to generate all the weights, it means that each combination from $\alpha_0 \cdots \alpha_k$ to $(1-\alpha_0)\cdots(1-\alpha_k)$ can be written as:

$$\prod_{i=1}^{k} b_i\alpha_{i-1} + (1-b_i)(1-\alpha_{i-1})$$

for each binary vector $b \in \{0,1\}^k$.

In addition, iteration (1) is going to perform a weighted average of two other nodes that depend on which network dynamics is selected. Following this, we can define a function $\varphi(i, b, k, \eta)$ used to determine the indices of the nodes selected for the average at node i, corresponding to the weight combination b and for k time instants after the initial time using a connectivity parameter η. For the example of a base network, going from $k-1$ to k means that this function either selects node $i+\eta$ (the weight corresponds to the maximum node in (1)) or $i-\eta$ (the weight corresponds to the minimum node in (1)). Since a node index cannot be smaller than 1 or higher than n, the φ function should saturate for each recursive iteration in k.

Using these two facts enables rewriting $V^\eta(k)$ as a function of the initial state $x(0) = x^{\eta_1}(0) = x^{\eta_2}(0)$ as:

$$\begin{aligned} V^\eta(k) = \\ \sum_{b\in\{0,1\}^k} \Big[\prod_{i=1}^{k} b_i\alpha_{i-1} + (1-b_i)(1-\alpha_{i-1})\Big] \\ \big[x_{\varphi(n,b,k,\eta)}(0) - x_{\varphi(1,b,k,\eta)}(0)\big] \end{aligned} \tag{2}$$

where

$$\varphi(c, b, \ell, \eta) = \begin{cases} \text{sat}(c + (-1)^{b_1}\eta), & \text{if } \ell = 1 \\ \varphi(\text{sat}(c + (-1)^{b_\ell}\eta), b, \ell-1, \eta), & \text{otherwise} \end{cases}$$

using the saturation function

$$\text{sat}(c) = \begin{cases} 1, & \text{if } c \le 1 \\ n, & \text{if } c \ge n \\ c, & \text{otherwise.} \end{cases}$$

The presented function $\varphi(\cdot)$ is for Definition 2 and similar functions can be given for the remaining definitions of network dynamics by adding to η the number of nodes with equal state. Nevertheless, the important feature of this function is stated next and is sufficient for proving the result.

The form in (2) means that $V^{\eta_1}(k)$ and $V^{\eta_2}(k)$ represent a sum of terms multiplied by weights that are equal. Even though the weight associated with a given $x_i(0)$ state might be different in $V^{\eta_1}(k)$ and $V^{\eta_2}(k)$, the approach herein is to directly compare each term $x_{\varphi(n,b,k,\eta)}(0) - x_{\varphi(1,b,k,\eta)}(0)$ for the two values η_1 and η_2, since the weight that multiplies each of these terms is independent of η.

Assuming the labeling of the nodes as the relative ordering at the initial state, to prove $x_{\varphi(n,b,k,\eta_1)}(0) - x_{\varphi(1,b,k,\eta_1)}(0) \ge x_{\varphi(n,b,k,\eta_2)}(0) - x_{\varphi(1,b,k,\eta_2)}(0)$ it is only required to show the equivalent for the indices, i.e.,

$$\varphi(n, b, k, \eta_2) - \varphi(1, b, k, \eta_2) \le \varphi(n, b, k, \eta_1) - \varphi(1, b, k, \eta_1). \tag{3}$$

We shall prove (3) by induction for any k.

Let us start with the base case of $k = 1$ and prove that

$$\varphi(n, b, 1, \eta_2) - \varphi(n, b, 1, \eta_2) \le \varphi(n, b, 1, \eta_1) - \varphi(1, b, 1, \eta_1). \tag{4}$$

If $b_1 = 0$, the left-hand side of (4) can be simplified as follows:

$$\begin{aligned} \varphi(n, 0, 1, \eta_2) - \varphi(n, 0, 1, \eta_2) &= \text{sat}(n + \eta_2) - \text{sat}(1 + \eta_2) \\ &= n - 1 - \eta_2 \end{aligned}$$

and similarly the right-hand side becomes $n-1-\eta_1$. Since $\eta_1 \le \eta_2$, it implies that $n-1-\eta_1 \ge n-1-\eta_2$.

If $b_1 = 1$, the left-hand side of (4) can be simplified to:

$$\begin{aligned} \varphi(n, 1, 1, \eta_2) - \varphi(n, 1, 1, \eta_2) &= \text{sat}(n - \eta_2) - \text{sat}(1 - \eta_2) \\ &= n - \eta_2 - 1 \end{aligned}$$

and the right-hand side to $n - \eta_1 - 1$. Thus, $\eta_1 \le \eta_2 \implies n - \eta_1 - 1 \ge n - \eta_2 - 1$.

To prove the induction step, assume (3) is true for some k and let us prove the induction step

$$\begin{aligned} \varphi(n, b, k+1, \eta_2) - \varphi(1, b, k+1, \eta_2) \le \\ \varphi(n, b, k+1, \eta_1) - \varphi(1, b, k+1, \eta_1). \end{aligned} \tag{5}$$

The first step is noting $\varphi(n, b, k, \eta_1) = \varphi(1, b, k, \eta_1)$ can only be true if and only if $\varphi(n, b, k, \eta_2) = \varphi(1, b, k, \eta_2)$ or it contradicts (3) since $\varphi(\cdot)$ is non-negative and $\varphi(n, b, k, \eta) \ge \varphi(1, b, k, \eta)$. Unless $\varphi(n, b, k, \eta_2) = \varphi(1, b, k, \eta_2)$ holds, in which case (5) is proved trivially because the function $\varphi(\cdot)$ is non-negative, the terms in (5) can always be written in the following form for $\eta \in \{\eta_1, \eta_2\}$:

$$\varphi(n, b, k, \eta) = n - \kappa_n \eta \quad , \quad \varphi(1, b, k, \eta) = 1 + \kappa_1 \eta$$

for some non-negative integers κ_1 and κ_n[1]. This is a result of $\varphi(n, b, k, \eta)$ being equal to the constant n with η added or subtracted multiple times and the equivalent for $\varphi(1, b, k, \eta)$. Three cases are possible: *a)* no saturation happens for time $k+1$; *b)* $\varphi(n, b, k, \eta)$ for $\forall \eta \in \{\eta_1, \eta_2\}$ saturates; *c)* $\varphi(1, b, k, \eta)$ for $\forall \eta \in \{\eta_1, \eta_2\}$ saturates.

a) the left-hand side of (5) for $k+1$ simplifies to

$$\begin{aligned} n - &\kappa_n \eta_2 + (-1)^{b_{k+1}} \eta_2 - 1 - \kappa_1 \eta_2 - (-1)^{b_{k+1}} \eta_2 \\ &= n - \kappa_n \eta_2 - 1 - \kappa_1 \eta_2 \\ &= \varphi(n, b, k, \eta_2) - \varphi(1, b, k, \eta_2) \end{aligned}$$

and equivalently for η_1. Then, the left- and right-hand side of (5) simplify to those in (3) and the conclusion follows.

b) if in this case, it means that $b_{k+1} = 0$ and (5) becomes

$$n - 1 - (\kappa_1 + 1)\eta_2 \leq n - 1 - (\kappa_1 + 1)\eta_1$$
$$\Longleftrightarrow$$
$$(\kappa_1 + 1)(\eta_1 - \eta_2) \leq 0,$$

which is true due to $\kappa_1 + 1 > 0$ and $\eta_1 - \eta_2 \leq 0$ by assumption.

c) if in this case, it means that $b_{k+1} = 1$ and (5) becomes

$$n - (\kappa_n + 1)\eta_2 - 1 \leq n - (\kappa_n + 1)\eta_1 - 1$$
$$\Longleftrightarrow$$
$$(\kappa_n + 1)(\eta_1 - \eta_2) \leq 0,$$

which is true $\kappa_n + 1 > 0$ and $\eta_1 - \eta_2 \leq 0$ by assumption.

