Stability of Delay Impulsive Systems with Application to Networked Control Systems

Payam Naghshtabrizi, João P. Hespanha, Andrew R. Teel

Abstract

We establish asymptotic and exponential stability theorems for delay impulsive systems by employing Lyapunov functionals with discontinuities. Our conditions have the property that when specialized to linear delay impulsive systems, the stability tests can be formulated as Linear Matrix Inequalities (LMIs). Then we consider Networked Control Systems (NCSs) consisting of an LTI process and a static feedback controller connected through a communication network. Due to the shared and unreliable channels, sampling intervals become uncertain and variable. Moreover, samples may be dropped or experience uncertain and variable delays before arriving at the destination. We show that the resulting NCSs can be modeled by linear delay impulsive systems and we provide conditions for stability of the closed-loop system in terms of LMIs. By solving these LMIs, one can find a positive constant that determines an upper bound between each sampling time and the subsequent input update time, for which stability of the closed-loop system is guaranteed.

Index Terms

Delay Impulsive systems, Stability, Networked Control Systems, delay hybrid systems

I. Introduction

Impulsive dynamical systems are special class of hybrid systems which exhibit continuous evolutions described by Ordinary Differential Equations (ODEs) and instantaneous state jumps

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Naghshtabrizi and Hespanha were supported by the Institute for Collaborative Biotechnologies through grant DAAD19-03-D-0004 from the U.S. Army Research Office and by the National Science Foundation under Grant no. CNS-0720842. Teel was supported by the Air Force Office of Scientific Research under Grant no. FA9550-06-1-0134, and National Science Foundation under Grant no. ECS-0622253 .

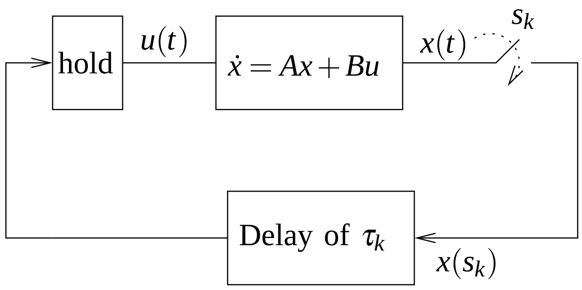


Fig. 1. NCSs with delay in the feedback loop where $u(t) = x(s_k)$, $\forall t \in [s_k + \tau_k, s_{k+1} + \tau_{k+1})$

or impulses. Motivated by systems with delay, we are interested in studying delay impulsive systems. We establish stability, asymptotic stability, and exponential stability theorems for delay impulsive systems by employing functionals with discontinuities at a countable set of times.

By defining the time lag space and other related concepts, criteria for the uniform stability and uniform asymptotic stability for Hybrid Dynamical Systems (HDSs) with time delays were constructed in [1, 2] using Razumikhin's Theorem. The same authors apply these results to analyze the stability of delay impulsive systems and nonlinear sampled-data feedback control systems with time delay. Michel et al. [3] present Lyapunov-Krasovskii type stability results and converse theorems for HDSs with time delay. Based on Lyapunov-Krasovskii functionals, [4] analyzes a class of HDSs consisting of delay differential equations with discontinuities. The authors consider a unified framework for wide classes of HDSs and provide different types of stability and converse theorems by employing a positive definite discontinuous functional. If the functional is bounded between discontinuities and "appropriately" decreases at the point of discontinuities then one obtains an "appropriate" notion of stability (such as uniform stability, asymptotic stability or exponential stability).

This paper provides novel sufficient conditions for different notions of stability: these conditions require the Lyapunov functional to have a negative derivative between impulses and to be non-increase at the points of discontinuity. An advantage of these results over those in [3, 4] is that to verify the conditions in this paper, one does not need to compute the solution of the system between discontinuities. In fact, a distinguishing feature of the stability conditions in this paper is that, when specialized to linear impulsive systems, the stability tests can be formulated as LMIs that can be solved efficiently.

As a special case of general delay impulsive systems, we study *linear delay impulsive systems* such as the one in Figure 1, which can be expressed by

$$\dot{x}(t) = Ax(t) + Bx(s_k), \qquad t_k \le t < t_{k+1}, k \in \mathbb{N}, \tag{1}$$

where s_k denotes the *k-th sampling time* and t_k the so called *k-th input update time*, which is the time instant at which the k-th sample arrives to the destination. In particular, denoting by τ_k the total delay that the k-th sample experiences in the loop, then $t_k := s_k + \tau_k$. Figure 1 and equation (1) can be viewed as modeling an NCS in which a linear process $\dot{x}(t) = Ax(t) + B_u u(t)$ is in feedback with a static state-feedback remote controller with gain K. This would correspond to $B := B_u K$ in (1).

We introduce a new *discontinuous Lyapunov functional* to establish the stability of (1) based on the theorems developed here for general nonlinear time-varying delay impulsive systems. The Lyapunov functional is discontinuous at the input update times, but its decrease is guaranteed by construction. We provide an inequality that guarantees the decrease of the Lyapunov functional between the discontinuities, from which stability follows. This inequality is expressed as a set of LMIs that can be solved numerically using software packages such as MATLAB. By solving these LMIs, one can find a positive constant that determines an upper bound between the sampling time s_k and the next input update time t_{k+1}, for which the stability of the closed-loop system is guaranteed for given lower and upper bounds on the total delay τ_k. When there is no delay, this upper bound corresponds to the maximum sampling interval, which is often called τ_{MATI} in the NCS literature. We use the τ_{MATI} terminology also for the case when there are delays in the system, which allows us to state our result in the form: the system (1) is exponentially stable for any sampling-delay sequence satisfying $t_{k+1} - s_k \le \tau_{MATI}$ and $\tau_{\min} \le \tau_k \le \tau_{\max}$ for $\forall k \in \mathbb{N}$, where $\tau_{\min}$, $\tau_{\max}$, and τ_{MATI} appear in our LMIs.

To reduce network traffic in NCSs, significant work has been devoted to finding values for τ_{MATI} that are not overly conservative (see [5] and references therein). First we review the related work in which there is no delay in the control loop. In [6], τ_{MATI} is computed for linear and nonlinear systems with Round-Robin (static) or Try-Once-Discard (TOD) (dynamic) protocols. Nesic et al. [7, 8] study the input-output stability properties of nonlinear NCSs based on a small gain theorem to find τ_{MATI} for NCSs. [9–11] consider linear NCSs and formulate the problem of finding τ_{MATI} as LMIs. In the presence of variable delays in the control loop, [12–14] show that for a given lower bound $\tau_{\min}$ on the delay in the control loop, stability can be guaranteed for a less conservative τ_{MATI} than in the absence of the lower bound.

