Stability of Switched Systems with Average Dwell-Time*

Technical Report

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Abstract

It is shown that switching among stable linear systems results in a stable system provided that switching is "slow-on-the-average." In particular, it is proved that exponential stability is achieved when the number of switches in any finite interval grows linearly with the length of the interval, and the growth rate is sufficiently small. Moreover, the exponential stability is uniform over all switchings with the above property. For switched systems with inputs this guarantees that several input-to-state induced norms are bounded uniformly over all slow-on-the-average switchings. These results extend to classes of nonlinear switched systems that satisfy suitable uniformity assumptions. In this paper it is also shown that, in a supervisory control context, scale-independent hysteresis can produce switching that is slow-on-the-average and therefore the results mentioned above can be used to study the stability of hysteresis-based adaptive control systems.

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1 Introduction

Consider a family of $n \times n$ stability matrices $\mathcal{A} \triangleq \{A_p : p \in \mathcal{P}\}$ indexed by the parameter p taking values on a set $\mathcal{P}$. It is well known that, even if each matrix in $\mathcal{A}$ is asymptotically stable, the time-varying switched system

$$\dot{x} = A_\sigma x \tag{1}$$

may have unbounded solutions for some "switching signals" σ. Here, a *switching signal* is simply a piecewise constant signal taking values on the index set $\mathcal{P}$. However, it is also known that (1) is exponentially stable if the interval between any two consecutive discontinuities of σ is sufficiently large.

Given a positive constant τ_D, let $\mathcal{S}[\tau_D]$ denote the set of all switching signals with interval between consecutive discontinuities no smaller than τ_D. The constant τ_D is called the *(fixed) dwell-time*. It was shown, e.g., in [1, Lemma 2], that one can pick τ_D sufficiently large so that (1) is exponentially stable for every $\sigma \in \mathcal{S}[\tau_D]$. In fact, there exist positive constants c, λ such that, for each $\sigma \in \mathcal{S}[\tau_D]$,

$$\|\Phi_\sigma(t,\tau)\| \leq ce^{-\lambda(t-\tau)}, \qquad \forall t \geq \tau \geq 0, \tag{2}$$

where[1] $\Phi_\sigma(t,\tau)$ denotes the state transition matrix of (1). In this paper we show that a similar result still holds when $\mathcal{S}[\tau_D]$ is enlarged to contain signals that occasionally have consecutive discontinuities separated by less than τ_D, but for which the average interval between consecutive discontinuities is no less than τ_D. We proceed to formalize this concept of "average dwell-time". For each switching signal σ and each $t \geq \tau \geq 0$, let $N_\sigma(t,\tau)$ denote the number of discontinuities of σ in the open interval (τ, t). For given $N_0, \tau_D > 0$, we denote by $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$ the set of all switching signals for which

$$N_\sigma(t,\tau) \leq N_0 + \frac{t-\tau}{\tau_D}.$$

The constant τ_D is called the *average dwell-time* and N_0 the *chatter bound*. Clearly,

$$\mathcal{S}[\tau_D] \subset \mathcal{S}_{\text{ave}}[\tau_D, 1].$$

In this paper we show that, if τ_D is sufficiently large, a bound like (2) actually holds for every $\sigma \in \mathcal{S}_{\text{ave}}[\tau_D, N_0]$, with arbitrary N_0, and not only for the switching signals in $\mathcal{S}[\tau_D]$.

Systems like (1) arise in an adaptive context when a high-level, logic-based supervisor orchestrates the switching between a family of candidate controllers so as to achieve some desired behavior for the closed-loop system [2, 3, 4, 1, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. The need for switching usually arises from the fact that no single candidate controller would be capable, by itself, of guaranteeing stability and good performance when connected with a poorly modeled process. In several of these algorithms the supervisor guarantees, by construction, that there is a minimum time τ_D between consecutive switchings [2, 3, 4, 1, 5, 6, 7, 8]. The dwell-time τ_D is then a design parameter[2] chosen so that (1) is exponentially stable.

Dwell-time switching supervisors force every candidate controller to remain in the loop for, at least, τ_D units of time, thus guaranteeing a fixed dwell-time of τ_D. Unfortunately, with nonlinear system this may lead to finite escape of the closed-loop. Adaptive switching algorithms for nonlinear system have therefore avoided a fixed dwell-time, and have been mostly[3] based on hysteresis switching [16, 17], or on its more recent scale-independent version [10, 18].

To date, the analysis of algorithms based on hysteresis switching relied heavily on showing that switching stops in finite time [9, 10, 11, 12, 13, 14]. However, in the presence of noise and disturbance inputs, this is

[1] Given a matrix A, we denote by $\|A\|$ the largest singular value of A.

[2] In [1, 6] the dwell-time is actually chosen arbitrarily and (1) is then "massaged" by output injection to make it exponentially stable for that particular dwell-time.

[3] An exception is the supervisor based on dwell-time switching described in [15]. However, because of finite escape, the stability results given are only semi-global.

hardly the case. In fact, the only known switching algorithms for which switching can be proved to stop in finite time, even in the presence of noise/disturbances, are those for which an upper bound on these signals is known a priori, or effectively estimated online [4, 19]. Unfortunately, even in the noiseless case, these algorithms usually lead to bad transient responses.

It turns out that, making use of the results in [18] (cf. Appendix), one can show that, although scale-independent hysteresis switching does not guarantee the existence of a fixed dwell-time between switchings, it can produce switching that is slow-on-the-average. This allows us to analyze supervisory control switching algorithms based on scale-independent hysteresis, even in the presence of noise. To the best of our knowledge, this is the first time that an hysteresis-based switching algorithm is analyzed without relying on switching stopping.

This paper is organized as follows. In Section 2, we formalize the notion of exponential stability, uniform over a class of switching signals. We show that with this type of stability one can compute bounds on several induced input-to-state norms that are uniform over the same class of switching signals. Section 3 contains the main result of this paper, namely that, for linear systems, slow average dwell-time guarantees uniform exponential stability. In Section 4, this result is extended to certain classes of nonlinear systems. Section 5 makes use of previous results in the analysis of a supervisory control algorithm that uses scale-independent hysteresis switching logic. A summary of the results in [18] that are needed for the analysis of this type of logic are recalled in the Appendix. Finally, Section 6 contains some concluding remarks and directions for future research.

2 Uniform Stability

Consider again the switched system

$$\dot{x} = A_\sigma x. \tag{1}$$

Given some family of piecewise constant switching signals $\mathcal{S}$, we say that (1) is *uniformly exponentially stable over* $\mathcal{S}$ if there exist positive constants c and λ such that, for each $\sigma \in \mathcal{S}$,

$$\|\Phi_\sigma(t,\tau)\| \leq ce^{-\lambda(t-\tau)}, \qquad \forall t \geq \tau \geq 0, \tag{3}$$

where $\Phi_\sigma(t,\tau)$ denotes the state transition matrix of (1). When we want to emphasize the rate of decay in the above bound we add that (1) has *stability margin* λ.