Therefore, each term in the summation in (2) for η_1 is going to be greater than or equal to the same term in (2) for η_2, thus implying the conclusion. Notice that the relationship above for $\varphi(\cdot)$ is valid for Definitions 2, 3 and 4. ∎

[1]If $\varphi(n, b, k, \eta_1) = n$ then $\varphi(n, b, k, \eta_2) = n$ because κ_n does not depend on η. The same rationale applies to $\varphi(1, b, k, \eta)$

B. Base Network

The next results asserts the conditions for convergence of the base definition.

Theorem 1: Consider a social network as in Definition 2 with update rule (1) and any sequence $\{\alpha_k\}$. Then,

(i) If $\eta \geq n - 1$, the network is guaranteed to have finite-time convergence;
(ii) If $\eta < n - 1$, the network achieves at least asymptotic convergence.

We omit the proof of the theorem as it follows the same lines of [28].

Remark 1 (Distinct state values): In any of the graph dynamics considered in this paper, two nodes with the same state value are not neighbors following the definitions (essentially since they are not going to affect one another). In addition, any two nodes i and j with the same state value have $N_i(k) = N_j(k), \forall k \geq 0$. Therefore, the number of (distinct) node values, i.e.,

$$\Phi(k) = |\{x_1(k), \cdots, x_n(k)\}|$$

is a non-increasing function. Whenever, the in the initial state there are repeated opinion nodes, the same results are valid by replacing n by $n - \Phi(0)$. Also remark that if $\alpha_k = 0$ or $\alpha_k = 1$, then $\Phi(k+1) = \Phi(k) - 1$, which means finite-time convergence in n time instants.

The next theorem provides a result for the base social network which is based on the eigenvectors of a matrix representing the interaction in a time step.

Theorem 2 (Base Network Final Opinion): Consider a social network as in Definition 2 with n nodes with distinct initial condition $x_i(0), 1 \leq i \leq n$ and a constant parameter α in (1). The final opinion of the network is given by

$$x_\infty = \frac{1_n w_1^\intercal}{\sqrt{n}} x(0),$$

where w_1 is the normalized left-eigenvector associated with the eigenvalue 1 of matrix $A \in \mathbb{R}^{n \times n}$ defined by

$$[A]_{ij} := \begin{cases} \alpha, \text{ if } j = \max(1, i - \eta) \\ 1 - \alpha, \text{ if } j = \min(n, i + \eta) \\ 0, \text{ otherwise} \end{cases}.$$

Proof. An iteration for the base social network as in Definition 2 can be given by $x(k+1) = Ax(k)$ when labeling the nodes according to their relative ordering, which remains constant by Lemma 1. Therefore, $x_\infty = \lim_{k\to\infty} A^k x(0)$. Notice that matrix A is row stochastic, so the eigenvalue 1 has corresponding right eigenvector $\frac{1_n}{\sqrt{n}}$ and all the remaining eigenvalues have magnitude smaller than 1, which concludes the proof. ∎

In Theorem 2, the final convergence value depends on the vector w_1, which is the so-called *PageRank* for matrix A [32]. This connection comes from the fact that the base social network, for constant parameter α, becomes a linear iteration for a fixed network structure. Similarly, it points out to the importance of nodes based on the left-eigenvector, which is a centrality measure for this network (see [33] for a connection between centrality measures and social networks).

Theorem 2 only required i) the ordering of the nodes to remain constant, which is ensured by Lemma 1; ii) the matrix A to be constant, which is valid for all cases when only asymptotic convergence is achieved and α is constant.

C. Nearest Distinct Values

Theorem 1 states that the base network is only guaranteed to achieve finite-time convergence for all initial conditions if $\eta = n-1$ (complete topology). The following theorem addresses the study of topology dynamics as in Definition 3, where we use the ceiling operator $\lceil . \rceil$ to denote the smallest integer greater or equal than the argument.

Theorem 3: Consider the social network as defined in Section II, with the graph dynamics as in Definition 3, and any sequence $\{\alpha_k\}$. Then,

(i) If $\eta \geq \frac{n}{2}$, the network is guaranteed to have finite-time convergence in no more than $\lceil \log_2 n \rceil$ steps;

(ii) If $\eta < \frac{n}{2}$, the network achieves at least asymptotic convergence.

Proof. (i) Without loss of generality, we assume $n = 2\eta$, the initial states are all distinct as in Remark 1, and that the numbers of the nodes are sorted according to their state ordering, so as to shorten the notation by identifying the minimum and maximum value nodes with x_1 and x_n, respectively. Since $n = 2\eta$, there exist at least two nodes reaching the minimum and maximum nodes, i.e., there are i, j :

$$\min_{\ell \in N_i(0)} x_\ell = \min_{\ell \in N_j(0)} x_\ell = x_1(0)$$
$$\max_{\ell \in N_i(0)} x_\ell = \max_{\ell \in N_j(0)} x_\ell = x_n(0)$$

Thus, $\Phi(1) = \Phi(0) - 1$. In the subsequent iterations the cardinality reduces by $2, 4, \cdots$ by nodes fulfilling the previous conditions, which leads to $\Phi_k = n - (2^k - 1)$. Hence, $\Phi(k) \leq 1 \Leftrightarrow k \geq \log_2 n$, thus leading to the conclusion.

(ii) Using the previous argument, one determines that if $\eta < \frac{n}{2}$, it is not possible to find at least a pair of nodes communicating with the whole network and guarantee finite-time convergence. Asymptotic convergence is achieved following the same argument from the proof of Theorem 1 given in [28] and by noticing that the graph dynamics in Definition 3 also imply a strongly connected graph. ∎

Theorem 2 provides a categorization of the final opinion for the base social network which depends on a left eigenvector of a matrix, but it is not straightforward to understand how the steady state is influenced by the initial conditions. In the sequel, closed-form results are presented that describe the dependence on the initial conditions when finite-time convergence is achieved for the network dynamics as in Definition 3 and Definition 4. The case of distinct values is presented next.

Theorem 4: Consider a social network with dynamics as described in Definition 3 and distinct initial conditions $x_i(0), 1 \leq i \leq n$, with parameters $\alpha = \frac{1}{2}$ and $\eta = \lceil \frac{n}{2} \rceil$. The network opinion converges to

$$x_\infty = \frac{\Gamma}{2^{\lceil \log_2 n \rceil}} 1_n,$$

with

$$\Gamma = \sum_{j=1}^{\lceil \log_2 n \rceil} \lceil 2^{\lceil \log_2 n \rceil - 1 - j} \rceil (x_{1+\theta_j} + x_{n-\theta_j})$$

using the following definitions for the indices

$$\theta_j = \begin{cases} 0, & \text{if } j = 1 \\ \sum_{i=1}^{j-2} \left[(-1)^{i+1}\Phi(i)\right] + \eta, & \text{if even } j \\ \sum_{i=1}^{j-1} \left[(-1)^{i+1}\Phi(i)\right] - 1, & \text{if odd } j > 1 \end{cases},$$

where recall that $\Phi(k) := |\{x_1(k), \cdots, x_n(k)\}|$.