Our stability conditions depends both on the lower bound ($\tau_{\min}$) and the upper bound ($\tau_{\max}$) of the loop delay, which can be estimated (perhaps conservatively) for most networks [15]. Through an example we show that considering a finite $\tau_{\max}$ can significantly reduce conservativeness. This improvement is achieved by the use of Lyapunov functionals with discountinuties and the judicious introduction of slack matrices. The introduction of discontinuities in the Lyapunov functionals seems natural in NCSs because sampling and control updates lead to state discontinuities. Slack [16, 17] or free weight matrices [18] introduce degrees of freedom that can be exploited to minimize conservativeness. When the delay in the feedback loop is small (i.e., as $\tau_{\min}, \tau_{\max} \to 0$), our LMIs reduce to the ones presented in [11], which are less conservative than those in [9, 10]. This observation shows that the results in [11] are robust with respect to small delays.

The remainder of this paper is organized as follows: In Section II we present asymptotic and exponential stability tests for time-varying nonlinear delay impulsive systems. In Section III we model NCSs as delay impulsive systems and based on the theorems from Section II we provide stability conditions for (linear) NCSs in the form of LMIs. In Section IV we apply our method to a benchmark example and we compare our results to the ones in the literature. Section V is devoted to conclusions and future work.

Notation: We denote the transpose of a matrix P by P' and its smallest and the largest eigenvalues by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively. We write $P > 0$ (or $P < 0$) when P is a symmetric positive (or negative) definite matrix and we write a partitioned symmetric matrix $\left[\begin{smallmatrix} A & B \\ B' & C \end{smallmatrix}\right]$ as $\left[\begin{smallmatrix} A & B \\ * & C \end{smallmatrix}\right]$. The notations 0_{ij} and I_k are used to denote the $i \times j$ matrix with zero entries and the $k \times k$ identity matrix, respectively. When there is no confusion about the size of such matrices we drop the subscript.

A function $\alpha \in [0,\infty) \to [0,\infty)$ is of class $\mathscr{K}$, and we write $\alpha \in \mathscr{K}$ when α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then we say it is of class $\mathscr{K}_\infty$ and we write $\alpha \in \mathscr{K}_\infty$. A (continuous) function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of class $\mathscr{K}\mathscr{L}$, and we write $\beta \in \mathscr{K}\mathscr{L}$ when, $\beta(.,t)$ is of class $\mathscr{K}$ for each fixed $t \geq 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each fixed s.

Given an interval $I \subset \mathbb{R}$, $B(I,\mathbb{R}^n)$ denotes the space of real functions from I to $\mathbb{R}^n$ with norm

$\|\phi\| := \sup_{t \in I} |\phi(t)|, \phi \in B(I, \mathbb{R}^n)$, where $|.|$ denotes any one of the equivalent norms in $\mathbb{R}^n$. Given a time function $x : [0,\infty) \to \mathbb{R}^n$, x_t denotes the function $x_t : [-r, 0] \to \mathbb{R}^n$ defined by $x_t(\theta) = x(t+\theta)$, where r is a fixed positive constant. We denote the limit from below of a signal $x(t)$ by $x^-(t)$, i.e., $x^-(t) := \lim_{\tau \uparrow t} x(\tau)$; and $\dot{x}(t)$ denotes the right-hand side derivative of x with respect to t, i.e., $\dot{x}(t) = \lim_{\tau \downarrow t} \frac{x(\tau)-x(t)}{\tau - t}$.

II. Stability of delay impulsive systems

Consider the following delay impulsive system

$$\dot{x}(t) = f_k(x(t), t), \qquad \forall t \neq t_k,\ k \in \mathbb{N}, \tag{2a}$$

$$x(t_k) = g_k(x^-(t_k), x^-(s_k), t_k), \qquad \forall t = t_k,\ k \in \mathbb{N}, \tag{2b}$$

where f_k and g_k are locally Lipschitz functions [19] such that $f_k(0,t) = 0$, $g_k(0,0,t) = 0$, $\forall t \in [0,\infty)$; $\{t_k : k \in \mathbb{N}\}$ is a monotone increasing sequence of times at which the state x is updated through (2b); and $\{s_k : k \in \mathbb{N}\}$ is a monotone increasing sequence of times at which the state is sampled for the update law in (2b). The sequence of update times is assumed finite or unbounded. We assume causality in the sense that each sampling time s_k must precede the corresponding update time t_k (although not necessarily strictly) and call $\{\tau_k := t_k - s_k \geq 0 : k \in \mathbb{N}\}$ the delay sequence. We call the system (2) a delay impulsive system since the reset map (2b) depends on the past value of state. It should be emphasized that we allow the delays τ_k to grow larger than the sampling intervals $s_{k+1} - s_k$, provided that the sequence of input update times $\{t_k\}$ remains strictly increasing.

We can view (2) as an infinite dimensional system whose state contains the past history of $x(\cdot)$ so that $x(s_k)$ can be recovered from the state $x_{t_k} := x(t_k + \theta), -\tau_{MATI} \leq \theta \leq 0$ in order to apply the reset map in (2b). This motivates the use of Lyapunov-Krasovskii tools in the analysis of (2). In this framework, it is possible to analyze (2) even when the delays grow much larger than the sampling intervals, which is not easy in methods based on a discretization of (2) between update times [20, 21].

We assume that the pair of impulse-delay sequences $(\{s_k\}, \{\tau_k\})$ belongs to a given set $\mathscr{S}$ and we consider different notions of stability definitions for (2):

(a) The system (2) is said to be *Globally Uniformly Stable* (GUS) over $\mathscr{S}$, if there exists some $\alpha \in \mathscr{K}$ such that for every pair $(\{s_k\}, \{\tau_k\}) \in \mathscr{S}$ and every initial condition x_{t_0} the solution to

(2) is globally defined and satisfies $|x(t)| \leq \alpha(\|x_{t_0}\|), \forall t \geq t_0$.

(b) The system (2) is said to be *Globally Asymptotically Stable* (GAS) over $\mathscr{S}$, if in addition to (a), every solution converges to zero as $t \to \infty$.

(c) The system (2) is said to be *Globally Uniformly Asymptotically Stable* (GUAS) over $\mathscr{S}$, if there exists some $\beta \in \mathscr{K}\mathscr{L}$ such that for every $(\{s_k\},\{\tau_k\}) \in \mathscr{S}$ and every initial condition x_{t_0} the solution to (2) is globally defined and satisfies $|x(t)| \leq \beta(\|x_{t_0}\|, t-t_0), \forall t \geq t_0$.