Take now a bounded family of $n \times m$ matrices $\mathcal{B} \stackrel{\triangle}{=} \{B_p : p \in \mathcal{P}\}$, also parameterized by the elements of the index set $\mathcal{P}$, and consider the m-input system

$$\dot{z} = A_\sigma z + B_\sigma u \tag{4}$$

where $u : [0,\infty) \to \mathbb{R}^m$ denotes some piecewise continuous input and σ some switching signal in $\mathcal{S}$. We show next that uniform exponential stability of (1) over $\mathcal{S}$ implies that several induced norms of (4) are uniformly bounded over $\mathcal{S}$.

Given a nonnegative constant λ, we say that (4) has *input-to-state* $e^{\lambda t}$*-weighted,* $\mathcal{L}_2$*-induced norm uniformly bounded over* $\mathcal{S}$ if there exist finite constants $\mathfrak{g}$, $\mathfrak{g}_0$ such that, for each piecewise continuous input u and each $\sigma \in \mathcal{S}$,

$$\left(\int_0^t e^{2\lambda\tau}\|z(\tau)\|^2\right)^{\frac{1}{2}} \leq \mathfrak{g}\left(\int_0^t e^{2\lambda\tau}\|u(\tau)\|^2\right)^{\frac{1}{2}} + \mathfrak{g}_0\|z(0)\|, \qquad t \geq 0, \tag{5}$$

along solutions to (4) [4]. If (5) is replaced by

$$e^{\lambda t}\|z(t)\| \leq \mathfrak{g}\sup_{[0,t)} e^{\lambda\tau}\|u(\tau)\| + \mathfrak{g}_0\|z(0)\|, \qquad t \geq 0, \tag{6}$$

[4] Given a vector x, we denote by $\|x\|$ the Euclidean norm of x.

we say that (4) has *input-to-state $e^{\lambda t}$-weighted, $\mathcal{L}_\infty$-induced norm uniformly bounded over $\mathcal{S}$*, and if (5) is replaced by

$$e^{\lambda t}\|z(t)\| \leq \mathfrak{g}\left(\int_0^t e^{2\lambda\tau}\|u(\tau)\|^2\right)^{\frac{1}{2}} + \mathfrak{g}_0\|z(0)\|, \qquad t \geq 0, \tag{7}$$

we say that (4) has *input-to-state $e^{\lambda t}$-weighted, $\mathcal{L}_2$-to-$\mathcal{L}_\infty$-induced norm uniformly bounded over $\mathcal{S}$*. The finiteness of the three induced norms above is crucial to the analysis of adaptive switching algorithms, especially when noise and unmodeled dynamics are taken into account (cf. Section 5 and [3, 4, 1, 5, 6, 7, 8]). The following Lemma is proved in the Appendix.

Lemma 1. *Given a family $\mathcal{S}$ of piecewise constant switching signals, if* (1) *is uniformly exponentially stable over $\mathcal{S}$, with stability margin λ_0, then, for any $\lambda \in [0, \lambda_0)$,* (4) *has input-to-state $e^{\lambda t}$-weighted, $\mathcal{L}_2$-induced norm uniformly bounded over $\mathcal{S}$. Similarly for the $\mathcal{L}_\infty$ and $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ induced norms.*

3 Average Dwell Time

Let us recall that, for given $N_0, \tau_D > 0$, $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$ denotes the set of all switching signals with average dwell time τ_D and chatter bound N_0, i.e., the set of piecewise constant signals σ for which

$$N_\sigma(t,\tau) \leq N_0 + \frac{t-\tau}{\tau_D},$$

where $N_\sigma(t,\tau)$ denote the number of discontinuities of σ on the interval (τ, t). The main result of this paper is that, for any family of $n \times n$ stability matrices $\mathcal{A} \stackrel{\triangle}{=} \{A_p : p \in \mathcal{P}\}$, there is always an average dwell time τ_D such that (1) is uniformly exponentially stable over $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$, for any chatter bound N_0. The next theorem formalizes this observation.

Theorem 2. *Given a compact set of $n \times n$ matrices $\mathcal{A} \stackrel{\triangle}{=} \{A_p : p \in \mathcal{P}\}$ and a positive constant λ_0 such that $A_p + \lambda_0 I$ is asymptotically stable for each $p \in \mathcal{P}$, then, for any $\lambda \in [0, \lambda_0)$, there is a finite constant τ_D^* such that* (1) *is uniformly exponentially stable over $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$, with stability margin λ, for any average dwell-time $\tau_D \geq \tau_D^*$ and any chatter bound $N_0 > 0$.*

Before proving Theorem 2 let us note that, since each matrix $A_p + \lambda_0 I$ is asymptotically stable, there is a set of $n \times n$, symmetric, positive definite matrices $\mathcal{Q} \stackrel{\triangle}{=} \{Q_p : p \in \mathcal{P}\}$ such that

$$Q_p(A_p + \lambda_0 I) + (A_p + \lambda_0 I)' Q_p = -I, \qquad p \in \mathcal{P}. \tag{8}$$

Moreover, since $\mathcal{A}$ is compact, $\mathcal{Q}$ is also compact. We therefore conclude that there is a family of Lyapunov functions

$$\mathcal{V} \stackrel{\triangle}{=} \{V_p : V_p(x) \stackrel{\triangle}{=} x' Q_p x,\ p \in \mathcal{P}\}$$

for the time-invariant systems $\dot{z} = A_p z$, $p \in \mathcal{P}$, with the following properties:

(i) Each V_p is continuous and decreases exponentially along solutions to the time-invariant system $\dot{z} = A_p z$. In particular,

$$\frac{\partial V_p}{\partial x} A_p x \leq -2\lambda_0 V_p, \qquad \forall x \in \mathbb{R}^n,\ p \in \mathcal{P}. \tag{9}$$

(ii) There are functions $\alpha, \bar{\alpha}$ of class[5] $\mathcal{K}_\infty$ such that

$$\alpha(\|x\|) \leq V_p(x) \leq \bar{\alpha}(\|x\|), \qquad \forall x \in \mathbb{R}^n,\ p \in \mathcal{P}. \tag{10}$$

[5] We denote by $\mathcal{K}$ the set of all continuous functions $\alpha : [0, \infty) \to [0, \infty)$ that are zero at zero, strictly increasing, and continuous, and by $\mathcal{K}_\infty$ the subset of $\mathcal{K}$ consisting of those functions that are unbounded.

