Examples of GES systems that can be driven to infinity by arbitrarily small additive decaying exponentials

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Abstract

Examples are given of nonlinear, time-invariant systems in continuous-time, discrete-time, and of hybrid type, that have linear sector growth, the origin globally exponentially stable (GES), and that can be driven to infinity by arbitrarily small additive decaying exponentials. Resulting observations about additive cascades and Lyapunov functions are discussed. These observations extend those derived from an example, which recently appeared in the literature, of a globally asymptotically stable continuous-time system destabilized by a piecewise constant, integrable disturbance.

I. Introduction

In the recent paper [7], an example is given, of the form $\dot{x} = f(x)+d$, having the following two properties: 1) when $d \equiv 0$ the origin is globally asymptotically stable, 2) there exist disturbances d that are arbitrarily small in $\mathcal{L}_1$ leading to unbounded solutions. The contribution here is to provide a simpler example, to make explicit that the example can be chosen to have the origin globally *exponentially* stable (GES) and so that $f(x)$ has linear sector growth (defined in Section II), and to show that the disturbance can be a simple decaying exponential. We also give discrete-time and hybrid system versions of the example. These examples are used to make the following observations:

This material is based upon work supported by the Army Research Office under grant DAAD19-03-1-0144 and the National Science Foundation under grant under Grant No. CCR-0311084.

1) the additive cascade of globally exponentially stable systems with linear sector growth may exhibit unbounded solutions; in particular, this has implications for the observer-based controller design problem when relying on a separation principle, like in [5], which motivated [7].
2) It is not possible in general, even for systems with linear sector growth, to construct radially unbounded Lyapunov functions for globally exponentially stable systems so that the gradient of the Lyapunov function is bounded by a constant plus linear growth in the Lyapunov function. In particular, as it was also pointed out in [7], it is not possible to find a Lyapunov function for such systems such that the gradient of the Lyapunov function is bounded.

Related to these observations are the following facts that have been observed in the literature:

1) a) The additive cascade of globally exponentially stable systems, the driven system having a *globally Lipschitz* right-hand side, is globally exponentially stable (e.g., see [3, Proposition 2.1]);
   b) The nonlinear cascade of globally exponentially stable systems, the driven system having a *globally Lipschitz* right-hand side when its input is zero and having certain growth restrictions on the connection terms, is globally asymptotically stable, possibly only when the trajectories of the driving system decay fast enough (e.g., see [6, Proposition 5][1] and [8, Theorems 6.2 and 6.4]).
   c) The additive cascade of globally exponentially stable systems for which the driven system is integral-input bounded-state is globally asymptotically stable (e.g., see [4] and the references therein).
2) a) A globally exponentially stable system with a *globally Lipschitz* right-hand side admits a C^1 Lyapunov function with upper and lower bounds that are quadratic in the state, with derivative along solutions bounded by a negative definite quadratic function of the state, and with its gradient bounded with linear sector growth in the size of the state (see, e.g., [1, Theorem 56.1] or [2, Theorem 3.12]; for discrete-time, see, e.g., [2, Exercise 3.54]).

[1] A global Lipschitz condition on $x \mapsto f(x,0)$, required to invoke the converse Lyapunov theorem used in the proof, was inadvertently omitted from this result.

b) A globally exponentially stable system with a *globally Lipschitz* right-hand side admits a C^1 Lyapunov function with a bounded gradient.

We emphasize that our examples do not exhibit unboundedness for additive input exponentials that decay arbitrarily quickly. The behavior of nonlinear systems in the presence of fast decaying exponentials received much attention in the controls literature in the late 1980's and early 1990's. The paper [8] contains several significant results. However, as far as we are aware it is still an open question whether globally exponentially stable systems that satisfy only a linear sector growth condition (in contrast to a global Lipschitz condition) exhibit only bounded trajectories in the presence of fast enough additive decaying exponentials.

The rest of this paper is organized as follows: We use one main idea to present examples of continuous-time (Section II), discrete-time (Section III) and hybrid (Section IV) systems where an additive decaying exponential produces unbounded trajectories. In each section, we make observations about additive cascades of globally exponentially stable systems. We comment on Lyapunov functions and their gradients in the continuous-time section, and Lyapunov functions and their differences in the discrete-time section.