(d) The system (2) is said to be *Globally Uniformly Exponentially Stable* (GUES) over $\mathscr{S}$, when the function β in (c) is of the form $\beta(s,r) = ce^{-\lambda r}s$ for some $c, \lambda > 0$.

Theorem 1. *Suppose that there exist $\psi_1, \psi_2 \in \mathscr{K}_\infty$, $\psi_3 \in \mathscr{K}$ and a functional $V : B([-r,0],\mathbb{R}^n) \times [0,\infty) \to [0,\infty)$, such that*

$$\psi_1(|\phi(0)|) \leq V(\phi,t) \leq \psi_2(\|\phi\|), \quad \forall \phi \in B(I,\mathbb{R}^n),\ t \geq 0, \tag{3}$$

and, for every $(\{s_k\},\{\tau_k\}) \in \mathscr{S}$, any solution x to (2) *is globally defined for $t \geq t_0$, $V(x_t,t)$ is continuously differentiable between update times, and*

$$\frac{\mathrm{d}V(x_t,t)}{\mathrm{d}t} \leq -\psi_3(|x(t)|), \qquad t \neq t_k, \forall k \in \mathbb{N}, \tag{4}$$

$$V(x_{t_k},t_k) \leq \lim_{t \uparrow t_k} V(x_t,t), \qquad \forall k \in \mathbb{N}. \tag{5}$$

Then the system (2) *is GUS over $\mathscr{S}$. In addition, the following statements hold:*

(a) The system (2) *is GUAS over $\mathscr{S}$ if there is a constant $h_{\min} > 0$ for which $t_{k+1} - t_k \geq h_{\min}, \forall k \in \mathbb{N}$ for every $(\{s_k\},\{\tau_k\}) \in \mathscr{S}$.*

(b) The system (2) *is GUES over $\mathscr{S}$ if the functions ψ_1, ψ_2 are of the following forms:*

$$\psi_1(|\phi(0)|)) := c_1|\phi(0)|^b, \quad \psi_2(\|\phi\|) := c_2\|\phi\|^b, \tag{6}$$

and the following condition holds instead of (4)*:*

$$\frac{\mathrm{d}V(x_t,t)}{\mathrm{d}t} \leq -c_3\|x_t\|^b, \qquad \forall t_k \leq t < t_{k+1},\ k \in \mathbb{N} \tag{7}$$

for some positive constants c_1, c_2, c_3, and b.

(c) The system (2) *is GUES over $\mathscr{S}$ if the functions ψ_1, ψ_3 are of the following forms:*

$$\psi_1(|\phi(0)|) := d_1|\phi(0)|^b, \quad \psi_3(|x(t)|) := d_3|x(t)|^b,$$

and the upper bound $\psi_2(\|\phi\|)$ *in* (3) *is replaced by*

$$d_2|\phi(0)|^b + \bar{d_2}\int_{t-r}^{t}|\phi(s)|^b ds,$$

for some positive constants $d_1, d_2, \bar{d_2}, d_3$ *and* b. $\square$

Items (b) and (c) in Theorem 1 both provide alternative conditions to guarantee GUES over $\mathscr{S}$. The former poses milder conditions on the Lyapunov functional than the latter, but it poses a more strict condition on the time derivative of the functional. We shall see shortly that item (c) will lead to sufficient conditions in terms of LMIs for linear impulsive systems.

Proof of Theorem 1: For an arbitrary pair $(\{s_k\},\{\tau_k\}) \in \mathscr{S}$, let us define

$$v(t) := V(x_t, t), \quad \forall t \geq 0,$$

along the corresponding solution to (2). Between update times, $v(t)$ is continuous differentiable and we have $\dot{v}(t) \leq 0$ for $\forall t \in (t_k, t_{k+1})$, $k \in \mathbb{N}$, therefore

$$\psi_1(|x(t)|) \leq v(t) \leq v(t_k), \quad t \in [t_k, t_{k+1}). \tag{8}$$

Based on the conditions (4) and (5), we also have

$$v^-(t_{k+1}) \leq v(t_k) \leq v^-(t_k), \quad \forall k \in \mathbb{N}. \tag{9}$$

Combining (8) and (9), we conclude that

$$\psi_1(|x(t)|) \leq v(t) \leq v^-(t_k) \leq \cdots \leq v^-(t_1) \leq V(x_{t_0}, t_0) \leq \psi_2(\|x_{t_0}\|), \quad \forall k \in \mathbb{N}. \tag{10}$$

From (10), Lyapunov stability follows since $|x(t)| \leq \alpha(\|x_{t_0}\|)$, $\forall t \geq t_0$ for $\alpha(.) := \psi_1^{-1}(\psi_2(.))$.

(a) for every $\varepsilon > 0$ let $\delta_1 > 0$ be such that $\psi_2(\delta_1) \leq \psi_1(\varepsilon)$. Then $\|x_{t_0}\| \leq \delta_1$ implies that $|x(t)| < \varepsilon, t \geq t_0$ because of (10). For this δ_1 and any $\eta > 0$, we show that there exists a $T = T(\delta_1, \eta)$ such that $|x(t)| \leq \eta$ for $\forall t \geq t_0 + T$. Choose $\delta_2 > 0$ such that $\psi_2(\delta_2) \leq \psi_1(\eta)$ for $t \geq t_0 + T$. Then it suffices to show that $\|x_{t_0+T}\| < \delta_2$ which implies $|x(t)| < \eta, \forall t \geq 0$. By contradiction we assume that such a T does not exist and therefore there exists an infinite sequence $c_k, k \in \mathbb{N}$ such that $\|x_{c_k}\| > \delta_2$. Each c_k is in an interval $[t_{k_i}, t_{k_{i+1}})$ where t_{k_i} is a subsequence of t_k. Since $t_{k+1} - t_k \geq h_{\min}, \forall k \in \mathbb{N}$ then either $c_k - t_{k_i} \geq \frac{h_{\min}}{2}$ or $t_{k_{i+1}} - c_k \geq \frac{h_{\min}}{2}$. We define intervals

$$I_k := \begin{cases} [c_k - \frac{\delta_2}{2L_1}, c_k] & \text{if } c_k - t_{k_i} \geq \frac{h_{\min}}{2} \\ [c_k, c_k + \frac{\delta_2}{2L_1}] & \text{if } t_{k_{i+1}} - c_k \geq \frac{h_{\min}}{2} \end{cases},$$

where $L_1 > \max(L, \frac{\delta_2}{h_{\min}})$ and $|f_k(x,t)| < L$ for $\forall k \in \mathbb{N}$ (since f_k is Lipschitz, there exists $L > 0$ such that $|f_k(x,t)| < L$). By construction, $x(t)$ is continuous for any $t \in I_k$ and we can use the Mean Value Theorem. So for any $t \in I_k$ there exists a $\theta \in [0,1]$ such that

$$\begin{aligned}|x(t)| &= |x(c_k) + \dot{x}(c_k + \theta(t - c_k))(t - c_k)| \\ &\geq |x(c_k)| - |\dot{x}(c_k + \theta(t - c_k))|\,(|t - c_k|) \geq \delta_2 - L\frac{\delta_2}{2L_1} \geq \frac{\delta_2}{2}.\end{aligned}$$