(iii) There is a positive constant μ such that

$$V_p(x) \leq \mu V_q(x), \qquad \forall x \in \mathbb{R}^n,\ p, q \in \mathcal{P}. \tag{11}$$

Equation (9) is a straightforward consequence of (8), because

$$\frac{\partial V_p}{\partial x} A_p x + 2\lambda_0 V_p = x' \Big(Q_p (A_p + \lambda_0 I) + (A_p + \lambda_0 I)' Q_p \Big) x = -\|x\|^2,$$

and equations (10) and (11) hold with

$$\alpha(s) \triangleq s^2 \inf_{Q \in \mathcal{Q}} \sigma_{\min}[Q], \qquad \bar{\alpha}(s) \triangleq s^2 \sup_{\bar{Q} \in \mathcal{Q}} \sigma_{\max}[\bar{Q}], \qquad \mu \triangleq \sup_{Q, \bar{Q} \in \mathcal{Q}} \frac{\sigma_{\max}[\bar{Q}]}{\sigma_{\min}[Q]}, \tag{12}$$

where $\sigma_{\min}[Q]$ denotes the smallest singular value of $Q \in \mathcal{Q}$ and $\sigma_{\max}[\bar{Q}]$ the largest singular value of $\bar{Q} \in \mathcal{Q}$. In the above definitions the supremum and infimum are actually a maximum and minimum, respectively, due to the compactness of $\mathcal{Q}$. The existence of a family of Lyapunov functions with the above properties is used below.

Proof of Theorem 2. In order to prove this theorem we construct an exponentially decaying bound on an arbitrary solution to (1). This bound will be independent of the switching signal σ, giving the desired uniformity. To this effect pick an arbitrary piecewise constant switching signal σ and let x denote some solution to (1). Given two time instants $T > t_0 \geq 0$, let

$$t_1 < t_2 < \cdots < t_{N_\sigma(t_0, T)}$$

be the discontinuities of σ on the interval (t_0, T) and $t_{N_\sigma(t_0,T)+1} \triangleq T$. Since σ is piecewise constant, the signal

$$v(t) \triangleq e^{2\lambda_0 t} V_{\sigma(t)}\big(x(t)\big), \qquad t \geq 0,$$

is piecewise continuously differentiable and

$$\dot{v}(t) = 2\lambda_0 v + e^{2\lambda_0 t} \frac{\partial V_{p_i}}{\partial x}(x) A_{p_i}(x), \qquad t \in (t_i, t_{i+1}), \quad i \in \{0, 1, \ldots, N_\sigma(t_0, T)\}, \tag{13}$$

because $\sigma = p_i \triangleq \sigma(t_i)$ on $[t_i, t_{i+1})$. From (9) and (13) one concludes that $\dot{v} \leq 0$ on (t_i, t_{i+1}), and therefore that

$$v(t) \leq v(t_i), \qquad t \in [t_i, t_{i+1}), \quad i \in \{0, 1, \ldots, N_\sigma(t_0, T)\}. \tag{14}$$

Moreover, because of (11),

$$v(t_{i+1}) = e^{2\lambda_0 t_{i+1}} V_{p_{i+1}}\big(x(t_{i+1})\big) \leq \mu e^{2\lambda_0 t_{i+1}} V_{p_i}\big(x(t_{i+1})\big), \qquad i \in \{0, 1, \ldots, N_\sigma(t_0, T)\}.$$

Since V_{p_i} and x are continuous, we then conclude that

$$v(t_{i+1}) \leq \mu \lim_{\tau \uparrow t_{i+1}} e^{2\lambda_0 \tau} V_{p_i}\big(x(\tau)\big) = \mu \lim_{\tau \uparrow t_{i+1}} v(\tau), \qquad i \in \{0, 1, \ldots, N_\sigma(t_0, T)\}.$$

But $\lim_{\tau \uparrow t_{i+1}} v(\tau) \leq v(t_i)$ because of (14), therefore

$$v(t_{i+1}) \leq \mu v(t_i), \qquad i \in \{0, 1, \ldots, N_\sigma(t_0, T)\}.$$

Iterating the above inequality from $i = 0$ to $i = N_\sigma(t_0, T) - 1$ yields

$$v(t_{N_\sigma(t_0,T)}) \leq \mu^{N_\sigma(t_0,T)} v(t_0)$$

and therefore, because of (14),

$$\lim_{\tau \uparrow T} v(\tau) \leq v(t_{N_\sigma(t_0,T)}) \leq \mu^{N_\sigma(t_0,T)} v(t_0).$$

From this and the definition of v one concludes that

$$e^{2\lambda_0 T} V_{\sigma(T^-)}(x(T)) \leq \mu^{N_\sigma(t_0,T)} e^{2\lambda_0 t_0} V_{\sigma(t_0)}(x(t_0)),$$

where $\sigma(T^-) \triangleq \lim_{\tau \uparrow T} \sigma(\tau)$. Multiplying both sides of the above inequality by $e^{-2\lambda_0 T}$ yields

$$V_{\sigma(T^-)}(x(T)) \leq e^{-2\lambda_0(T-t_0)+N_\sigma(t_0,T)\log\mu} V_{\sigma(t_0)}(x(t_0)).$$

From this and (10) one concludes that

$$\|x\| \leq \alpha^{-1}\left(e^{-2\lambda_0(T-t_0)+N_\sigma(t_0,T)\log\mu}\bar{\alpha}(\|x(t_0)\|)\right), \tag{15}$$

and therefore, because of (12),

$$\|x\| \leq q\, e^{-\lambda_0(T-t_0)+N_\sigma(t_0,T)\frac{\log\mu}{2}} \|x(t_0)\|,$$

with $q \triangleq \left(\frac{\sup_{\bar{Q}\in\mathcal{Q}} \sigma_{\max}[\bar{Q}]}{\inf_{Q\in\mathcal{Q}} \sigma_{\min}[Q]}\right)^{\frac{1}{2}}$. To get a stability margin λ, it is then enough to have

$$-\lambda_0(T-t_0) + N_\sigma(t_0,T)\frac{\log\mu}{2} \leq k - \lambda(T-t_0), \tag{16}$$

for some finite k. Since (16) is equivalent to

$$N_\sigma(t_0,T) \leq N_0 + \frac{T-t_0}{\tau_D^*},$$

with $N_0 \triangleq \frac{2k}{\log\mu}$ and $\tau_D^* \triangleq \frac{\log\mu}{2(\lambda_0-\lambda)}$, we conclude that (16) holds, and therefore that

$$\|x\| \leq q\, e^{k-\lambda(T-t_0)} \|x(t_0)\|,$$

whenever $\sigma \in \mathcal{S}_{\text{ave}}[\tau_D, N_0]$ with $\tau_D \geq \tau_D^*$. Uniform exponential stability of (1) over $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$, with stability margin λ, follows from this. Note that by choosing $k = \frac{N_0 \log\mu}{2}$ sufficiently large we can accommodate for any chatter bound. □

4 Nonlinear Switched Systems

In the sequel we extend the results in the previous section to certain classes of nonlinear switched systems. To this effect consider a family $\mathcal{F} \triangleq \{F_p : p \in \mathcal{P}\}$ of nonlinear maps from $\mathbb{R}^n$ to itself such that $F_p(0) = 0$, $p \in \mathcal{P}$. For each piecewise constant switching signal σ we can then define the following switched non-linear system

$$\dot{x} = F_\sigma(x). \tag{17}$$

The objective of this section is to derive conditions under which (17) is asymptotically stable in a uniform way over all slow-on-the-average switchings.