Proof. We start our proof by showing that θ is the set of indices of the initial states that contribute to the final opinion value. At time instant $k = 1$, the minimum node will have a state equal to the weighted average between x_1 (i.e., the node with minimum state at time $k = 0$) and $x_{1+\eta}$ and, conversely, the maximum state will be the weighted average between x_n and $x_{n-\eta}$, thus obtaining the second term η.

In the next time instant, the minimum value node contacts the node that is the η-th smaller value which corresponds to adding the node $x_{1+(1+2\eta) \bmod n} = x_{1+n-\Phi(1)}$ and conversely to the maximum value getting $x_{\Phi(1)}$. The key aspect to notice is that $\Phi(1)$ was added to take into account that the cardinality of nodes with distinct values has decreased. By following the same pattern, we obtain the expression for θ.

To finalize the proof, we must compute the weights associated with each index. We notice that the aggregation is a binary tree and the weights double after each time instant that the index was added to θ. Thus, the weights are given by $2^{\lceil \log_2 n \rceil - 1 - j}$ where we must subtract 1 since the time starts at $k = 0$ and j accounts for the time instant it enters in the index set θ. ■

To illustrate Theorem 4, consider a network with $n = 16$ and $\eta = 8$ for $\alpha = 0.5$ where the aim is to compute the final social opinion. Using Theorem 4, the final state is given by

$$x_\infty = \frac{4x_1 + x_2 + x_6 + 2x_8 + 2x_9 + x_{11} + x_{15} + 4x_{16}}{32}$$

while if $\eta = n - 1$ the solution is

$$x_\infty = \frac{x_1 + x_{16}}{2},$$

which indicates that the minimum and maximum opinion nodes are the most influential in the final network belief and as η increases their preponderance follows.

D. Nearest Circular Value

The following result studies Definition 5 as the dynamics of the social network.

Theorem 5: Consider the social network with graph dynamics as in Definition 5, update rule (1) and any sequence $\{\alpha_k\}$. Then, for any $\eta \geq 1$, the network has finite-time convergence in no more than $\lceil \frac{n-(2\eta+1)}{2\eta-1} \rceil + 1$ time steps.

Proof. Without loss of generality, we assume distinct initial states as in Remark 1 and that the nodes labels are sorted according to their state ordering. If $\Phi(k) \leq 2\eta + 1$, then we have the complete network and finite-time consensus is achieved in a single time instant.

At each time k, there are 2η nodes that have access both to $x_1(k)$ and $x_n(k)$. Thus, $\Phi(k) = n - (2\eta - 1)k$ and we need to have $\Phi(k) \leq 2\eta + 1 \Leftrightarrow k \geq \frac{n-(2\eta+1)}{2\eta-1}$ to get to a configuration where finite-time convergence is achieved in a single time instant, which concludes the proof. ■

Remark 2: In a first analysis, the convergence time provided in Theorem 3, i.e., $\log_2 n$, could appear significantly faster when compared to $\lceil \frac{n-(2\eta+1)}{2\eta-1} \rceil + 1$ from Theorem 5. However, we stress that, in Theorem 3, such a rate is achieved when $n = 2\eta$, which would lead to convergence in a single instant in the conditions of Theorem 5.

E. Nearest Distinct Neighbors

The next theorem proves the conditions for convergence when the selected neighbor selection follows Definition 4.

Theorem 6: Consider the social network with graph dynamics as in Definition 4, update rule (1) and any sequence $\{\alpha_k\}$. Then, for any $\eta \geq 1$, the network has finite-time convergence in no more than $\lceil \frac{n-(2\eta+1)}{2\eta} \rceil + 1$ time steps.

Proof. Without loss of generality, we assume distinct initial states as in Remark 1 and that the nodes labels are sorted according to their state ordering. Similarly to the previous theorem, if $\Phi(k) \leq 2\eta + 1$ then the network is complete between all the nodes with distinct values and finite-time consensus is achieved in a single time instant.

At each time k, there are $\eta+1$ nodes that have access to $x_1(k)$ and $x_{1+\eta}(k)$, and $\eta+1$ nodes receive the information $x_{n-\eta}(k)$ and $x_n(k)$. Thus, $\Phi(k) = n - 2\eta k$ and we need to have $\Phi(k) \leq 2\eta+1 \Leftrightarrow k \geq \frac{n-(2\eta+1)}{2\eta}$ to get to a configuration where finite-time convergence is achieved in a single instant, which concludes the proof. ■

The clustering behavior observed when using the previous definition for selecting neighbors is very different from what is obtained when using Definition 4, where the median nodes play the key role. The result is summarized in the following theorem.

Theorem 7: Consider a social network with graph dynamics as in Definition 4, update rule (1), and distinct initial conditions $x_i(0), 1 \leq i \leq n$, with parameter $\alpha = \frac{1}{2}$. The network opinion when $n = 1 + 2\eta\ell, \ell = \{0, 1, \cdots\}$ converges to

$$x_\infty = \frac{\sum_{j=0}^{\tau} \binom{\tau}{j} x_{1+j2\eta}}{2^\tau}, \tag{6}$$

where $\tau = \lceil \frac{n}{2\eta} \rceil - 1$. For the remaining values of n we get

$$x_\infty = \frac{\sum_{j=0}^{\tau} \binom{\tau}{j} \left[x_{1+j2\eta} + x_{n-(\tau-j)2\eta}\right]}{2^{\tau+1}}. \tag{7}$$

Proof. For the trivial case of $n = 2$, the expression is straightforward to verify. For the general case, we use induction to prove the result. Start by noticing that for $k = 1$, we get weighted averages of pairs of variables x_i and $x_{i+2\eta}$. When $k = 2$, the averages are among nodes x_i, $2x_{i+\eta}$, and $x_{i+2\eta}$ since $n > 2\eta$, or otherwise an additional communication step would not be necessary and k would be one.

Using the same reasoning, we need to consider 3 cases: when $n = 1 + 2\eta\ell$, when $1 + 2\eta\ell < n < 2\eta(\ell+1)$, and when $n = 2\eta\ell$.

(i) When $n = 1 + 2\eta\ell$, there exists an instant k such that $\Phi(k-1) = 2\eta + 1$. For time instant k, $\Phi(k) = 1$ and for n nodes, by assumption, all of their values is a weighted average in the form of (6) for time k, i.e.,

$$\frac{\sum_{j=0}^{k} \binom{k}{j} x_{1+j2\eta}}{2^k}. \tag{8}$$

If we consider $n+1$, from the previous observation, there will be a node at time k with the value of

$$\frac{\sum_{j=0}^{k} \binom{k}{j} x_{2+j2\eta}}{2^k}$$

and where the last term of the sum is, by definition, dependent on x_n. Thus, we can rewrite it as

$$\frac{\sum_{j=0}^{k} \binom{k}{j} x_{n-(k-j)2\eta}}{2^k}. \tag{9}$$

By combining equations (8) and (9), we get that all nodes at time $k+1$ achieve (7).

(ii) For this case, the proof is similar to the previous one by noting that, since $n < 2\eta(\ell+1)$, at time $k-1$, $\Phi(k-1) < 2\eta$ for the case of n nodes. Thus, when considering $n+1$, the same setting as before is achieved.