Therefore $\dot{v}(t) \leq -\psi_3(\frac{\delta_2}{2})$ for any $t \in I_k$ and elsewhere v cannot increase. By integration we conclude that

$$V(x_{c_k}, c_k) \leq V(x_{t_0}, t_0) - \psi_3\Big(\frac{\delta_2}{2}\Big)\,\frac{k\delta_2}{2L_1},$$

but this would imply that $V(x_{c_k}, c_k) < 0$ for a sufficiently large k. By contradiction, we then conclude that the system is GUAS over $\mathscr{S}$.

(b) Inequalities (3) with the choice of (6) and (7) implies

$$\dot{v}(t) \leq -\frac{c_3}{c_2}v(t).$$

By the Comparison Lemma [19] and (5), we conclude that $v(t) \leq V(x_{t_0}, t_0)e^{-\frac{c_3}{c_2}(t-t_0)}$. Hence

$$\begin{aligned}|x(t)| &\leq \Big(\frac{v(t)}{c_1}\Big)^{1/b} \leq \Big(\frac{V(x_{t_0}, t_0)e^{-\frac{c_3}{c_2}(t-t_0)}}{c_1}\Big)^{1/b} \\ &\leq \Big(\frac{c_2\|x_{t_0}\|^b e^{\frac{-c_3}{c_2}(t-t_0)}}{c_1}\Big)^{1/b} = (\frac{c_2}{c_1})^{1/b}\|x_{t_0}\|e^{-\frac{c_3}{c_2 b}(t-t_0)}.\end{aligned}$$

Thus, the origin is GUES over $\mathscr{S}$.

(c) Defining $w(t) := e^{\varepsilon(t-t_0)}v(t)$, $\forall t$ we conclude from (3)–(4) that for $\forall t \neq t_k, k \in \mathbb{N}$

$$\begin{aligned}\dot{w}(t) &= \varepsilon e^{\varepsilon(t-t_0)}v(t) + e^{\varepsilon(t-t_0)}\dot{v}(t) \\ &\leq \varepsilon e^{\varepsilon(t-t_0)}\Big(d_2|x(t)|^b + \bar{d_2}\int_{t-r}^{t}|x(v)|^b dv\Big) - d_3 e^{\varepsilon(t-t_0)}|x(t)|^b.\end{aligned} \tag{11}$$

By integration (11) over each interval (t_k, t_{k+1}) and using (5) and (11), we obtain

$$\begin{aligned}w(t) - w(t_0) &\leq \int_{t_k}^{t}\dot{w}(t)dt + (w(t_k) - w^-(t_k)) + \int_{t_{k-1}}^{t_k}\dot{w}(t)dt + \cdots + (w(t_1) - w^-(t_1)) + \int_{t_0}^{t_1}\dot{w}(t)dt \\ &\leq \varepsilon d_2\int_{t_0}^{t}e^{\varepsilon(s-t_0)}|x(s)|^b ds + \varepsilon\bar{d_2}\int_{t_0}^{t}\int_{s-r}^{s}e^{\varepsilon(s-t_0)}|x(\sigma)|^b d\sigma ds - d_3\int_{t_0}^{t}e^{\varepsilon(s-t_0)}|x(s)|^b ds.\end{aligned} \tag{12}$$

By changing the order of integration, one can show that

$$\int_{t_0}^{t}\int_{s-r}^{s} e^{\varepsilon(s-t_0)}|x(\sigma)|^b d\sigma ds \leq \int_{t_0-r}^{t_0}\int_{t_0}^{\sigma+r} e^{\varepsilon(s-t_0)}|x(\sigma)|^b ds d\sigma +$$
$$\int_{t_0}^{t}\int_{\sigma}^{\sigma+r} e^{\varepsilon(s-t_0)}|x(\sigma)|^b ds d\sigma \leq re^{\varepsilon r}\int_{t_0-r}^{t_0}|x(\sigma)|^b d\sigma + re^{\varepsilon r}\int_{t_0}^{t} e^{\varepsilon(\sigma-t_0)}|x(\sigma)|^b d\sigma. \quad (13)$$

Combining (12), (13) and the fact that

$$w(t_0) \leq d_2|x(t_0)|^b + \bar{d_2}\int_{t_0-r}^{t_0}|x(s)|^b ds$$

we conclude that

$$w(t) \leq d_2|x(t_0)|^b + \bar{d_2}(1+\varepsilon re^{\varepsilon r})\int_{t_0-r}^{t_0}|x(\sigma)|^b d\sigma + (\varepsilon\bar{d_2}re^{\varepsilon r} - d_3)\int_{t_0}^{t} e^{\varepsilon(\sigma-t_0)}|x(\sigma)|^b d\sigma.$$

and, for sufficiently small $\varepsilon > 0$, this leads to

$$w(t) \leq d_2|x(0)|^b + \bar{d_2}(1+\varepsilon re^{\varepsilon r})\int_{t_0-r}^{t_0}|x(\sigma)|^b d\sigma. \quad (14)$$

We thus finally conclude that if (14) holds then there exists a $d_4 > 0$ such that $w(t) \leq d_4\|x_{t_0}\|^b$, which means that $v(t) \leq d_4 e^{-\varepsilon(t-t_0)}\|x_{t_0}\|^b$ and consequently $x(t) \leq (\frac{d_4}{d_1})^{1/b} e^{-\frac{\varepsilon}{b}(t-t_0)}\|x_{t_0}\|$. ■

III. NCSs WITH VARIABLE SAMPLING AND DELAY

Consider an NCS consisting of an LTI process with state space model of the form $\dot{x}(t) = Ax(t) + B_u u(t)$, $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and a state-feedback controller with constant gain K connected through sample and hold blocks. At time s_k, $k \in \mathbb{N}$ the process's state $x(s_k)$ is sent to the controller and some time after the control command $Kx(s_k)$ is sent back to the process. This command should be used as soon as it arrives and held constant until the next control command update. The total delay in the control loop that the k-th sample experiences is denoted by τ_k. We allow the delays τ_k to grow larger than the sampling intervals $s_{k+1} - s_k$, provided that the sequence of input update times $\{t_k\}$ remains strictly increasing. In essence, this means that if a sample gets to the destination out of order (i.e., an old sample gets to the destination after the most recent one), it should be dropped. The resulting closed-loop system can written as

$$\dot{x}(t) = Ax(t) + Bx(s_k), \qquad t_k \leq t < t_{k+1}, \quad (15)$$

where $t_k := s_k + \tau_k, B := B_u K$ and this system is depicted in Figure 1.