Throughout the paper, $|\cdot|$ denotes the Euclidean norm.

II. Continuous-time

A. GES system driven by a decaying exponential

Consider the system

$$\begin{aligned} \dot{x}_1 &= g(x_1 x_2) x_1 \\ \dot{x}_2 &= -2x_2 + d \ . \end{aligned} \tag{1}$$

where $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$, $d \in \mathbb{R}$. We assume the following:

Assumption 1: The function $s \mapsto g(s)$ is such that

a) it is continuous;
b) $|g(s)| \leq 1$ for all $s \in \mathbb{R}$;
c) $g(s) = -1$ for all $s \in (-\infty, 1/2] \bigcup [3/2, \infty)$;
d) $g(1) = 1$.

Let $f(x)$ denote the right-hand side of (1) with $d = 0$. Then, under Assumption 1, $|f(x)| \leq 2|x|$ and $x^T f(x) \leq |x_1|^2$ for all x, i.e., f satisfies a linear sector growth condition. However, the function f is not globally Lipschitz, even if g is smooth. Indeed, let $x_1 \neq 0$ and consider two points $x_a = (x_1, 1/(2x_1))$ and $x_b = (x_1, 1/x_1)$. Then

$$|f(x_a) - f(x_b)| \geq |g(1/2) - g(1)||x_1| = 2|x_1| \ . \tag{2}$$

Now let $|x_1| \to \infty$ and note that $|x_a - x_b| \to 0$ while $|f(x_a) - f(x_b)| \to \infty$.

Proposition 1: Under Assumption 1, when $d \equiv 0$ the solutions of (1) satisfy

$$|x(t)| \leq 9e^{-t}|x(0)| \ . \tag{3}$$

Proof. Clearly $|x_2(t)| = e^{-2t}|x_2(0)| \leq 9e^{-t}|x_2(0)|$. So it is enough to establish the bound (3) with x_1 in place of x. Define $\xi := x_1 x_2$ and note that

$$\dot{\xi} = -\left[2 - g(\xi)\right]\xi \ . \tag{4}$$

Since $|g(\xi)| \leq 1$ for all $\xi \in \mathbb{R}$, it follows that $|\xi(t)|$ is monotonically decreasing and $|\xi(t)| \leq e^{-t}|\xi(0)|$. It then follows from $g(s) = -1$ for all $s \leq 1/2$ that for all initial conditions satisfying $\xi(0) \leq 1/2$ the bound (3) holds. For $\xi(0) > 1/2$, define t_1 to satisfy $\xi(t_1) = 3/2$, or take $t_1 = 0$ if $\xi(0) < 3/2$, and define t_2 to satisfy $\xi(t_2) = 1/2$. From the monotonicity of $\xi(t)$, these times are well-defined, $t_1 < t_2$ and satisfy $\exp(t_2 - t_1) \leq 3$ since

$$\xi(t_2) = \frac{1}{2} = \exp\left(\int_{t_1}^{t_2} (g(\xi(s)) - 2)ds\right)\xi(t_1) \leq \exp(-(t_2 - t_1))\frac{3}{2} \ . \tag{5}$$

First consider the case where $t \geq t_2$. We then have

$$x_1(t) = \exp\left(\int_0^{t_1} g(\xi(\tau))d\tau + \int_{t_1}^{t_2} g(\xi(\tau))d\tau + \int_{t_2}^{t} g(\xi(\tau))d\tau\right)x_1(0) \ . \tag{6}$$

Then using the properties of g,

$$\begin{aligned} |x_1(t)| &\leq \exp(-t_1 - (t - t_2) + t_2 - t_1)|x_1(0)| \\ &= \exp(2(t_2 - t_1))\exp(-t)|x_1(0)| \\ &\leq 9e^{-t}|x_1(0)| \ . \end{aligned} \tag{7}$$