Given some family of piecewise constant switching signals $\mathcal{S}$, we say that (17) is *uniformly asymptotically stable over* $\mathcal{S}$ if there exists a function β of class[6] $\mathcal{KL}$ such that, for each $\sigma \in \mathcal{S}$,

$$\|x(t)\| \leq \beta(\|x(\tau)\|, t-\tau), \qquad \forall t \geq \tau \geq 0, \tag{18}$$

[6] We denote by $\mathcal{KL}$ the set of continuous functions $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ which, for each fixed value of the second argument, are of class $\mathcal{K}$ when regarded as functions of the first argument, and that have $\lim_{\tau\to\infty} \beta(s,\tau) = 0$ for each fixed $s \geq 0$.

along solutions to (17). When $\beta(s,t)$ is of the form $ce^{-\lambda t}s$ for some $c, \lambda > 0$ we say that (1) is *uniformly exponentially stable over* $\mathcal{S}$ which, for the linear case, reduces to the definition in Section 2.

Clearly, for (17) to be uniformly asymptotically stable over any set $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$, $\tau_D, N_0 > 0$, the origin must be a globally, asymptotically stable equilibrium point of each time-invariant system $\dot{z} = F_p(z)$, $p \in \mathcal{P}$. Here we actually demand more of the F_p:

Assumption 3. There exist continuously differentiable functions $V_p : \mathbb{R}^n \to \mathbb{R}$, $p \in \mathcal{P}$, positive constants λ_0, μ, and functions $\alpha, \bar{\alpha}$ of class $\mathcal{K}_\infty$ such that

$$\frac{\partial V_p}{\partial x} F_p(x) \leq -2\lambda_0 V_p, \tag{19}$$

$$\alpha(\|x\|) \leq V_p(x) \leq \bar{\alpha}(\|x\|), \tag{20}$$

$$V_p(x) \leq \mu V_q(x), \tag{21}$$

for each $x \in \mathbb{R}^n$ and $p, q \in \mathcal{P}$.

Equations (19)–(20) are the standard conditions for V_p to be a Lyapunov function of $\dot{z} = F_p(z)$. It should be noted that, in light of [20, 21], the exponential decay suggested by (19) does not really introduce loss of generality. As for (21), it may, in fact, reduce the set of systems to which the results in this section can be applied. Equation (21) can be violated, for example, if the value of the parameter changes significantly the nature of the Lyapunov function, e.g., $\mathcal{P} = \{1, 2\}$ and

$$V_p(x) = \begin{cases} x^2 & p = 1 \\ x^4 & p = 2 \end{cases}.$$

In this paper we do not consider such cases.

Since equations (19)–(21) match exactly equations (9)–(11), by proceeding as in the proof of Theorem 2 we conclude that for every switching signal σ and every two time instants $T > t_0 \geq 0$,

$$\|x\| \leq \alpha^{-1}\left(e^{-2\lambda_0(T-t_0)+N_\sigma(t_0,T)\log\mu}\bar{\alpha}(\|x(t_0)\|)\right), \tag{22}$$

(cf. equation (15)). Take now some $\tau_D \geq \tau_D^* \triangleq \frac{\log\mu}{2(\lambda_0-\lambda)}$, with $\lambda \in (0, \lambda_0)$ and $\sigma \in \mathcal{S}_{\text{ave}}[\tau_D, N_0]$, with $N_0 \triangleq \frac{2k}{\log\mu}$, $k > 0$. We then have

$$N_\sigma(t_0, T) \leq N_0 + \frac{2(\lambda_0 - \lambda)}{\log\mu}(T - t_0).$$

From this and (22) we conclude that

$$\|x\| \leq \beta(\|x(t_0)\|, T - t_0),$$

with $\beta(s,t) \triangleq \alpha^{-1}\left(e^{N_0\log\mu - 2\lambda t}\bar{\alpha}(s)\right) \in \mathcal{KL}$. The following was proved:

Theorem 4. *Given a set $\mathcal{F} \triangleq \{F_p : p \in \mathcal{P}\}$ of nonlinear maps from $\mathbb{R}^n$ to itself for which Assumption 3 holds, there is a finite constant τ_D^* such that (1) is uniformly asymptotically stable over $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$, for any average dwell-time $\tau_D \geq \tau_D^*$ and any chatter bound $N_0 > 0$.*

5 Supervisory Control

In this section we show how the previous results can be used in the context of supervisory control. We follow closely the formulation in [1, 6].

The problem addressed here is the set-point control of an imprecisely modeled process $\mathbb{P}$. In particular, we want to generate the control input u to the process so as to drive its output y to a constant reference r.

The process has two other exogenous inputs that cannot be measured: a bounded measurement noise signal $\mathbf{n}$ and a bounded disturbance $\mathbf{d}$. For simplicity the signals u, y, $\mathbf{n}$, and $\mathbf{d}$ are scalar. $\mathbb{P}$ is assumed linear, time-invariant, with a stabilizable (through u) and detectable realization

$$\dot{x}_{\mathbb{P}} = A_{\mathbb{P}}x + B_{\mathbb{P}}u + D_{\mathbb{P}}\mathbf{d}, \qquad y = C_{\mathbb{P}}x + \mathbf{n}, \tag{23}$$

but precise values for $A_{\mathbb{P}}, B_{\mathbb{P}}, C_{\mathbb{P}}, D_{\mathbb{P}}$ are not known. It is known, however, that $\mathbb{P}$'s transfer function τ, from u to y, belongs to a family of transfer functions of the form

$$\mathcal{N} \stackrel{\triangle}{=} \bigcup_{p\in\mathcal{P}} \mathcal{N}_p,$$

where p is an unknown parameter taking values in some parameter set $\mathcal{P}$ and each $\mathcal{N}_p$ denotes a family of transfer functions centered around a known, *nominal transfer function* ν_p, e.g.,

$$\mathcal{N}_p \stackrel{\triangle}{=} \{(1+\delta)\nu_p : \|\delta\|_\infty < \epsilon\}.$$

Here, ϵ denotes some small positive constant and δ a stable transfer function with $\mathcal{H}_\infty$-norm smaller than ϵ. For simplicity, in the sequel we assume that $\epsilon = 0$ and the set $\mathcal{P}$ is finite and equal to $\{1, 2, \ldots, m\}$.

The solution proposed in [1, 6] to solve this problem is based on certainty equivalence and starts with the selection of a family of linear, time-invariant *candidate controllers* $\mathcal{C} \stackrel{\triangle}{=} \{\kappa_p : p \in \mathcal{P}\}$. Each κ_p would make the feedback closed-loop system in Figure 1 asymptotically stable if the process transfer function τ was known to belong to $\mathcal{N}_p$. To avoid pole-zero cancellations it is assumed that $\mathbb{P}$ does not have transmission zeros at the origin.

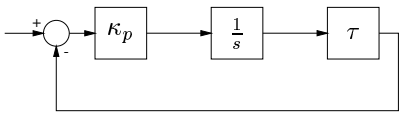

Figure 1: Feedback configuration.