As in [12–14], we consider sets $\mathscr{S}$ of sampling-delay sequences $(\{s_k\},\{\tau_k\})$ characterized by the following inequalities

$$\tau_{\min} \leq \tau_k \leq \tau_{\max}, \qquad s_{k+1}+\tau_{k+1}-s_k \leq \tau_{MATI}, \tag{16}$$

where $\tau_{\min} \geq 0$ and $\tau_{\max} \geq \tau_{\min}$ denote, respectively, the minimum and maximum delays encountered by the samples; and τ_{MATI} denotes the maximum time span[1] between the time s_k at which the state is sampled and the time $t_{k+1} := s_{k+1}+\tau_{k+1}$ at which the *next* update arrives at the controller.

The closed-loop NCS given by (15) can be modeled by the following delay impulsive system

$$\dot{\xi}(t) = F\xi(t), \qquad t_k \leq t < t_{k+1}, \tag{17a}$$

$$\xi(t_{k+1}) = \begin{bmatrix} x^-(t_{k+1}) \\ x(s_{k+1}) \end{bmatrix}, \qquad k \in \mathbb{N}, \tag{17b}$$

where

$$F := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \qquad \xi(t) := \begin{bmatrix} x(t) \\ z_1(t) \end{bmatrix}, \qquad z_1(t) := x(s_k), \quad t_k \leq t < t_{k+1}.$$

Consider the Lyapunov functional

$$\begin{aligned} V := x'Px + \int_{t-\rho_1}^{t} (\rho_{1\max} - t + s)\dot{x}'(s)R_1\dot{x}(s)ds + \\ \int_{t-\rho_2}^{t} (\rho_{2\max} - t + s)\dot{x}'(s)R_2\dot{x}(s)ds + \int_{t-\tau_{\min}}^{t} (\tau_{\min} - t + s)\dot{x}'(s)R_3\dot{x}(s)ds + \\ \int_{t-\rho_1}^{t-\tau_{\min}} (\rho_{1\max} - t + s)\dot{x}'(s)R_4\dot{x}(s)ds + (\rho_{1\max} - \tau_{\min}) \int_{t-\tau_{\min}}^{t} \dot{x}'(s)R_4\dot{x}(s)ds + \\ \int_{t-\tau_{\min}}^{t} x'(s)Zx(s)ds + (\rho_{1\max} - \rho_1)(x-w)'X(x-w), \end{aligned} \tag{18}$$

with $P,,X,Z,R_i, i = 1,..,4$ positive definite matrices and

$$w(t) := x(t_k), \qquad \rho_1(t) := t - s_k, \qquad \rho_2(t) := t - t_k, \quad t_k \leq t < t_{k+1},$$

$$\rho_{1\max} := \sup_{t\geq 0} \rho_1(t), \qquad \rho_{2\max} := \sup_{t\geq 0} \rho_2(t).$$

[1]The quantity $s_{k+1}+\tau_{k+1}-s_k$ can also be viewed as the sum of the delay τ_k incurred by the sample plus the length $t_{k+1}-t_k = s_{k+1}+\tau_{k+1}-s_k-\tau_k$ of the time interval during which the corresponding control is held.

In Section VI we show that, when the LMIs in the next theorem are feasible, there exists a constant $d_3 > 0$ such that $\frac{dV(x_t,t)}{dt} \le -d_3|x(t)|^2$. Since it is straightforward to show that the Lyapunov functional (18) satisfies the remaining conditions in Theorem 1, we conclude that the NCS modeled by the delay impulsive system (15) is GUES over the set $\mathscr{S}$ of sampling-delay sequences characterized by (16).

Theorem 2. *The system* (15) *is GUES over the set $\mathscr{S}$ of sampling-delay sequences characterized by* (16) *if there exist positive definite matrices $P, X, Z, R_i, i \in \{1,..,4\}$, (not necessarily symmetric) matrices $N_i, i \in \{1,..,4\}$, and $\varepsilon > 0$ that satisfy the following LMIs:*

$$\begin{bmatrix} M_1+\beta_{\max}(M_2+M_3) & \tau_{\max}N_1 & \tau_{\min}N_3 \\ * & -\tau_{\max}R_1 & 0 \\ * & * & -\tau_{\min}R_3 \end{bmatrix} \le 0, \tag{19a}$$

$$\begin{bmatrix} M_1+\beta_{\max}M_2 & \tau_{\max}N_1 & \tau_{\min}N_3 & \beta_{\max}(N_1+N_2) & \beta_{\max}N_4 \\ * & -\tau_{\max}R_1 & 0 & 0 & 0 \\ * & * & -\tau_{\min}R_3 & 0 & 0 \\ * & * & * & -\beta_{\max}(R_1+R_2) & 0 \\ * & * & * & * & -\beta_{\max}R_4 \end{bmatrix} \le 0, \tag{19b}$$

where

$$\begin{aligned} M_1 :=& \bar{F}'\begin{bmatrix} P & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} P \\ 0 \\ 0 \\ 0 \end{bmatrix}\bar{F} + \tau_{\min}F'(R_1+R_3)F - \begin{bmatrix} I \\ 0 \\ -I \\ 0 \end{bmatrix} X \begin{bmatrix} I \\ 0 \\ -I \\ 0 \end{bmatrix}' + \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} Z \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}' - \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} Z \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix}' \\ & - N_1\begin{bmatrix} I & -I & 0 & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \\ 0 \\ 0 \end{bmatrix} N_1' - N_2\begin{bmatrix} I & 0 & -I & 0 \end{bmatrix} - \begin{bmatrix} I \\ 0 \\ -I \\ 0 \end{bmatrix} N_2' - N_3\begin{bmatrix} I & 0 & 0 & -I \end{bmatrix} - \begin{bmatrix} I \\ 0 \\ 0 \\ -I \end{bmatrix} N_3' \\ & - N_4\begin{bmatrix} 0 & -I & 0 & I \end{bmatrix} - \begin{bmatrix} 0 \\ -I \\ 0 \\ I \end{bmatrix} N_4' + \varepsilon \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}', \\ M_2 :=& \bar{F}'(R_1+R_2+R_4)\bar{F}, \\ M_3 :=& \begin{bmatrix} I \\ 0 \\ -I \\ 0 \end{bmatrix} X\bar{F} + \bar{F}'X\begin{bmatrix} I & 0 & -I & 0 \end{bmatrix}. \end{aligned} \tag{20}$$

with $\bar{F} := \begin{bmatrix} A & B & 0 & 0 \end{bmatrix}$ *and* $\beta_{\max} := \tau_{MATI} - \tau_{\min}$. □