Next consider $t \leq t_1$. In this case

$$x_1(t) = \exp\left(\int_0^{t} g(\xi(\tau))d\tau\right)x_1(0) \tag{8}$$

and thus

$$|x_1(t)| \leq e^{-t}|x_1(0)| \ . \tag{9}$$

Finally, consider $t_1 < t < t_2$. In this case

$$x_1(t) = \exp\left(\int_0^{t_1} g(\xi(\tau))d\tau + \int_{t_1}^{t} g(\xi(\tau))d\tau\right) x_1(0) \tag{10}$$

and thus

$$\begin{aligned} |x_1(t)| &\leq \exp(-t_1 + t - t_1)|x_1(0)| \\ &\leq \exp(-t_1 + 2t_2 - t - t_1)|x_1(0)| \\ &= \exp(2(t_2 - t_1))\exp(-t)|x_1(0)| \\ &\leq 9e^{-t}|x_1(0)| \ . \end{aligned} \tag{11}$$

This establishes the result. ■

Proposition 2: Under Assumption 1, when $x_1(0) \neq 0$, $x_2(0) = x_1(0)^{-1}$, and $d(t) = x_2(0)e^{-t}$ a solution of (1) satisfies

$$x_1(t) = e^t x_1(0) \qquad \forall t \geq 0 \ . \tag{12}$$

Proof. We show that the function proposed in (12), together with the function $x_2(t) = x_2(0)e^{-t}$, is a solution. It is straightforward to verify that the proposed $x_2(t)$ satisfies the differential equation. We also have, for the proposed solution, $x_1(t)x_2(t) = x_1(0)x_2(0) = 1$ for all $t \in \mathbb{R}_{\geq 0}$. Therefore, using $g(1) = 1$, $\dot{x}_1(t) = x_1(t) = g(x_1(t)x_2(t))x_1(t)$. Thus the proposed $x_1(t)$ also satisfies the differential equation. ■

B. Additive cascades

We now consider an additive cascade of globally exponentially stable systems where the right-hand side exhibits linear sector growth. In particular, we consider the system of the previous subsection driven by a scalar linear system:

$$\begin{aligned} \dot{x}_1 &= g(x_1x_2)x_1 \\ \dot{x}_2 &= -2x_2 + z \\ \dot{z} &= -z \ . \end{aligned} \tag{13}$$

The following corollary follows from the results of the previous subsection.

Corollary 1: Under Assumption 1, for the initial condition $x_1(0) \neq 0$, $x_2(0) = x_1(0)^{-1}$, $z(0) = x_2(0)$, a solution of (13) satisfies

$$x_1(t) = e^t x_1(0) \qquad \forall t \geq 0 \ . \tag{14}$$

C. Lyapunov functions

As another simple corollary, we have:

Corollary 2: Let $f(x)$ denote the right-hand side of the system (1) with $d = 0$. There does not exist a positive definite, radially unbounded, C^1 function V such that, for all $x \in \mathbb{R}^2$, $\langle \nabla V(x), f(x) \rangle \leq 0$ and $|\nabla V(x)| \leq \mu + V(x)$ for some $\mu \in \mathbb{R}_{\geq 0}$.

Proof. Suppose such a function V does exist. Then the trajectories of the system (1) would satisfy

$$\dot{V}(x(t)) \leq (\mu + V(x(t)))|d(t)| \ . \tag{15}$$

By a standard comparison theorem, we get that

$$V(x(t)) + \mu \leq (\mu + V(x(0))) \exp\left(\int_0^t |d(\tau)| dt \right) \tag{16}$$

and thus $V(x(t))$ is uniformly bounded if $d \in \mathcal{L}_1$, e.g., d converges to zero exponentially. Since V is radially unbounded in x, this contradicts the conclusion of Proposition 2. ■

III. Discrete-time

A. GES system driven by a decaying exponential

Now we present a similar discrete-time example. Consider the system

$$\begin{aligned} x_1^+ &= h(x_1 x_2) x_1 \\ x_2^+ &= d \ . \end{aligned} \tag{17}$$

Assume the following:

Assumption 2: The function $s \mapsto h(s)$ is such that

a) it is continuous;

b) $|h(s)| \leq 2$ for all $s \in \mathbb{R}$;

c) $h(0) = 0$;

d) $h(1) = 2$.