In case we knew to which set $\mathcal{N}_p$ the actual process transfer function τ belonged, stability of the closed loop could be achieved with a nonadaptive, linear, time-invariant controller with transfer function equal to $\frac{1}{s}\kappa_p$. Since the process transfer function is not known in advance we build a "multi-controller" $\mathbb{C}$ that effectively allows switching between all the controller transfer functions in $\mathcal{C}$. If $\{(\bar{A}_p, \bar{B}_p, \bar{C}_p, \bar{D}_p) : p \in \mathcal{P}\}$ is a family of n-dimensional, stabilizable and detectable realizations for the transfer functions in $\mathcal{C}$, the multi-controller $\mathbb{C}$ can be defined by

$$\dot{x}_{\mathbb{C}} = \bar{A}_\sigma x_{\mathbb{C}} + \bar{B}_\sigma \mathbf{e_T}, \qquad v = \bar{C}_\sigma x_{\mathbb{C}} + \bar{D}_\sigma \mathbf{e_T}, \qquad \dot{u} = v, \tag{24}$$

where $\mathbf{e_T} \stackrel{\triangle}{=} r - y$ and $\sigma : [0, \infty) \to \infty$ denotes a "switching signal" that, at each instant of time, determines which candidate controller is put into the feedback loop. The system that generates the switching signal σ is called a supervisor. Here we are interested in estimator-based supervisors like the one in Figure 2. An estimator-based supervisor consists of three blocks: a multi-estimator, a performance signal generator, and a switching logic.

The *multi-estimator* $\mathbb{E}$ is a linear, time-invariant system whose inputs are the outputs of the process and multi-controller and whose outputs are the *output estimation errors* e_p, $p \in \mathcal{P}$. Each e_p is a signal that would converge to zero if the process transfer function τ was equal to the nominal transfer function ν_p. The reader is referred to [1, 6] for the precise structure of $\mathbb{E}$. Denoting by x the combined state of the multi-estimator and multi-controller (excluding the integrator), the evolution of x is determined by

$$\dot{x} = A_\sigma x + d_\sigma e_\sigma, \tag{25}$$

$$\mathbf{e_T} = c_{p^*} x + e_{p^*}, \tag{26}$$

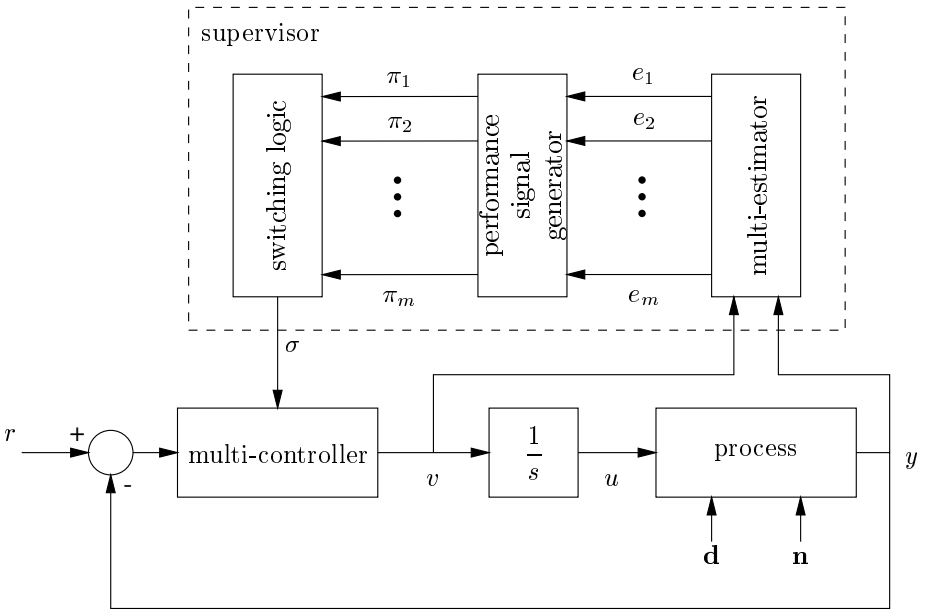

Figure 2: Supervisory control architecture.

where A_p, d_p, c_p, $p \in \mathcal{P}$ are appropriately defined matrices, and p^* is the element of $\mathcal{P}$ for which $\tau \in \mathcal{N}_{p^*}$. Equation (25) is obtained from equation (23) in [1] with $l = \sigma$ and (26) is obtained from equation (26) in [1] with $l = p^*$. Moreover, there is a positive constant λ_0 for which each $\lambda_0 I + A_p$ is asymptotically stable (cf. Remark 4 in [1]). From equation (28) in [1], one also concludes that e_{p^*} is bounded and

$$\int_0^t e^{2\lambda\tau} e_{p^*}(\tau)^2 d\tau \leq c_n e^{2\lambda t} + c_0, \qquad \|e_{p^*}(t)\| \leq d_n + d_0 e^{-\lambda t}, \qquad t \geq 0, \tag{27}$$

where λ is any constant in $(0, \lambda_0)$, c_0, d_0 are positive constants that depends only on initial conditions, and c_n, d_n are positive constants that depends only on upper bounds on the norms of $\mathbf{n}$ and $\mathbf{d}$.

The *performance signal generator* $\mathbb{G}$ takes as inputs the output estimation errors e_p, $p \in \mathcal{P}$, and produces the *performance signals* π_p, $p \in \mathcal{P}$ defined by

$$\dot{\tilde{\pi}}_p = -2\lambda\tilde{\pi}_p + e_p^2, \qquad \pi_p = \tilde{\pi}_p + \epsilon_\pi, \qquad p \in \mathcal{P}, \tag{28}$$

with $\lambda \in (0, \lambda_0)$ and $\epsilon_\pi > 0$ constant. $\mathbb{G}$ is initialized so that $\tilde{\pi}_p(0) \geq 0$, $p \in \mathcal{P}$.

The *switching logic* $\mathbb{S}$ generates the switching signal σ based on the values of the performance signals π_p, $p \in \mathcal{P}$. The logic used here is called a *scale-independent hysteresis switching logic* and can be regarded as a hybrid dynamical system $\mathbb{S}_\mathbb{H}$ whose state and output are both σ. To specify $\mathbb{S}_\mathbb{H}$ it is necessary to first pick a positive number $h > 0$ called a *hysteresis constant*. $\mathbb{S}_\mathbb{H}$'s internal logic is then defined by the computer diagram shown in Figure 3 where, at each time t, $q \stackrel{\triangle}{=} \operatorname{argmin}_{p\in\mathcal{P}} \pi_p(t)$. In interpreting this diagram it is to be understood that σ's value at each of its switching times $\bar{t}$ is equal to its limit from the right as $t \downarrow \bar{t}$. Thus if $\bar{t}_i$ and $\bar{t}_{i+1}$ are two consecutive switching times, then σ is constant on $[\bar{t}_i, \bar{t}_{i+1})$. The functioning of $\mathbb{S}_\mathbb{H}$ is roughly as follows. Suppose that at some time t_0, $\mathbb{S}_\mathbb{H}$ has just changed the value of σ to p. The signal σ is then held fixed at this value unless and until there is a time $t_1 > t_0$ at which $(1+h)\pi_q \leq \pi_p$ for some $q \in \mathcal{P}$. If this occurs, σ is set equal to q and so on.