Remark 1. When the delays are negligible (i.e., $\tau_{\min}, \tau_{\max} = 0$) we can show that the LMIs (19a) and (19b) hold if and only if the LMIs in Theorem 2 of [11] hold. This shows that in the absence of delays we recover the results from [11] in which there is no delay in the control loop and the sampling intervals are variable. To do so, we pick $R_3, R_4, Z, N_3, N_4 = 0$ and

$$N_1 = \begin{bmatrix} N_{11} & N_{12} & N_{13} & 0 \end{bmatrix}', \quad N_2 = \begin{bmatrix} N_{21} & N_{22} & N_{23} & 0 \end{bmatrix}',$$

and we omit the all zero rows and columns. Then we multiply the resulted LMIs by $\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & I \end{smallmatrix}\right]$ and its transpose from the left and the right, respectively, and we choose

$$N = \begin{bmatrix} N_{11}+N_{21} & N_{12}+N_{13}+N_{22}+N_{23} \end{bmatrix}', \qquad R_1+R_2 = R,$$

to obtain the LMIs in Theorem 2 of [11], $\square$

Often the lower bound on the delay in the network is very close to zero. This typically occurs when the load in the network is low and the computation delays are small, because in this scenario the total end-to-end delay in the loop is simply equal to the sum of the transmission and propagation delays which are typically small. This motivates considering the special case $\tau_{\min} = 0$. The stability conditions in the corollary below can be derived from the conditions in Theorem 2 with $\tau_{\min} = 0$ or they can be directly derived from Theorem 1 by employing the following Lyapunov functional

$$V := x'Px + \int_{t-\rho_1}^{t} (\rho_{1\max} - t + s)\dot{x}'(s)R_1\dot{x}(s)ds + \\ \int_{t-\rho_2}^{t} (\rho_{2\max} - t + s)\dot{x}'(s)R_2\dot{x}(s)ds + (\rho_{1\max} - \rho_1)(x-w)'X(x-w).$$

Corollary 1. *The system* (15) *is GUES over the set $\mathscr{S}$ of sampling-delay sequences characterized by* (16) *with $\tau_{\min} = 0$, if there exist positive definite matrices P, X, R_1, R_2, (not necessarily symmetric) matrices N_1, N_2, and $\varepsilon > 0$ that satisfy the following LMIs:*

$$\begin{bmatrix} M_1+\beta_{\max}(M_2+M_3) & \tau_{\max}N_1 \\ * & -\tau_{\max}R_1 \end{bmatrix} \le 0,$$

$$\begin{bmatrix} M_1+\beta_{\max}M_2 & \tau_{\max}N_1 & \beta_{\max}(N_1+N_2) \\ * & -\tau_{\max}R_1 & 0 \\ * & * & -\beta_{\max}(R_1+R_2) \end{bmatrix} \le 0,$$

where

$$M_1 := \bar{F}'\begin{bmatrix} P & 0 & 0 \end{bmatrix} + \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix}\bar{F} - \begin{bmatrix} I \\ 0 \\ -I \end{bmatrix} X \begin{bmatrix} I \\ 0 \\ -I \end{bmatrix}' - N_1\begin{bmatrix} I \\ -I \\ 0 \end{bmatrix}' - \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} N_1' - N_2\begin{bmatrix} I \\ 0 \\ -I \end{bmatrix}' - \begin{bmatrix} I \\ 0 \\ -I \end{bmatrix} N_2' + \varepsilon\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}'$$

$$M_2 := \bar{F}'(R_1+R_2+R_4)\bar{F},$$

$$M_3 := \begin{bmatrix} I \\ 0 \\ -I \end{bmatrix} X\bar{F} + \bar{F}'X\begin{bmatrix} I & 0 & -I \end{bmatrix}.$$

with $\bar{F} := \begin{bmatrix} A & B & 0 \end{bmatrix}$. $\square$

IV. Comparison with other results

Since there are no necessary and sufficient conditions for the stability of NCSs such as (15), we will compare our results using the benchmark example in [22] that has been extensively used to compare stability conditions. It consists of the following state-space plant model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u,$$

with state feedback gain $K = -\begin{bmatrix} 3.75 & 11.5 \end{bmatrix}$, for which we have

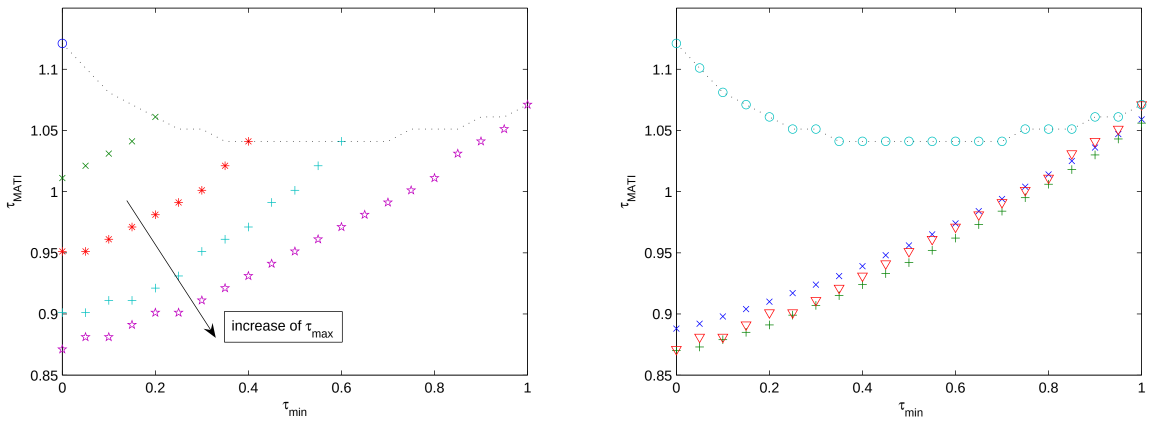
$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \qquad B = -\begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \times \begin{bmatrix} 3.75 & 11.5 \end{bmatrix}.$$

a) Comparison with exact methods for limiting cases: In the *absence of delay and with constant sampling time h* one can perform an exact discretization of (17). By checking the eigenvalues of $\left[\begin{smallmatrix} I & 0 \\ I & 0 \end{smallmatrix}\right] e^{Fh} < 0$ on a tight grid of h, we can show that the closed-loop system remains stable for any constant sampling interval smaller than 1.7, and becomes unstable for larger constant sampling intervals. Theorem 2 guarantees stability for sampling times up to 1.1137. However stability is guaranteed for any variable sampling time up to this upper bound.