(iii) When $n = 2\eta\ell$, we get that at time $k-1$, $\Phi(k-1) = 2\eta$. When considering the case of $n+1$, we will get at time k exactly $2\eta+1$ distinct values with the minimum

$$\frac{\sum_{j=0}^{k} \binom{k}{j} x_{1+j2\eta}}{2^k} \tag{10}$$

and maximum

$$\frac{\sum_{j=0}^{k} \binom{k}{j} x_{1+(1+j)2\eta}}{2^k}. \tag{11}$$

Since the last element of equation (11) must be x_n by definition, we can rewrite the equation as to count the variables from n instead of 1 and get

$$\frac{\sum_{j=0}^{k} \binom{k}{j} x_{n-j2\eta}}{2^k}. \tag{12}$$

Combining equations (10) and (12) and noticing that except for the first term in (10) and last term in (12), all of the remaining terms are repeated. Given that

$$\binom{k}{j} = \binom{k-1}{j-1} + \binom{k-1}{j},$$

the social network for $n+1$ final value is as in equation (6) when considering that it takes an additional time instant to converge. ■

As a small example to illustrate Theorem 7, let us consider a network with $n = 16$ and $\eta = 2$. Hence, one obtains

$$x_\infty = \frac{x_1 + x_4 + 3x_5 + 3x_8 + 3x_9 + 3x_{12} + x_{13} + x_{16}}{16}$$

which shows that under the network dynamics of Definition 4, the most influential nodes are close to the median and not the minimum and maximum nodes, as in the case of Definition 3. These results sustain the fact that given different objectives, it might be beneficial to choose one network over the other and scale the connectivity parameter η accordingly.

F. Stochastic Social Network

Section IV introduced the stochastic version of the social network presented in this article, which relaxes the condition that all the nodes are influenced deterministically at the same time instants. By considering the stochastic model of the network, we allow for the asynchronous case to be considered, which is closer to the actual dynamics that we are trying to model. Remark that while the convergence for the deterministic case was studied by the way each definition induces clusters of opinions, the stochastic counterparts have to be based on showing that the expected value of the function measuring the dispersion is decreasing. Moreover, the state-dependent graph dynamics prevents the use of techniques that typically explore the time-invariance property or the fact that the random variables are independent. We start by analyzing the case where the network dynamics is the base version and the case where nodes only select distinct opinions, in the following theorem.

Theorem 8: Consider a stochastic social network with graph dynamics as in Definition 2 or as in Definition 3 with connectivity parameter η, update rule (1), and initial conditions $x_i(0), 1 \leq i \leq n$, with parameter α_k following a probability distribution with mean $\bar{\alpha}$. Then, the network opinion converges to a consensus in the mean square sense.
In the proof, all inequalities and equalities involving random variables are valid for a arbitrary $\omega \in \Omega$ and occur with probability one.

Proof. We start by defining the shorter notation for the minimum and maximum as

$$x_{\min}(k,\omega) := \min_{\ell} x_\ell(k,\omega)$$

$$x_{\max}(k,\omega) := \max_{\ell} x_\ell(k,\omega)$$

and the limit random variable $c(\omega)$, for an outcome ω of the random selections i_k and all random variables α_k, is defined as

$$c(\omega) := \lim_{k\to\infty} x_{\min}(k,\omega),$$

which exists and is measurable by the Monotone Convergence Theorem, since $x_{\min}(k,\omega)$ is a monotonically increasing sequence and upper bounded by $x_{\max}(0)$ for all outcomes ω.

Also, given the definition of the function $V^\eta(\cdot)$ in Lemma 2, $\forall k \geq 0$

$$\begin{aligned}\|x(k,\omega) - x_\infty(\omega)\|^2 &= \sum_{\ell=1}^{n} (x_\ell(k,\omega) - c(\omega))^2 \\ &\leq V^\eta(x(0)) \sum_{\ell=1}^{n} |x_\ell(k,\omega) - c(\omega)| \\ &\leq V^\eta(x(0)) \sum_{\ell=1}^{n} x_{\max}(k,\omega) - x_{\min}(k,\omega),\end{aligned} \tag{13}$$

where the inequalities in (13) are given by the relationship $\forall \ell \in \mathcal{V}, k \geq 0 : |x_\ell(k,\omega) - c(\omega)| \leq x_{\max}(k,\omega) - x_{\min}(k,\omega)$, which comes directly from the definition of minimum and maximum. Note that the updating rule in (1) performs convex combinations, i.e., $x_\ell(k+1,\omega) = \sum_{q=1}^{n} a_q x_q(k,\omega)$ for some weights a_q with $\sum_{q=1}^{n} a_q = 1$. Therefore, $x_{\min}(k,\omega)$ and $x_{\max}(k,\omega)$ are respectively monotonically increasing and decreasing and $\forall \ell \in \mathcal{V}, k \geq 0 : |x_\ell(k,\omega) - c(\omega)| \leq x_{\max}(0) - x_{\min}(0)$ since $\forall k \geq 0 : x_{\max}(k,\omega) \leq x_{\max}(0)$ and $\forall k \geq 0 : x_{\min}(k,\omega) \leq x_{\min}(0)$.

Using (13), it follows

$$\mathbb{E}[\|x(k,\omega) - x_\infty(\omega)\|^2 | x(0)] \leq V^\eta(x(0)) \mathbb{E}[V^\eta(x(k,\omega)) | x(0)].$$

We shall prove, for $\eta = 1$, that

$$\mathbb{E}[V^\eta(x(k,\omega))|x(0)] \leq \bar{\gamma}^k V^\eta(x(0)) \tag{14}$$

from which stability in the mean square sense follows, because

$$\mathbb{E}[\|x(k,\omega) - x_\infty(\omega)\|^2 |x(0)] \leq \rho\bar{\gamma}^k V^\eta(x(0))^2.$$

for some positive constant ρ and $\bar{\gamma} < 1$.

Let us start with $\eta = 1$. In this case, when $\alpha_k \in (0,1)$, since $\eta = 1$, we can take the labeling of the nodes to be their relative order such that $x_1(0) \leq x_2(0) \leq \cdots \leq x_n(0)$. This labeling is not changed since $\forall \ell \in \mathcal{V} \setminus \{1, n\}, k \geq 0 : x_{\ell-1}(k,\omega) \leq x_\ell(k,\omega) \leq x_{\ell+1}(k,\omega)$ due to $x_\ell(k,\omega)$ being a convex combination of $x_{\ell-1}(k-1,\omega)$ and $x_{\ell+1}(k-1,\omega)$. For the nodes with the minimum and maximum state, the converse is true, i.e., $\forall k \geq 0 : x_1(k,\omega) \leq x_2(k,\omega)$ and $\forall k \geq 0 : x_{n-1}(k,\omega) \leq x_n(k,\omega)$. When considering some $\alpha_k = 0$ or $\alpha_k = 1$, one can take the relative order of the nodes at time k instead of their labeling, i.e., replace 1 by (1), 2 by (2), and conversely for the remaining nodes for all the expressions of this proof.

From the previous observation, the random variable $x(k,\omega)$ takes the form of a linear system of the type $x(k+1,\omega) = Q_{i_k}(\alpha_k)x(k,\omega)$, where matrices $Q_i(\alpha)$ are defined as

$$[Q_i(\alpha)]_{j\ell} := \begin{cases} \alpha, & \text{if } \ell = \max(1, j-1) \wedge j = i \\ 1-\alpha, & \text{if } \ell = \min(n, j+1) \wedge j = i \\ 1, & \text{if } j = \ell \wedge j \neq i \\ 0, & \text{otherwise.} \end{cases}$$

for nodes $i, j, \ell \in \mathcal{V}$ and a parameter $\alpha \in [0,1]$. Matrices $Q_i(\alpha)$ are equivalent to taking row i implementing the update of node i given by (1) and all the other rows from the identity matrix.

To prove (14) for $\eta = 1$, it is sufficient to show that

$$\mathbb{E}[V^1(x(k+\tau,\omega))|x(k,\omega)] - \gamma V^1(x(k,\omega)) \leq 0 \tag{15}$$

for time interval of size τ, constant $\gamma < 1$, which relates to $\bar{\gamma}$ through $\gamma^{\frac{k}{\tau}} = \bar{\gamma}^k$, and where $\mathbb{E}[\cdot|\cdot]$ is the conditional expected value operator.