Proposition 3: Under Assumption 2, when $d \equiv 0$ the solutions of (17) satisfy

$$|x(k)| \leq 4 \left(\frac{1}{2} \right)^k |x(0)| \tag{18}$$

for each nonnegative integer k.

Proof. Using the fact that $h(0) = 0$, the solution sequence for (17) is given by

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}, \begin{bmatrix} h(x_1(0)x_2(0))x_1(0) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \ldots \tag{19}$$

So the bound (18) follows from $|h(s)| \leq 2$ for all $s \in \mathbb{R}$. ■

Proposition 4: Under Assumption 2, when $x_1(0) \neq 0$, $x_2(0) = x_1(0)^{-1}$, and $d(k) = x_2(0)(1/2)^{k+1}$ for each nonnegative integer k, the solution of (17) satisfies

$$x_1(k) = 2^k x_1(0) \ . \tag{20}$$

Proof. We show that the function proposed in (20), together with the function $x_2(k) = x_2(0)(1/2)^k$, is the solution to the difference equation. It is straightforward to verify that the proposed $x_2(k)$ satisfies the difference equation. We also have, for the proposed solution, that $x_1(k)x_2(k) = x_1(0)x_2(0) = 1$ for all nonnegative integers k. Therefore, using $h(1) = 2$, we have

$$x_1(k+1) = 2x_1(k) = h(x_1(k)x_2(k))x_1(k) \ . \tag{21}$$

Thus the proposed $x_1(k)$ also satisfies the difference equation. ■

B. Additive cascades

Consider the system

$$\begin{aligned} x_1^+ &= h(x_1 x_2)x_1 \\ x_2^+ &= z \\ z^+ &= \frac{1}{2}z \ . \end{aligned} \tag{22}$$

Corollary 3: Under Assumption 2, for the initial condition $x_1(0) \neq 0$, $x_2(0) = x_1(0)^{-1}$, $z(0) = \frac{1}{2}x_2(0)$, the solution of (22) satisfies

$$x_1(k) = 2^k x_1(0) \tag{23}$$

for each nonnegative integer k.

C. Lyapunov functions

As another simple corollary, we have (cf. [2, Exercise 3.54]):

Corollary 4: Let $f(x)$ denote the right-hand side of the system (17) with $d = 0$. There does not exist a positive definite, radially unbounded function V such that, for all $x \in \mathbb{R}^2$, $V(f(x)) \leq V(x)$ and

$$V(x+d) - V(x) \leq (\mu + V(x))\rho(|d|)|d| \tag{24}$$

for some $\mu \in \mathbb{R}_{\geq 0}$ and continuous function ρ.

Proof. Suppose such a function V does exist. Then we would have

$$\begin{aligned} V(f(x)+d) &\leq V(f(x)) + (\mu + V(f(x)))\rho(|d|)|d| \\ &\leq V(x) + (\mu + V(x))\rho(|d|)|d| \end{aligned} \tag{25}$$

and thus the trajectories of the system (17) would satisfy

$$V(x(k+1)) + \mu \leq (V(x(k)) + \mu)\left[1 + \rho(|d(k)|)|d(k)|\right] \ . \tag{26}$$

By induction, we get that

$$\begin{aligned} V(x(k)) + \mu &\leq (V(x(0)) + \mu) \prod_{i=0}^{k-1} \left[1 + \rho(|d(i)|)d(i)\right] \\ &\leq (V(x(0)) + \mu) \exp\left(\sum_{i=0}^{k-1} \rho(|d(i)|)d(i)\right) \end{aligned} \tag{27}$$

and thus $V(x(k))$ is uniformly bounded if $d \in \ell_1$, e.g., d converges to zero exponentially. Since V is radially unbounded in x, this contradicts the conclusion of Proposition 4. ∎

IV. HYBRID

A. *GES system hit by exponentially decaying impulses*

First consider the system (1) with $d = 0$, i.e., the system

$$\begin{aligned} \dot{x}_1 &= g(x_1 x_2)x_1 \\ \dot{x}_2 &= -2x_2 \end{aligned} \tag{28}$$

and with g satisfying the following assumption:

Assumption 3: The function $s \mapsto g(s)$ is such that

a) it is continuous;

b) $|g(s)| \leq 1$ for all $s \in \mathbb{R}$;

c) $g(s) = -1$ for all $s \in (-\infty, 1/2] \bigcup [5/2, \infty)$;

d) $g(s) = 1$ for all $s \in [1, 2]$.