When the *sampling interval approaches zero,* the system is described by a DDE and we can find the maximum constant delay for which stability is guaranteed by looking at the roots of the characteristic function $\det(sI - A - Be^{-\tau_0 s})$. Using the Pade approximation $e^{-\tau_0 s} = \frac{1 - s\tau_0/2}{1 + s\tau_0/2}$ to compute the determinant polynomial, we conclude by the Routh-Hurwitz test that the system is stable for any constant delay smaller than 1.36. Theorem 2 guarantees stability for constant delay up to 1.0744.

b) No-delay and variable sampling: When there is no delay but the sampling intervals are variable, τ_{MATI} determines an upper bound on the variable sampling intervals $s_{k+1} - s_k$. The upper bound given by [9, 10, 13] (when $\tau_{\min} = 0$) is 0.8696 which was improved to 0.8871 in [14]. Theorem 2 and [11] give an upper bound equal to 1.1137.

c) Variable-delay and sampling: Figure 2(a) shows the value of τ_{MATI} obtained from Theorem 2 as a function of the minimum delay $\tau_{\min}$, for different values of the maximum delay τ_{max}. The dashed curves in both Figures 2(a) and 2(b) correspond to the largest possible τ_{MATI} for different values of $\tau_{\min}$, which is obtained for the constant delay case $\tau_{\max} = \tau_{\min}$. Figure 2(b)



(a) (b)

Fig. 2. (a) shows a plot of τ_{MATI} versus $\tau_{\min}$, for different values of $\tau_{\max} \in \{0, .2, .4, .6, 1\}$, obtained from Theorem 2. The dash line in both plots corresponds to the constant delay case $\tau_{\max} = \tau_{\min}$, which gives the largest possible value for τ_{MATI} that can be obtained for a given (fixed) delay. (b) shows a plot of τ_{MATI} versus $\tau_{\min}$ for arbitrary delay (bounded below by $\tau_{\min}$ and above by τ_{MATI}) obtained from results in [13] ('$+$') and in [14] ('$\times$'). In this figure we also see plots of τ_{MATI} versus $\tau_{\min}$ obtained from Theorem 2 for $\tau_{\max} = \tau_{\min}$ ('o') and for $\tau_{\max} = \tau_{MATI}$ ('∇'). The former is the most favorable case in terms of getting a large value for τ_{MATI} since it corresponds to the smallest variability in delays, whereas the latter leads to the smallest value for τ_{MATI} since it allows for the largest variability in the delays.

compares the values obtained from Theorem 2 with those obtained from previous results [12–14]. The values of τ_{MATI} obtained from [12] lie between those of [13] and [14] and are not shown explicitly in the figure. The results in [12–14] do not take into consideration specific knowledge on bounds for the delay and guarantee stability for any delays compatible with τ_{MATI}. This results in more conservative conditions and consequently lower values for τ_{MATI}. In Theorem 2, τ_{MATI} is a function of both $\tau_{\min}$ and $\tau_{\max}$ and the results obtained are significantly less conservative when one can explore this extra knowledge. For example, in the extreme case when the delay is known to be fixed ('o' plot in Figure 2(b), for $\tau_{\max} = \tau_{\min}$), the values for τ_{MATI} obtained from Theorem 2 are significantly larger than those obtained in [12–14]. However, when nothing is known about the delay ('∇' plot in Figure 2(b), for $\tau_{\max} = \tau_{MATI}$), then Theorem 2 gives little or no improvement. For most intermediate levels, we do see improvements with respect to prior work, which can be confirmed by comparing the plots in Figure 2(a) obtained from Theorem 2 with those in Figure 2(b) obtained from prior work.

The solution to these examples shows that none of the slack matrix variables introduced is redundant and that they cannot be omitted without introducing conservativeness.

V. CONCLUSION

We established stability, asymptotic stability, and exponential stability theorems for delay impulsive systems. Our stability conditions have the property that when specialized to linear impulsive systems, the stability tests can be formulated as LMIs. Then we considered NCSs consisting of an LTI process and a static feedback controller connected through a communication network. Due to the shared, unreliable channel that connects process and controller, the sampling intervals and delays are uncertain and variable. We showed that the resulting NCSs can be modeled by linear delay impulsive systems. We provided conditions for the stability of the closed-loop expressed in terms of LMIs. By solving these LMIs, one can find a positive constant that determines an upper bound between the sampling time and the next input update time, for which stability of the closed-loop system is guaranteed.

The analysis results presented are less conservative than the existing ones in the literature, however there might be ways to further reduce the conservativeness of the sampling and delay bounds. For instance numerical results suggest that the Lyapunov functional in (18) with the constant matrices $R_i, 1 \leq i \leq 4$ replaced by appropriately selected time-varying matrices $R_i(s), 1 \leq i \leq 4$ leads to LMI stability conditions that appear to be necessary or at least close to it. This type of Lyapunov functionals are inspired by the Lyapunov functionals used for the *discretized method* in the DDE literature [23].

Although in this paper we focused our attention on the issue of stability, it is possible to derive LMI conditions that allow one to determine closed-loop induced norms, leading to H_∞ designs [14]. We plan to extend these results to model more general NCSs, including two-channel NCSs with dynamic feedback controllers.