In order to upperbound the expected value in (15), notice by the definition of $V^1(\cdot)$, for all time instant k, $\exists i^\star < j^\star : x_{j^\star}(k,\omega) - x_{i^\star}(k,\omega) \geq \frac{V^1(x(k,\omega))}{n}$. In particular, there exists adjacent nodes $i^\star$ and $j^\star$, i.e., $j^\star = i^\star + 1$. Thus, $i^\star$ and $j^\star$ cannot be 1 and n at the same time. Assuming $i^\star$ and $j^\star$ are both different from n, we can define a finite sequence ρ, of size τ, such that $\rho_1 = i_{k+1}, \cdots, \rho_\tau = i_{k+\tau}$. With the objective of writing $x_1(k+\tau,\omega)$ with terms that include both $x_1(k,\omega)$ and $x_n(k,\omega)$, we consider the finite sequence $\rho_1^\star = n-1, \rho_2^\star = n-2, \cdots, \rho_\tau^\star = 1$. This sequence of updates, of size $\tau = n-1$ occurs with non-zero probability

$$p_{\text{good}} = \prod_{\ell=1}^{\tau} [P]_\ell.$$

Computing the product $Q_1(\alpha_{k+\tau-1}) \cdots Q_{n-1}(\alpha_k)x(k)$ allows one to write the expected value of function $V^1(\cdot)$ subject to the chosen sequence $\rho^\star$ to occur from time k to $k+\tau$ as

$$\begin{aligned} &\mathbb{E}[V^1(x(k+\tau,\omega)|x(k,\omega), \rho = \rho^\star] = x_n(k,\omega) \\ &-\mathbb{E}[(\alpha_{k+\tau-1} + \alpha_{k+\tau-2}(1-\alpha_{k+\tau-1}))x_1(k,\omega)|x(k,\omega)] \\ &-\mathbb{E}[\sum_{\ell=2}^{\tau-1}\left(\alpha_{k+\tau-\ell-1}\prod_{j=0}^{\ell-1} 1-\alpha_{k+\tau-j-1}\right)x_\ell(k,\omega)|x(k,\omega)] \\ &-\mathbb{E}[\left(\prod_{\ell=0}^{\tau-1}(1-\alpha_{k+\ell})\right)x_n(k,\omega)|x(k,\omega)], \end{aligned} \tag{16}$$

where the conditional expected values in (16) are over the random variables $\alpha_k, \alpha_{k+1}, \cdots, \alpha_{k+\tau-1}$. Since α_k is assumed to be independently selected in each time instant k, $\forall k \geq 0, \phi > 0 : \mathbb{E}[\alpha_k\alpha_{k+\phi}] = \mathbb{E}[\alpha_k]\mathbb{E}[\alpha_{k+\phi}]$. Thus, and due to linearity of the expected value operator, (16) can be simplified to

$$\begin{aligned} &\mathbb{E}[V^1(x(k+\tau,\omega)|x(k,\omega), \rho = \rho^\star] = x_n(k,\omega) - \\ &\quad\left[\bar{\alpha}(2-\bar{\alpha})x_1(k,\omega) + \sum_{\ell=2}^{\tau-1}\left(\bar{\alpha}(1-\bar{\alpha})^\ell\right)x_\ell(k,\omega)\right. \\ &\quad\left. + (1-\bar{\alpha})^\tau x_n(k,\omega)\right]. \end{aligned} \tag{17}$$

Lastly, due to the fact that the nodes labeling correspond to their relative ordering, we can upperbound

(17) and get

$$\begin{aligned}\mathbb{E}[V^1(x(k+\tau,\omega)|x(k,\omega),\rho=\rho^\star] &\leq x_n(k,\omega)\\ &-[(1-(1-\bar{\alpha})^\tau)x_1(k,\omega)+(1-\bar{\alpha})^\tau x_n(k,\omega)]\\ &\leq (1-(1-\bar{\alpha})^\tau)(x_n(k,\omega)-x_1(k,\omega))\\ &\leq (1-(1-\bar{\alpha})^\tau)V^1(x(k,\omega)),\end{aligned} \tag{18}$$

where all the $x_\ell(k,\omega)$ inside the summation in (17) were replaced by $x_1(k,\omega)$. Remark that

$$\begin{aligned}\mathbb{E}[V^1(x(k+\tau,\omega))|x(k,\omega)] &= \sum_\rho p_\rho \mathbb{E}[V^1(x(k+\tau,\omega))|x(k,\omega),\rho]\\ &= p_{\text{good}}\mathbb{E}[V^1(x(k+\tau,\omega))|x(k,\omega),\rho=\rho^\star]\\ &+\sum_{\rho_s\neq\rho^\star} p_{\rho_s}\mathbb{E}[V^1(x(k+\tau,\omega))|x(k,\omega),\rho=\rho_s],\end{aligned}$$

where p_ρ is the probability of occurring the finite sequence ρ out of all possible finite sequences of size τ. Given the upperbound in (18) for the chosen sequence and that for all the remaining ρ_s, $\forall k\geq 0,\tau\geq 0: V^1(x(k+\tau,\omega))\leq V^1(x(k,\omega))$, the expected value in (15) can be upper-bounded by

$$\begin{aligned}&\mathbb{E}[V^1(x(k+\tau,\omega))|x(k,\omega)] \leq\\ &p_{\text{good}}\left(1-(1-\bar{\alpha})^\tau\right)V^1(x(k,\omega))+(1-p_{\text{good}})V^1(x(k,\omega)).\end{aligned} \tag{19}$$

By simplifying (19), we get

$$\mathbb{E}[V^1(x(k+\tau,\omega))|x(k,\omega)] \leq \left[1-p_{\text{good}}(1-\bar{\alpha})^\tau\right]V^1(x(k,\omega)),$$

which satisfies (15) for $\gamma=1-p_{\text{good}}(1-\bar{\alpha})^\tau$.

For the other case where $i^\star$ and $j^\star$ are both different from 1, following a similar reasoning, we would select the finite sequence $\rho_1^\star=2,\rho_2^\star=3,\cdots,\rho_\tau^\star=n$. Following the same steps would lead to

$$\mathbb{E}[V^1(x(k+\tau,\omega))|x(k,\omega)] \leq \left[1-p_{\text{good}}\bar{\alpha}^\tau\right]V^1(x(k,\omega)),$$

which satisfies (15) for $\gamma=1-p_{\text{good}}\bar{\alpha}^\tau$. Inequality (15) holds for both cases by selecting $\gamma=1-p_{\text{good}}\max(\bar{\alpha}^\tau,(1-\bar{\alpha})^\tau)<1$ which confirms that (14) holds for $\eta=1$, from which convergence in mean square sense follows for $\eta=1$.

As for $\eta>1$, applying the same reasoning as in the proof of Lemma 2, we have

$$0\leq V^\eta(x(k,\omega))\leq V^1(x(k,\omega)), \tag{20}$$

which means that for a generic η, the function $V^\eta(\cdot)$ is upperbounded by $V^1(\cdot)$. Combining (14) and (20) leads to

$$\mathbb{E}[V^\eta(x(k,\omega))|x(0)]\leq\bar{\gamma}^k V^1(x(0))$$

from which convergence in mean square follows for $\eta>1$, thus concluding the proof. ∎

In the next theorem, we analyze the convergence for the case of distinct neighbors.

Theorem 9: Consider a stochastic social network with graph dynamics as in Definition 4, update rule (1) and initial conditions $x_i(0), 1\leq i\leq n$ with parameter α_k following a probabilistic distribution with mean $\bar{\alpha}$. The, the network opinion converges to a consensus in mean square sense.