Suppose now that we define *scaled performance signals* $\bar{\pi}_p \stackrel{\triangle}{=} \vartheta\pi_p$, with $\vartheta(t) \stackrel{\triangle}{=} e^{2\lambda t}$, $t \geq 0$. From (28) one

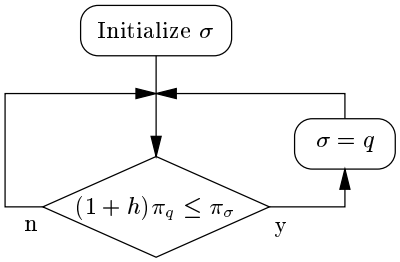

Figure 3: Computer Diagram of $\mathbb{S}_{\mathbb{SH}}$.

concludes that

$$\bar{\pi}_p(t) = \tilde{\pi}_p(t_0) + e^{2\lambda t}\epsilon_\pi + \int_{t_0}^{t} e^{2\lambda\tau} e_p(\tau)^2 d\tau, \qquad 0 \leq t_0 \leq t, \tag{29}$$

and therefore each $\bar{\pi}_p$ is always monotone increasing and never smaller than ϵ_π. By the Hysteresis Switching Theorem [18] (cf. Theorem 7 in the Appendix) we can then conclude that, for any $\ell \in \mathcal{P}$,

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{\bar{\pi}_\ell(t)}{\inf_{p\in\mathcal{P}} \bar{\pi}_p(t_0)}\right), \qquad 0 \leq t_0 \leq t < T, \tag{30}$$

where $N_\sigma(t_0, t)$ denotes the number of discontinuities of σ on the interval (t_0, t), and also that

$$\int_{t_0}^{t} \left(2\lambda e^{2\lambda\tau}\epsilon_\pi + e^{2\lambda\tau} e_\sigma^2\right) d\tau \leq m\left((1+h)\bar{\pi}_\ell(t) - \inf_{p\in\mathcal{P}} \bar{\pi}_p(t_0)\right), \qquad 0 \leq t_0 \leq t < T. \tag{31}$$

Here we used the fact that $\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\bar{\pi}_{\sigma(\tau)}(\tau)\right) = 2\lambda e^{2\lambda\tau}\epsilon_\pi + e^{2\lambda\tau} e_\sigma^2$, wherever the derivative exists. Now, from (27) and (29) we obtain

$$\bar{\pi}_{p^*}(t) \leq e^{2\lambda t}(\epsilon_\pi + c_n) + \bar{c}_0, \qquad t \geq 0,$$

where $\bar{c}_0 \triangleq c_0 + \tilde{\pi}_{p^*}(0)$. From this, (30)–(31) with $\ell \triangleq p^*$, and the fact that $\bar{\pi}_p(t_0) \geq e^{2\lambda t_0}\epsilon_\pi$, $p \in \mathcal{P}$, we conclude that

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log\left(e^{2\lambda(t-t_0)}\left(1 + \frac{c_n}{\epsilon_\pi}\right) + \frac{\bar{c}_0}{\epsilon_\pi}\right),$$

$$\int_{t_0}^{t} e^{2\lambda\tau} e_\sigma^2 d\tau \leq m\left((1+h)\left(e^{2\lambda t}(\epsilon_\pi + c_n) + \bar{c}_0\right) - e^{2\lambda t_0}\epsilon_\pi\right) - \left(e^{2\lambda t} - e^{2\lambda t_0}\right)\epsilon_\pi,$$

for every $t \geq t_0 \geq 0$. Since for $a, b > 0$, $\log(a+b) \leq \log(2a) + \log(2b)$, we also conclude that

$$N_\sigma(t, t_0) \leq N_0 + \frac{t - t_0}{\bar{\tau}_D}, \tag{32}$$

$$\int_{t_0}^{t} e^{2\lambda\tau} e_\sigma^2 d\tau \leq \bar{m} e^{2\lambda t} c_n + \bar{m}\bar{c}_0 - (m-1)e^{2\lambda t_0}\epsilon_\pi + (\bar{m} - 1)e^{2\lambda t}\epsilon_\pi, \tag{33}$$

with

$$\tau_D \triangleq \frac{\log(1+h)}{2\lambda m}, \qquad \bar{m} = m(1+h), \qquad N_0 \triangleq 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{4\bar{c}_0}{\epsilon_\pi}\left(1 + \frac{c_n}{\epsilon_\pi}\right)\right).$$

Now, because of Lemma 1 and Theorem 2, there is a finite constant τ_D^* such that (25) has input-to-state $e^{\lambda t}$-weighted $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ norm uniformly bounded over $\mathcal{S}_{\text{ave}}[\tau_D^*, N_0]$. If we then choose λ and h so that

$$\frac{\lambda}{\log(1+h)} \leq \gamma \triangleq \frac{1}{2m\tau_D^*},$$

we get $\tau_D \geq \tau_D^*$ and the output σ of the switching logic is guaranteed to be in $\mathcal{S}_{\text{ave}}[\tau_D^*, N_0]$. From this and (33) one concludes that x is bounded and, because of (24) and (26), $\mathbf{e_T}$ and v are also bounded. The boundedness of u and the internal state of the process follows from the detectability of the cascade formed by the integrator in (24) and the process (23). The following can then be stated.

Theorem 5. *There exists a positive constant γ such that, whenever $\frac{\lambda}{\log(1+h)} \leq \gamma$, all signals remain bounded, for any bounded $\mathbf{n}$ and $\mathbf{d}$, and any initialization of $\mathbb{P}$, $\mathbb{E}$, $\mathbb{C}$, $\mathbb{G}$, $\mathbb{S}$, with $\tilde{\pi}_p(0) \geq 0$, $p \in \mathcal{P}$.*

6 Conclusions

We showed that switching among stable linear systems results in a stable system provided that switching is sufficiently slow on the average. In particular, it was shown that exponential stability is achieved when the number of switches in any finite interval grows linearly with the length of the interval, and the growth rate is sufficiently small. This was used to analyze a supervisory control algorithm using scale-independent hysteresis switching. The analysis took noise and disturbances into account and, to the best of our knowledge, is the first analysis of an hysteresis switching logic that does not rely on switching stopping. It should be emphasized that the analysis presented here is far simpler than the one in [6], in which a dwell-time switching logic is used. In a paper under preparation, the analysis presented here is extended to processes with unmodeled dynamics.