VI. APPENDIX

Proof of Theorem 2 : It is easy to show that the Lyapunov functional (18) satisfies the condition (3) with

$$\psi_1(s) := d_1 s^2, \qquad \psi_2(\|\phi\|) := d_2|\phi(0)|^2 + \bar{d_2}\int_{t-r}^{t} |\phi(s)|^2 ds,$$

for $d_1, d_2, \bar{d_2} > 0$. Also the condition (5) is guaranteed by construction of the Lyapunov functional. The only remaining condition of Theorem 1 (part c) that is needed to guarantee GUES is (4) and therefore in the remainder of this proof we derive sufficient conditions for $\dot{V} \le -\varepsilon\|x\|^2$ to hold for some $\varepsilon > 0$ [condition in (4)]. Along the trajectory of the system (2), we have that

$$
\begin{aligned}
\dot{V} =& 2x'(t)P(Ax(t)+Bz) + \rho_{1\max}\dot{x}'(t)R_1\dot{x}(t) \\
& - \int_{t-\rho_1}^{t} \dot{x}'(s)R_1\dot{x}(s)ds + \rho_{2\max}\dot{x}'(t)R_2\dot{x}(t) \\
& - \int_{t-\rho_2}^{t} \dot{x}'(s)R_2\dot{x}(s)ds + \tau_{\min}\dot{x}'(t)R_3\dot{x}(t) \\
& - \int_{t-\tau_{\min}}^{t} \dot{x}'(s)R_3\dot{x}(s)ds \\
& + (\rho_{1\max} - \tau_{\min})\left(\dot{x}'(t-\tau_{\min})R_4\dot{x}(t-\tau_{\min})\right) \\
& - \int_{t-\rho_1}^{t-\tau_{\min}} \dot{x}'(s)R_4\dot{x}(s)ds + (\rho_{1\max} - \tau_{\min})\left(\dot{x}'(t)R_4\dot{x}(t)\right. \\
& \left. - \dot{x}'(t-\tau_{\min})R_4\dot{x}(t-\tau_{\min})\right) + x'(t)Zx(t) \\
& - x'(t-\tau_{\min})Zx(t-\tau_{\min}) - (x(t)-w)'X(x(t)-w) \\
& \qquad + 2(\rho_{1\max} - \rho_1)(x-w)'X(Ax+Bz).
\end{aligned} \tag{21}
$$

Defining $\bar{\xi}(t) := \begin{bmatrix} x'(t) & z' & w' & x'(t-\tau_{\min}) \end{bmatrix}'$, for any matrices $N_i, i = 1,..,4$ we have

$$
\begin{aligned}
& 2\bar{\xi}'N_1 \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix} \bar{\xi} + 2\bar{\xi}'N_2 \begin{bmatrix} I & 0 & -I & 0 \end{bmatrix} \bar{\xi} \\
& = 2\bar{\xi}'(N_1+N_2)\Big(\int_{t-\rho_2}^{t} \dot{x}(s)ds\Big) + 2\bar{\xi}'N_1\Big(\int_{t-\rho_1}^{t-\rho_2} \dot{x}(s)ds\Big) \\
& \le \rho_2\bar{\xi}'(N_1+N_2)(R_1+R_2)^{-1}(N_1+N_2)'\bar{\xi} \\
& + \int_{t-\rho_2}^{t} \dot{x}'(s)(R_1+R_2)\dot{x}(s)ds \\
& + (\rho_1-\rho_2)\bar{\xi}'N_1R_1^{-1}N_1'\bar{\xi} + \int_{t-\rho_1}^{t-\rho_2} \dot{x}'(s)R_1\dot{x}(s)ds,
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
& 2\bar{\xi}'N_3 \begin{bmatrix} I & 0 & 0 & -I \end{bmatrix} \bar{\xi} = 2\bar{\xi}'N_3\Big(\int_{t-\tau_{\min}}^{t} \dot{x}(s)ds\Big) \\
& \le \tau_{\min}\bar{\xi}'N_3R_3^{-1}N_3'\bar{\xi} + \int_{t-\tau_{\min}}^{t} \dot{x}'(s)R_3\dot{x}(s)ds,
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
& 2\bar{\xi}'N_4 \begin{bmatrix} 0 & -I & 0 & I \end{bmatrix} \bar{\xi} = 2\bar{\xi}'N_4\Big(\int_{t-\rho_1}^{t-\tau_{\min}} \dot{x}(s)ds\Big) \\
& \le (\rho_1-\tau_{\min})\bar{\xi}'N_4R_4^{-1}N_4'\bar{\xi} + \int_{t-\rho_1}^{t-\tau_{\min}} \dot{x}'(s)R_4\dot{x}(s)ds,
\end{aligned} \tag{24}
$$

which relies on the fact that $x(t)-z(t)=x(t)-x(t-\rho_1)$ and $x(t)-w(t)=x(t)-x(t-\rho_2)$. The matrix variables N_1,N_2,N_3,N_4 represent degrees of freedom that can be exploited to minimize conservativeness and we call them slack matrix variables. Let us define $\beta := \rho_1-\tau_{\min}$ and $\beta_{\max} := \rho_{1\max}-\tau_{\min}$. Note that $\tau_{\min}\le\rho_1-\rho_2\le\tau_{\max}$ and

$$\rho_{1\max} = \sup_k (s_{k+1}+\tau_{k+1}-s_k+\tau_k-\tau_k) \le \rho_{2\max}+\tau_{\max},$$

$$\rho_{2\max}+\tau_{\min} = \sup_k (s_{k+1}+\tau_{k+1}-s_k-\tau_k+\tau_{\min}) \le \rho_{1\max},$$

so we conclude that $\tau_{\min}\le\rho_{1\max}-\rho_{2\max}\le\tau_{\max}$, $\rho_{2\max}\le\beta_{\max}$, and $\rho_2\le\beta$. After combining (21), (22),(23), and (24) and replacing $\rho_{2\max},\rho_2,\rho_1-\rho_2$ with $\beta_{\max},\beta,\tau_{\max}$ we get

$$\dot{V}(\bar{\xi})+\varepsilon\|x\|^2 \le \bar{\xi}'\big(\Psi+\beta_{\max}(M_2+M_3)+\beta(M_4-M_3)\big)\bar{\xi}, \tag{25}$$

$$\Psi := M_1+\tau_{\max}N_1R_1^{-1}N_1'+\tau_{\min}N_3R_3^{-1}N_3',$$

$$M_4 := (N_1+N_2)(R_1+R_2)^{-1}(N_1+N_2)'+N_4R_4^{-1}N_4',$$

and M_1,M_2,M_3 are defined in (20). A necessary and sufficient condition to satisfy $\Psi+\beta_{\max}(M_2+M_3)+\beta(M_4-M_3)\le 0$ for $\forall\beta\in[0,\beta_{\max}]$ is

$$\Psi+\beta_{\max}(M_2+M_3)<0, \quad \Psi+\beta_{\max}(M_2+M_4)<0 \tag{26}$$

The proof is similar to the proof of Theorem 1 of [11]. By Schur complement, the matrix inequalities in (26), can be written as the LMIs in Theorem 2. If the LMIs in Theorem 2 are feasible, then the condition (4) is satisfied with $\psi_3(s):=\varepsilon s^2$ for $\varepsilon>0$. Hence all the conditions in Theorem 1 are satisfied so the system (15) is GUES over $\mathscr{S}$.

■

ACKNOWLEDGMENT

The first author would like to thank Professor Dong Yue for his helpful comments on the proof of Theorem 1.

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