The proof of the following proposition is just like the proof of Proposition 1 and thus is omitted.

Proposition 5: Under Assumption 3, the solutions of (28) satisfy

$$|x(t)| \leq 25e^{-t}|x(0)| \qquad \forall t \geq 0 \ . \tag{29}$$

Now consider the behavior of the system (28) under the influence of exponentially decaying, additive impulsive disturbances, i.e., at each time instance $t_k = kh$, where h is a positive constant, we reset x_2 according to

$$x_2^+(t_k) = x_2(t_k) + d_k \qquad \forall k \geq 0 \tag{30}$$

where $x_2^+(t)$, $t \geq 0$ denotes the limit from above of x_2 at time t and d_k is an exponentially decaying sequence. We then continue to integrate the differential equation (28) until the time t_{k+1}. We call h the "reset period" and we will refer to the resulting system as "the system (28) with additive impulses (30)".

Proposition 6: Under Assumption 3, for each reset period $h \in (0, \log(2)]$, when $x_1(0) \neq 0$, $x_2(0) = 2e^{-h}/x_1(0)$, and $d_k = (1 - e^{-h})e^{-(k-1)h}x_2(0)$ for each nonnegative integer k, a solution of (28) with additive impulses (30) satisfies $x_1(t) = e^t x_1(0)$ for all $t \geq 0$.

Proof. The solution for the x_2 component of the system with additive impulses can be written explicitly as

$$x_2(t) = e^{(k+1)h-2t}x_2(0) \qquad \forall t \in (kh, (k+1)h]. \tag{31}$$

Indeed, for this signal we have $\dot{x}_2 = -2x_2$ on the intervals $(kh, (k+1)h]$, $k \geq 0$; and at times kh, $k \geq 0$ we have that

$$x_2(kh) + d_k = e^{kh-2kh}x_2(0) + (1 - e^{-h})e^{-(k-1)h}x_2(0) = e^{-(k-1)h}x_2(0) = x_2^+(kh). \tag{32}$$

To see that $x_1(t) = e^t x_1(0)$ is a solution to the system (28) with additive impulses, note that this function satisfies $x_1(t)x_2(t) = 2e^{-(t-kh)} \in [1, 2]$ for all $t \in (kh, (k+1)h]$ since $e^{-h} \geq 1/2$. Therefore $g(x_1(t)x_2(t)) = 1$, $\forall t \geq 0$ and we have that $\dot{x}_1 = x_1 = g(x_1x_2)x_1$. ■

B. Additive cascades

Consider the hybrid system

$$\begin{array}{rcl} \dot{x}_1 & = & g(x_1x_2)x_1 \\ \dot{x}_2 & = & -2x_2 \qquad\qquad \dot{z} = -z \\ x_2^+ & = & x_2 + z \end{array} \tag{33}$$

with reset times $t_k = kh$ where k ranges over the nonnegative integers. This system is the additive cascade of GES subsystems since the z-subsystem is GES and the (x_1, x_2) subsystem is GES when $z = 0$. The following result follows from Proposition 6.

Corollary 5: Under Assumption 3, for each reset period $h \in (0, \log(2)]$, and initial conditions $x_1(0) \neq 0$, $x_2(0) = 2e^{-h}/x_1(0)$, $z(0) = (1 - e^{-h})e^h x_2(0)$, a solution of (33) satisfies $x_1(t) = e^t x_1(0)$ for all $t \geq 0$.

V. Conclusion

We have extended the results in [7] by constructing an example where the disturbance can be a simple decaying exponential, and by making explicit that the system can be globally exponentially stable (GES) with zero input and can have linear sector growth. This enables making observations about additive cascades of globally exponentially stable systems, and about the gradients of Lyapunov functions for globally exponentially stable systems with linear sector growth. We have also provided similar examples for discrete-time and hybrid systems.

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