Proof. The proof follows a similar reasoning as that of Theorem 8 and focuses on establishing (15). Similarly to Theorem 8, taking $\eta=1$ makes possible to write the random variable $x(k,\omega)$ in the form of a linear system of the type $x(k+1,\omega)=Q_{i_k}(\alpha_k)x(k,\omega)$, but with matrices $Q_i(\alpha)$ being defined as

$$[Q_i(\alpha)]_{j\ell}:=\begin{cases}\alpha, & \text{if } \ell=\max(1,\min(j-1,n-2))\wedge j=i\\ 1-\alpha, & \text{if } \ell=\min(n,\max(j+1,3))\wedge j=i\\ 1, & \text{if } j=\ell\wedge j\neq i\\ 0, & \text{otherwise.}\end{cases}$$

for nodes $i,j,\ell\in\mathcal{V}$ and $\alpha\in[0,1]$. Matrices $Q_i(\alpha)$ are equivalent to taking row i from the matrix defining the network dynamics in Definition 4 of the deterministic case, and all the other rows are taken from the identity matrix.

For $\eta=1$ and $i^\star$ and $j^\star$ both different from n, we can select $\rho^\star$ of length $\tau=n-2$ such that $\rho_1^\star=n-1,\rho_2^\star=n-2,\cdots,\rho_{\tau-1}^\star=3,\rho_\tau^\star=1$ since the update of node 2 is irrelevant due to node 1 having as neighbor both node 2 and 3 for $\eta=1$. In doing so, (16) becomes

$$\begin{aligned}&\mathbb{E}[V^1(x(k+\tau,\omega)|x(k,\omega),\rho=\rho^\star]=x_n(k,\omega)\\ &-\mathbb{E}[\alpha_{k+\tau-1}x_1(k,\omega)|x(k,\omega)]\\ &-\mathbb{E}[\sum_{\ell=2}^{\tau}\left(\alpha_{k+\tau-\ell}\prod_{j=0}^{\ell-2}1-\alpha_{k+\tau-j-1}\right)x_\ell(k,\omega)|x(k,\omega)]\\ &-\mathbb{E}[\left(\prod_{\ell=0}^{\tau-1}(1-\alpha_{k+\ell})\right)x_n(k,\omega)|x(k,\omega)].\end{aligned}$$

Following that, equation (17) becomes

$$\mathbb{E}[V^1(x(k+\tau,\omega)|x(k,\omega),\rho=\rho^\star]=x_n(k,\omega)- \left[\bar{\alpha}x_1(k,\omega)+\sum_{\ell=2}^{\tau}\left(\bar{\alpha}(1-\bar{\alpha})^{\ell-1}\right)x_\ell(k,\omega) + (1-\bar{\alpha})^\tau x_n(k,\omega)\right].$$

However, by replacing all $x_\ell(k,\omega)$ inside the summation by $x_1(k,\omega)$, we get the same expression for (18) but with $\tau = n-2$ instead of $n-1$. Following the same steps for $i^\star$ and $j^\star$ both different from 1 would lead to the same expression as in Theorem 8. Thus, by following the remaining steps in the proof of Theorem 8, the conclusion follows. ■

Another interesting case of the stochastic social network is the random neighbors version, which is analyzed in the next theorem.

Theorem 10: Consider a random neighbors social network and initial conditions $x_i(0), 1 \le i \le n$, with parameter α_k following a probabilistic distribution with mean $\bar{\alpha}$. Then, the network opinion converges in mean square sense to consensus.

Proof. Let us recall the random variables i_k to represent the node whose clock ticked and is going to update its state and define the random variables j_k as the minimum node selected by node i_k at time k, and ℓ_k as the maximum node selected by node i_k at time k. The social network takes the form of a linear system of the type $x(k+1) = Q^{i_k}_{j_k \ell_k}(\alpha_k)x(k)$, where matrices $Q^i_{j\ell}(\alpha)$ are defined as

$$[Q^i_{j\ell}(\alpha)]_{qr} := \begin{cases} \alpha, & \text{if } q = i \wedge r = j \\ 1-\alpha, & \text{if } q = i \wedge r = \ell \\ 1, & \text{if } q \neq i \wedge q = r \\ 0, & \text{otherwise.} \end{cases}$$

for nodes $i, j, \ell, q, r \in \mathcal{V}$ and $\alpha \in [0,1]$ as the parameter for (1). In the remainder of the proof we will omit the dependence of $x(\cdot)$ on ω to shorten the notation and all inequalities and equalities involving random variables hold for an arbitrary ω with probability one.

Let us compute the probabilities associated with each of the matrices $Q^i_{j\ell}(\cdot)$ for a given value of i, j and ℓ. Let us define matrices Π_i, where $[\Pi_i]_{j\ell}$ is the probability that after selecting node i its update uses the minimum as node j and the maximum as node ℓ

$$[\Pi_i]_{j\ell} := \begin{cases} \frac{2^{\ell-j}}{2^n-1}, & \text{if } j = i \wedge j \le \ell \\ \frac{2^{\ell-j-1}}{2^n-1}, & \text{if } j < i \wedge i < \ell \\ \frac{2^{\ell-j}}{2^n-1}, & \text{if } j < i \wedge i = \ell \\ 0, & \text{otherwise.} \end{cases}$$

The probability of each $Q^i_{j\ell}(\alpha)$ is going to be the probabilities in $[\Pi_i]_{j\ell}$ multiplied by the probability distribution function of α. Let us also define matrix

$$R = \mathbb{E}[Q^i_{j\ell}(\alpha)].$$

Then,

$$\mathbb{E}[x(k+1)] = R^k \mathbb{E}[x(0)]$$

due to the probability distribution of selecting each matrix $Q^i_{j\ell}(\alpha)$ and the corresponding parameter α being independent. The expected value matrix R can be written as

$$R = \frac{1}{n}\big((n-1)I + (1-\bar{\alpha})(I \otimes 1_n^\intercal)\Omega + \bar{\alpha}\Upsilon\big)$$

where

$$\Omega := \begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \vdots \\ \Pi_n \end{bmatrix}, \; [\Upsilon]_{ij} := \begin{cases} \frac{2^{n-j}}{2^n-1}, & \text{if } i > j \\ \frac{2^{n-j+1}-1}{2^n-1}, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Matrices Π_i have all entries summing to 1, making each $1_n^\intercal \Pi_i$ sums to 1, leading to $(I \otimes 1_n^\intercal)\Omega$ being row stochastic and upper triangular. In addition, matrix Υ is also row stochastic but lower triangular. As a consequence, matrix R is a full matrix with all positive entries and row stochastic as it is a convex combination of row stochastic matrices. Thus, according to the Gershgorin's disk theorem, it has all eigenvalues within the unit circle. Since R is full, it is irreducible and, by the Perron-Frobenius theorem, it only has one eigenvalue equal to 1, showing that the limit of the expected value converges. These properties are required for the proof of convergence in the mean square sense.