Appendix

Proof of Lemma 1. Let c be a constant for which (3) holds with $\lambda = \lambda_0$, for every $\sigma \in \mathcal{S}$, and take an arbitrary switching signal $\sigma \in \mathcal{S}$ and a piecewise continuous input u. By the variation of constants formula

$$z(t) = \Phi(t,0)z(0) + \int_0^t \Phi(t,\tau)B_{\sigma(\tau)}u(\tau)d\tau, \qquad t \geq 0,$$

and, because of (3) with $\lambda = \lambda_0$,

$$\|z(t)\| \leq ce^{-\lambda_0 t}\|z(0)\| + cb\int_0^t e^{-\lambda_0(t-\tau)}\|u(\tau)\|d\tau, \qquad t \geq 0,$$

with $\|\bar{B}\| \leq b$, for every $\bar{B} \in \mathcal{B}$. Denoting by λ a nonnegative constant strictly smaller than λ_0, we can then write

$$e^{\lambda t}\|z(t)\| \leq ce^{-(\lambda_0-\lambda)t}\|z(0)\| + cb\int_0^t e^{-(\lambda_0-\lambda)(t-\tau)}e^{\lambda\tau}\|u(\tau)\|d\tau. \tag{34}$$

Using the Schwartz inequality[7] to bound the integral in (34) we conclude that

$$e^{\lambda t}\|z(t)\| \leq ce^{-(\lambda_0-\lambda)t}\|z(0)\| + cb\left(\int_0^t e^{-2(\lambda_0-\lambda)(t-\tau)}d\tau\right)^{\frac{1}{2}}\left(\int_0^t e^{2\lambda\tau}\|u(\tau)\|^2 d\tau\right)^{\frac{1}{2}},$$

and therefore (7) holds with $\mathfrak{g}_0 \triangleq c$ and $\mathfrak{g} \triangleq \frac{cb}{\sqrt{2(\lambda_0-\lambda)}}$. Here we used the fact that

$$\int_0^t e^{-2(\lambda_0-\lambda)(t-\tau)}d\tau \leq \frac{1}{2(\lambda_0-\lambda)}, \qquad t \geq 0.$$

On the other hand, if we use the Hölder inequality[8] to bound the integral in (34), we obtain

$$e^{\lambda t}\|z(t)\| \leq ce^{-(\lambda_0-\lambda)t}\|z(0)\| + cb\left(\int_0^t e^{-(\lambda_0-\lambda)(t-\tau)}d\tau\right)\sup_{\tau\in[0,t)} e^{\lambda\tau}\|u(\tau)\|,$$

[7] Given that $\int_\Omega |x|^2 < \infty$ and $\int_\Omega |y|^2 < \infty$, $\int_\Omega |xy| \leq \left(\int_\Omega |x|^2\right)^{\frac{1}{2}}\left(\int_\Omega |y|^2\right)^{\frac{1}{2}}$ [22, p. 548].
[8] Given that $\int_\Omega |x| < \infty$, $\int_\Omega |xy| \leq \left(\int_\Omega |x|\right)\left(\sup_\Omega |y|\right)$ [22, p. 551].

and therefore (6) holds with $\mathfrak{g}_0 \triangleq c$ and $\mathfrak{g} \triangleq \frac{cb}{\lambda_0 - \lambda}$. From (34) and the Minkowski inequality[9] one also concludes that

$$\left(\int_0^T e^{2\lambda t}\|z(t)\|^2 dt\right)^{\frac{1}{2}} \leq c\left(\int_0^T e^{-2(\lambda_0-\lambda)t}dt\right)^{\frac{1}{2}}\|z(0)\| + cb\left(\int_0^T\left(\int_0^t e^{-(\lambda_0-\lambda)(t-\tau)}e^{\lambda\tau}\|u(\tau)\|d\tau\right)^2 dt\right)^{\frac{1}{2}}. \quad (35)$$

Now, because of the Schwartz inequality,

$$\left(\int_0^t e^{-(\lambda_0-\lambda)(t-\tau)}e^{\lambda\tau}\|u(\tau)\|d\tau\right)^2 dt \leq \left(\int_0^t e^{-(\lambda_0-\lambda)(t-\tau)}d\tau\right)\left(\int_0^t e^{-(\lambda_0-\lambda)(t-\tau)}e^{2\lambda\tau}\|u(\tau)\|^2 d\tau\right),$$

and therefore

$$\int_0^T\left(\int_0^t e^{-(\lambda_0-\lambda)(t-\tau)}e^{\lambda\tau}\|u(\tau)\|d\tau\right)^2 dt \leq \frac{1}{\lambda_0-\lambda}\int_0^T\int_0^t e^{-(\lambda_0-\lambda)(t-\tau)}e^{2\lambda\tau}\|u(\tau)\|^2 d\tau dt$$
$$= \frac{1}{\lambda_0-\lambda}\int_0^T e^{2\lambda\tau}\|u(\tau)\|^2\left(\int_\tau^T e^{-(\lambda_0-\lambda)(t-\tau)}dt\right)d\tau \leq \frac{1}{(\lambda_0-\lambda)^2}\int_0^T e^{2\lambda\tau}\|u(\tau)\|^2 d\tau.$$

From this and (35) one concludes that (5) holds with $\mathfrak{g}_0 \triangleq \frac{c}{\sqrt{2(\lambda_0-\lambda)}}$ and $\mathfrak{g} \triangleq \frac{cb}{\lambda_0-\lambda}$. □

Scale-Independent Hysteresis Switching

Consider the hybrid dynamical system

$$\dot{x} = f_\sigma(x,t), \qquad x(0) = x_0, \quad (36)$$

where $\{f_p : p \in \mathcal{P}\}$ is an indexed family of locally Lipschitz functions taking values on a finite dimensional space $\mathcal{X}$ and defined on $\mathcal{X} \times [0,\infty)$ and σ is a piecewise constant *switching signal* taking values in $\mathcal{P}$. The switching signal σ is chosen so as to cause the *performance signals*

$$\pi_p \triangleq \Pi(p,x,t), \qquad p \in \mathcal{P}, \quad (37)$$

to have certain desired properties. Here, Π is a *performance function* from $\mathcal{P} \times \mathcal{X} \times [0,\infty)$ to $\mathbb{R}$ that is continuous with respect to the second and third arguments for frozen values of the first.

The algorithm used to generate σ is called a *scale-independent hysteresis switching logic* and can be regarded as a hybrid dynamical system $\mathbb{S}_\mathbb{H}$ whose input is x and whose state and output are both σ. To specify $\mathbb{S}_\mathbb{H}$ it is necessary to first pick a positive number $h > 0$ called a *hysteresis constant*. $\mathbb{S}_\mathbb{H}$'s internal logic is then defined by the computer diagram shown in Figure 3, where the π_p are defined by (37) and, at each time t, $q \triangleq \arg\min_{p\in\mathcal{P}} \Pi(p,x,t)$. In interpreting this diagram it is to be understood that σ's value at each of its switching times $\bar{t}$ is equal to its limit from the right as $t \downarrow \bar{t}$. Thus if $\bar{t}_i$ and $\bar{t}_{i+1}$ are two consecutive switching times, then σ is constant on $[\bar{t}_i, \bar{t}_{i+1})$.