Similarly, let us introduce the matrix

$$R_2 := \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\ell=1}^{n}[\Pi_i]_{j\ell} Q^i_{j\ell} \otimes Q^i_{j\ell}.$$

Manipulating the expression, and given that the distributions are independent, we can write

$$\mathbb{E}[x(k+1)x(k+1)^\intercal] = R_2^k \mathbb{E}[x(0)x(0)^\intercal].$$

Due to the structure of matrices $Q^i_{j\ell}$, the second moment matrix can be written as

$$\begin{bmatrix} \Gamma_1 & \Lambda_{1,2} & \Lambda_{1,3} & \cdots & \Lambda_{1,n} \\ \Lambda_{2,1} & \Gamma_2 & \Lambda_{2,3} & \cdots & \Lambda_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda_{n-1,1} & \Lambda_{n-1,2} & \cdots & \Gamma_{n-1} & \Lambda_{n-1,n} \\ \Lambda_{n,1} & \Lambda_{n,2} & \cdots & \Lambda_{n,n-1} & \Gamma_n \end{bmatrix}$$

where

$$\Gamma_\ell = R - \sum_{i \neq \ell} \left[(1-\bar{\alpha})[\Pi_i]_{j\ell} Q^\ell_{\ell i} + \bar{\alpha}[\Pi_i]_{j\ell} Q^\ell_{i\ell}\right] - \sum_{i\neq \ell} \sum_{j \neq \ell, j\neq i} Q^\ell_{ij}$$

and

$$\Lambda_{\ell j} = \sum_{i \neq j} \left[(1-\bar{\alpha})[\Pi_i]_{j\ell} Q^\ell_{ij} + \bar{\alpha}[\Pi_i]_{j\ell} Q^\ell_{ji}\right].$$

Matrix R_2 is still a row stochastic matrix but with non-negative entries. In order to show that R_2 is irreducible, consider its support graph given by having n^2 nodes corresponding to the dimension of R_2 and having an edge (i, j) for each $[R_2]_{ij} \neq 0$. Notice that in the block diagonal we have full matrices and, therefore, have n complete graphs of n nodes each. Since the support graph of $\Lambda_{\ell j}$ has a link (ℓ, j) which connects the ℓ of one of the clusters with j of another, the overall graph is still connected. Following the same reasoning, all the eigenvalues are within the unit circle with only one eigenvalue 1 and the conclusion follows. ∎

The proofs regarding the convergence of the considered social network use similar steps and tools that can be used for addressing other network dynamics. However, the focus of this work is on these specific network dynamics, as they reflect the observation of social networks in real-life.

VI. Simulation Results

In this section, we aim to compare the performance of the four network dynamics with an example consisting of $n = 100$ agents and $x_i(0) = i^2, i = 1, \cdots, n$ with $\alpha_k = \frac{1}{2}, \forall k \geq 0$.

Figure 3 depicts the values of function $V(k)$ in each time step for the base social network. The

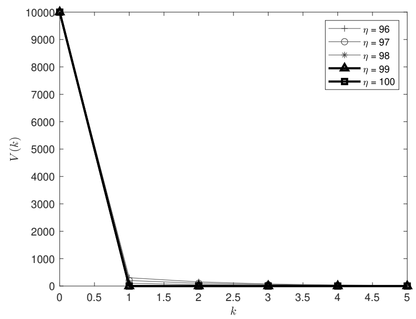


Fig. 3: Evolution of $V(k)$ for the case of a base social network for values of $\eta = 96, \cdots, 100$.

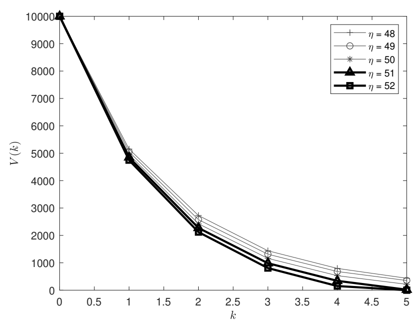


Fig. 4: Evolution of $V(k)$ for the case of a social network with agents communicating with nodes with distinct opinions for values of $\eta = 8, \cdots, 12$.

function $V(k)$, defined in the statement of Lemma 2, measures the spread between the minimum and maximum opinions. In the figure, the example with $\eta = 99$ is overlapped by the one where $\eta = 100$ since in both cases the topology is complete.

In Fig. 4 is presented the simulation for the distinct value policy. To have a clear representation, the thick lines correspond to the finite-time convergence cases which satisfy $\eta \geq \frac{n}{2}$. The number of time steps before settling corresponds to the results in Theorem 3. In comparison with the base network where all the nodes are completely reachable, in this case, only two are needed.

Figures 5 and 6 present the evolution using the circular graph dynamics and the closest distinct neighbor policy, respectively. We point out the fact that both rules achieve finite-time convergence for

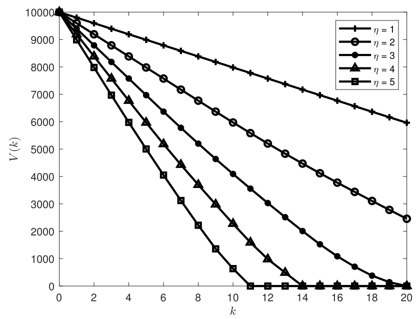


Fig. 5: Evolution of $V(k)$ for the case of a social network with agents with strong opinion looking for opposite opinions for values of $\eta = 1, \cdots, 5$.

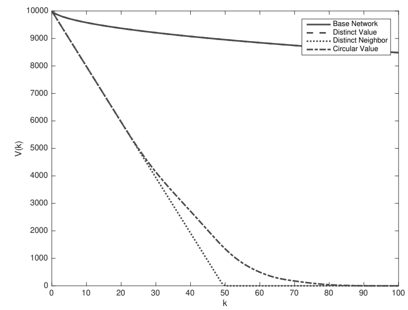


Fig. 7: Comparison of the evolution of $V(k)$ for the four cases with $\eta = 1$.

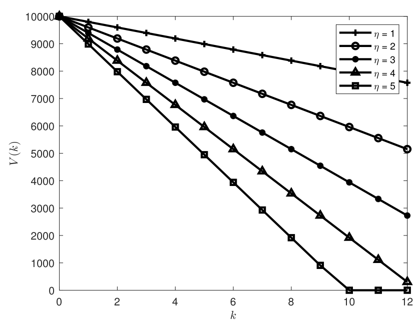


Fig. 6: Evolution of $V(k)$ for the case of a social network with agents contacting the 2η closest distinct neighbors for values of $\eta = 1, \cdots, 5$.

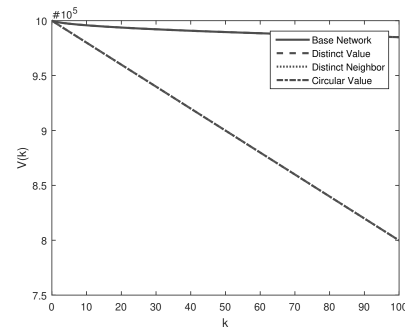


Fig. 8: Comparison of the evolution of $V(k)$ for the four cases with $\eta = 1$ and a large network of 1000 nodes.

any selection of η, although the distinct neighbor has a faster rate. The difference is justified by the way they operate. The circular strategy forms a cluster of nodes connecting the strongest opinions in each iteration. As opposed, the distinct neighbor builds two clusters with the same opinion in the first time step and new nodes are added in subsequent steps.

The above simulations illustrate the main theorems in this paper but fail to truly compare the speed of these techniques. We run a new simulation with $n = 100$ and $\eta = 1$ to make the results comparable, since the first two scenarios have a number of links equal to $\eta(2n - \eta - 1)$ and the following two have $2n\eta$ links.

Figure 7 depicts function $V(k)$ for the different topology dynamics. As expected, the circular and distinct neighbor achieve finite-time convergence while the graph dynamics in Definition 2 and Definition 3 present an asymptotic convergence with the same behavior (the lines overlap) since we have set $\eta = 1$.

Figure 8 depicts the evaluation of function $V(k)$ for a larger network of 1000 nodes. The convergence and behavior conform with the predictions of the theoretical result.