Let $\mathcal{X}_0$ denote a given subset of $\mathcal{X}$, and $\mathcal{S}$ the class of all piecewise constant functions $s : [0,\infty) \to \mathcal{P}$. In what follows, for each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$, $T_{\{x_0,s\}}$ denotes the length of the maximal interval of existence of solution to the equations

$$\dot{x} = f_{s(t)}(x,t), \qquad x(0) = x_0,$$

and $x_{\{x_0,s\}}$ is the corresponding solution. The following "open-loop" assumptions are made:

[9] Given that $\int_\Omega |x|^2 < \infty$ and $\int_\Omega |y|^2 < \infty$, $\left(\int_\Omega |x+y|^2\right)^{\frac{1}{2}} \leq \left(\int_\Omega |x|^2\right)^{\frac{1}{2}} + \left(\int_\Omega |y|^2\right)^{\frac{1}{2}}$ [22, p. 548].

Assumption 6 (Relaxed Open-Loop). There exists a positive time-function ϑ such that for each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$, the scaled performance signals $\bar{\pi}_p \stackrel{\triangle}{=} \vartheta\Pi(p, x_{\{x_0,s\}}, t)$, $p \in \mathcal{P}$, are such that:

1. There exists a positive constant ϵ such that $\bar{\pi}_p(0) \geq \epsilon$, for each $p \in \mathcal{P}$.
2. Each performance signal $\bar{\pi}_p$, $p \in \mathcal{P}$, is monotone increasing on $[0, T_{\{x_0,s\}})$.

The following was proved in [18].

Theorem 7 (Scale-Independent Hysteresis Switching). *Let $\mathcal{P}$ be a finite set with m elements and assume that the Relaxed Open-Loop Assumptions hold. For fixed initial state $\{x_0, \sigma_0\} \in \mathcal{X}_0 \times \mathcal{P}$, let $\{x, \sigma\}$ denote the unique solution to* (36) *with σ generated by $\mathbb{S}_{\mathbb{H}}$ and $[0, T)$ the largest interval on which this solution is defined. For any $\ell \in \mathcal{P}$,*

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log\left(\frac{\bar{\pi}_\ell(t)}{\min_{p\in\mathcal{P}} \bar{\pi}_p(t_0)}\right), \qquad 0 \leq t_0 \leq t < T,$$

where $N_\sigma(t_0, t)$ denotes the number of discontinuities of σ on the interval (t_0, t), and[10]

$$\int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\bar{\pi}_{\sigma(\tau)}(\tau)\big) d\tau \leq m\left((1+h)\bar{\pi}_\ell(t) - \min_{p\in\mathcal{P}} \bar{\pi}_p(t_0)\right), \qquad 0 \leq t_0 \leq t < T. \tag{38}$$

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References

[1] A. S. Morse, "Supervisory control of families of linear set-point controllers—part 1: exact matching," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1413–1431, Oct. 1996.

[2] A. S. Morse, "Control using logic-based switching," in *Trends in Control: A European Perspective* (A. Isidori, ed.), pp. 69–113, London: Springer-Verlag, 1995.

[3] J. Balakrishnan, *Control Systems Design Using Multiple Models, Switching, and Tuning.* PhD thesis, Yale University, 1995.

[4] M. Chang and E. J. Davison, "Switching control of a family of plants," in *Proc. of the 14th Amer. Contr. Conf.*, vol. 1, pp. 1015–1020, June 1995.

[5] K. S. Narendra and J. Balakrishnan, "Adaptive control using multiple models," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 171–187, Feb. 1997.

[6] A. S. Morse, "Supervisory control of families of linear set-point controllers—part 2: robustness," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1500–1515, Nov. 1997.

[7] F. M. Pait and F. Kassab, Jr., "Parallel algorithms for adaptive control: Robust stability," in *Control Using Logic-Based Switching* (A. S. Morse, ed.), no. 222 in Lecture Notes in Control and Information Sciences, pp. 262–276, London: Springer-Verlag, 1997.

[10] In general, the signal $\pi_{\sigma(\tau)}(\tau)$ is discontinuous but piecewise differentiable. The integral in the left-hand-side of (38) should then be interpreted as

$$\sum_{k=0}^{N_\sigma(t,t_0)} \int_{t_k}^{t_{k+1}} \dot{\pi}_{\sigma(\tau)}(\tau) d\tau$$

where $t_1 < t_2 < \cdots < t_{N_\sigma(t,t_0)}$ denote the discontinuities of π_σ on (t_0, t) and $t_{N_\sigma(t,t_0)+1} \stackrel{\triangle}{=} t$.

[8] J. Hockerman-Frommer, S. R. Kulkarni, and P. J. Ramadge, "Controller switching based on output prediction errors," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 596–607, May 1998.

[9] G. Chang, J. P. Hespanha, A. S. Morse, M. Netto, and R. Ortega, "Supervisory field-oriented control of induction motors with uncertain rotor resistance," in *Proc. of the 1998 IFAC Workshop on Adaptive Control and Signal Processing, Glasgow, Scotland*, Aug. 1998.

[10] J. P. Hespanha, *Logic-Based Switching Algorithms in Control*. PhD thesis, Yale University, New Haven, CT, 1998.

[11] S. Fujii, J. P. Hespanha, and A. S. Morse, "Supervisory control of families of noise suppressing controllers," in *Proc. of the 37th Conf. on Decision and Contr.*, Dec. 1998.

[12] J. P. Hespanha and A. S. Morse, "Certainty equivalence implies detectability," *Syst. & Contr. Lett.*, vol. 36, pp. 1–13, Jan. 1999.

[13] J. P. Hespanha, D. Liberzon, and A. S. Morse, "Towards the supervisory control of uncertain nonholonomic systems," in *Proc. of the 1999 American Contr. Conf.*, pp. 3520–3524, June 1999.

[14] J. P. Hespanha and A. S. Morse, "Supervisory control of integral-input-to-state stabilizing controllers," in *Proc. of the 1999 European Contr. Conf.*, Aug. 1999.

[15] J. P. Hespanha and A. S. Morse, "Towards the supervisory control of families of nonlinear regulators," in *Proc. of the 30th Annual Conf. on Information Sciences and Syst.*, vol. 2, pp. 765–768, Mar. 1996.

[16] R. H. Middleton, G. C. Goodwin, D. J. Hill, and D. Q. Mayne, "Design issues in adaptive control," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 50–58, Jan. 1988.

[17] A. S. Morse, D. Q. Mayne, and G. C. Goodwin, "Applications of hysteresis switching in parameter adaptive control," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1343–1354, Sept. 1992.

[18] J. P. Hespanha and A. S. Morse, "Bounds on switching with scale-independent hysteresis." To be submitted for publication., Jan. 1999.

[19] S. R. Kulkarni and P. J. Ramadge, "Model and controller selection policies based on output prediction errors," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1594–1604, Nov. 1996.

[20] E. D. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1283–1294, Sept. 1996.

[21] L. Praly and Y. Wang, "Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability," *Math. Contr., Signals, Syst.*, vol. 9, pp. 1–33, 1996.

[22] A. W. Naylor and G. R. Sell, *Linear Operator Theory in Engineering and Science*. No. 40 in Applied Mathematical Sciences, New York: Springer-Verlag, 1982.