In order to compare the four policies regarding the final consensus, we consider a social network with $n = 100$ agents and three different cases for the initial conditions:

- initial conditions are drawn from independent normal distributions with expected value 100 and variance 1;

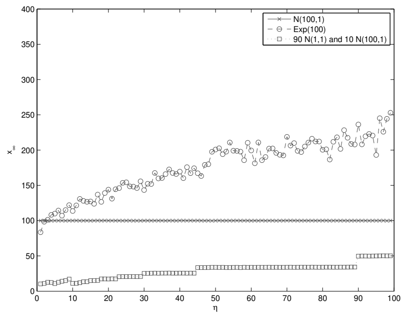


Fig. 9: Evolution of the final state x_∞ as function of η for the case of the base network dynamics.

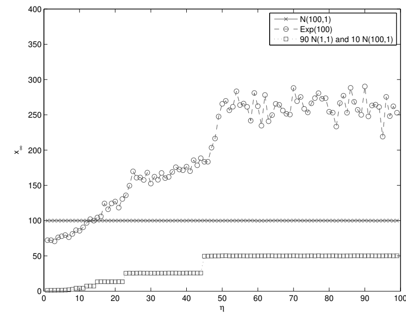


Fig. 10: Evolution of the final state x_∞ as function of η for the case of the Neighbor Network dynamics.

- initial states are chosen from independent exponential distributions with $\lambda = 100$;
- and a final example where 90 nodes are chosen from a normal distribution with expected value 1 and 10 agents are selected from normal distributions with expected value 100.

Figure 9 depicts the evolution of the final opinion value x_∞ as a function of η when considering $\alpha_k = \frac{1}{2}, \forall k \geq 0$ and network dynamics as in Definition 2 (the distinct value case is very similar). The first interesting point (observed in all cases) is that, when considering all the initial states drawn from independent normal variables with expected value equal to 100 and variance 1, the final opinion converges to the expected value. Such a result can be explained by the fact that the final belief is a convex combination of the initial states, which are normally distributed. We point out that for the geometric distribution, the social opinion is smaller than what is achieved for the Circular and Neighbor dynamics. An interesting aspect for small values of η is that the final value is greater than what can be achieved using other dynamics since the minimum and maximum values have a higher weight as suggested by Theorem 4.

Figure 10 depicts the final opinion of the network when using the Neighbor network dynamics. The final opinion increases with η except for the case of the normal distribution with expected value equal to 100. In the geometric distribution case, it is possible to achieve a higher opinion by selecting η close to n and a smaller value if we consider η close to 1. In the case where the population is divided into two groups, we see that the social opinion can approximate that of the majority by selecting $\eta = 1$, since this policy places higher weights in the median nodes, as shown in Theorem 7. We omit the plot for the circular network since for the cases using normal distributions they were different by a factor of 10^{-2} (except when $\eta < 9$ which was around 1).

VII. Experimental Data

The previous section simulated the behavior of the network when people follow *exactly* the proposed model. Such a case is of particular interest when applying the current models to networks of robots or multi-agent systems. In order to assess how much do people comply with the model, we have conducted an experimental trial with computer science students. Each participant were left with the choice of assigning a value in the interval $[0, 1]$ to translate the choice for a computational machine for the department. A value of zero would correspond to machine A whereas a value of 1 was a certain vote on machine B. Each participant were randomly assigned one of the characteristics of the machines shown in Table I. As future work, we would like to develop research in order to have stochastic interactions based on semi Markov models as the ones in [34].

The students were then asked to input a vote in each iteration corresponding to how likely they would advise on the purchase. After each vote, the system would match them with other people based

Machine A	Machine B
Is a All-in-one Desktop	Is a All-in-one Desktop
Intel i7 processor with Quad Core	Intel i5 processor with Dual Core
Processor works at 2.9 GHz	Processor works at 2.5 GHz
Clock turbo of 3.9 GHz	Clock turbo of 3.1 GHz
Cache of 8 Mb	Cache of 3 Mb
32 GB of RAM	8 GB of RAM
No information about RAM speed	2133 MHz
1 TB of SSD	256 GB of SSD
GeForce GTX 1070	Intel Graphics 620
Same audio system	Same audio system
No available HDMI	Available HDMI
USB fast port	Standard USB ports
28 in touch screen	Standard 23 in screen
4500 x 3000 resolution	1920 x 1080 resolution
Touch screen pen	n.a.
Costs 5000 euros	Costs 1120 euros

TABLE I: Information relating characteristics from machine A and machine B.

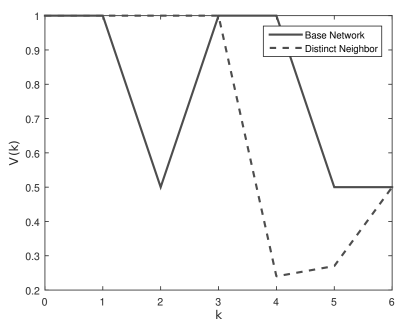


Fig. 11: Evolution of function $V(k)$ for the real data.

on their vote and exchanged known characteristics by the neighbors. The evolution of function $V(k)$ for the experimental data is depicted in Figure 11.

An interesting remark is that some participants did not send the last vote (i.e., the last vote means that no more information can be shared) as an integer. This prevented convergence since we always had at least a student voting 0 to signal selection of machine A. In order to better assess the proposed model against the Degroot model, in Figure 12, it is depicted the mean value of the votes for the proposed model using the Distinct Neighbors. Notice that using the Degroot model, as it resembles a consensus algorithm and given the ring network, the mean of the state is always going to be 0.25, i.e., equal to the average of the initial votes. In the experiment, this mean approaches 0.05

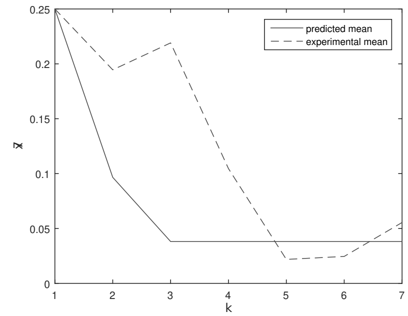


Fig. 12: Evolution of the mean of the votes in the experiment using the distinct neighbors against the predicted by the model with $\alpha_k = 0.95$ for all nodes and time instants.

as most participants voted 0, which motivates the use of more advanced models.

VIII. Conclusions

In this paper, the problem of studying the evolution of the opinion in a social network associated with a political party or an association is firstly tackled in a deterministic setting by modeling it as a distributed iterative algorithm with different topology dynamics that translate how agents interact. The dynamics considered are motivated by the fact that people tend to engage discussion with those with opinions close to their own.

For the deterministic setup, we have presented stability results for the base social network and remark that discarding repeated opinions improves the speed (requiring half the links). Two novel strategies are presented to reduce the number of necessary links η, namely one where nodes with extreme opinions seek the belief of the opposite opinion agents; another where 2η connections are established by selecting the closest agents without forcing to be greater or smaller. It is shown that albeit finite-time convergence is obtained through distinct processes, both algorithms place different weights on each agent initial opinion.

The circular strategy does not maintain the relative order of the nodes according to their opinion. Evaluating this strategy in simulation revealed that it follows the same general behavior of the distinct neighbors policy. The distinct neighbor policy results in a social opinion where the nodes closer to the median are more influential, and the weights are given by the entries of the Pascal triangle.

Future Work

Directions of future research include the collection of experimental data and the use of the model for different weight parameters for each individual. Under this scenario, the model would be comprehensive to include agents that are rational but value differently the positive and negative arguments relating a given topic. Another interesting topic is the application of the current models as guidance algorithms for mobile autonomous agents to achieve fast convergence with a small number of exchanged messages.

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