Bounds on the Number of Switchings with Scale-Independent Hysteresis: Applications to Supervisory Control*

Technical Report

João P. Hespanha* and A. Stephen Morse**

*Dept. Electrical Engineering–Systems, Univ. of Southern California
3740 McClintock Ave., Room 318, MC 2563, Los Angeles, CA 90089-2563
phone: (213) 740-9137, fax: (213) 821-1109
hespanha@usc.edu

**Dept. Electrical Engineering, Yale University
P.O. Box 208267, New Haven, CT 06520-8267
phone: (203) 432-4295, fax: (203) 432-7481
morse@sysc.eng.yale.edu

February 29, 2000

Abstract

In this paper we analyze the Scale-Independent Hysteresis Switching Logic introduced in [1, 2]. We show that, under suitable "open-loop" assumptions, one can establish an upper bound on the number of switchings produced by the logic on any given interval. This bound comes as a function of the variation of the inputs to the logic on that interval. In this paper it is also shown that, in a supervisory control context, this leads to switching that is slow-on-the-average [3], allowing us to study the stability of hysteresis-based adaptive control systems in the presence of measurement noise.

*This research was supported by the Office of Naval Research, Air Force Office of Scientific Research, the Army Research Office, and the National Science Foundation.

1 Introduction

Adaptive control algorithms that employ a logic-based supervisor to orchestrate the switching between a family of candidate controllers have been quite successful in numerous applications [4, 5, 6, 7, 8, 9, 10, 11, 1, 12, 13, 14, 15]. The need for switching usually arises from the fact that no single candidate controller would be capable, by itself, of guaranteeing stability and good performance when connected with a poorly modeled process. This type of supervisory control results in a switched closed-loop system of the form

$$\dot{x} = f_\sigma(x,t), \qquad x(0) = x_0, \qquad x \in \mathcal{X},\ t \geq 0, \tag{1}$$

where $\mathcal{X}$ is a finite dimensional space, $\{f_p : p \in \mathcal{P}\}$ an indexed family of locally Lipschitz functions, and σ a piecewise constant *switching signal* taking values on $\mathcal{P}$. σ is generated by a *hybrid* switching logic that combines continuous dynamics with discrete logic. Typically, the switching logic is designed to make the *monitoring signals*

$$\mu_p(t) := \Pi(p,x,t), \qquad t \geq 0,\ p \in \mathcal{P},\ x \in \mathcal{X} \tag{2}$$

have certain desired properties. Here, Π is a *monitoring function* from $\mathcal{P} \times \mathcal{X} \times [0,\infty)$ to $\mathbb{R}$ that is continuous with respect to the second and third arguments, for frozen values of the first. The points of discontinuity of σ are called *switching times*. For a given σ and $0 \leq t_0 < t$, it is convenient to denote by $N_\sigma(t,\tau)$ the number of switching times—i.e., discontinuities of σ—on the interval (τ, t).

For some switching logics the supervisor guarantees, by construction, that there is a minimum time τ_D between switchings [4, 5, 6, 7, 8, 9, 10]. Any switching signal generated by such logics thus satisfy

$$N_\sigma(\tau,t) = 0, \qquad \forall\ t - \tau \leq \tau_D.$$

The *dwell-time* τ_D is then a design parameter[1] chosen so that (1) remains stable. Unfortunately, with nonlinear systems this may lead to finite escape of the closed-loop. Adaptive switching algorithms for nonlinear systems have therefore avoided a fixed dwell-time, and have been mostly[2] based on hysteresis switching [17, 18], or on its more recent scale-independent version [1, 2].

To date, the analysis of algorithms based on hysteresis switching relied heavily on showing that switching stops in finite time [11, 1, 12, 13, 14, 15]. However, in the presence of noise and disturbance inputs, this is hardly the case. In fact, the only known algorithms for which switching can be proved to stop in finite time, even in the presence of noise/disturbances, are those for which an upper bound on these signals is known a priori, or effectively estimated online [5, 19]. Unfortunately, even in the noiseless case, these algorithms usually lead to bad transient responses. The objective of this paper is to analyze the behavior of the Scale-Independent Hysteresis Switching Logic introduced in [1, 2], under assumptions compatible with the existence of noise. We show that, although switching may not stop, it is possible to derive an upper bound on the number of switchings on any given interval. This bound comes as a function of the variation of the inputs to the logic on that interval.

The algorithm used to generate σ considered in this paper is called a *Scale-Independent Hysteresis Switching Logic* and can be regarded as a hybrid dynamical system $\mathbb{S}_\mathbb{H}$ whose input is x and whose state and output are both σ. To specify $\mathbb{S}_\mathbb{H}$ it is necessary to first pick a positive number $h > 0$ called a *hysteresis constant*. $\mathbb{S}_\mathbb{H}$'s internal logic is then defined by the computer diagram shown in Figure 1, where the μ_p are defined by (2) and, at each time t, $q := \arg\min_{p\in\mathcal{P}} \Pi(p,x,t)$. In interpreting this diagram it is to be understood that σ's value at each of its switching times $\bar{t}$ is equal to its limit from the right as $t \downarrow \bar{t}$. Thus if $\bar{t}_i$ and $\bar{t}_{i+1}$ are two consecutive switching times, then σ is constant on $[\bar{t}_i, \bar{t}_{i+1})$. The functioning of $\mathbb{S}_\mathbb{H}$ is roughly as follows. Suppose that at some time t_0, $\mathbb{S}_\mathbb{H}$ has just changed the value of σ to p. σ is then held fixed at

[1] In [6, 7] the dwell-time is actually chosen arbitrarily and (1) is then "massaged" by output injection to make it exponentially stable for that particular dwell-time.

[2] An exception is the supervisor based on dwell-time switching described in [16]. However, because of finite escape, the stability results given are only semi-global.

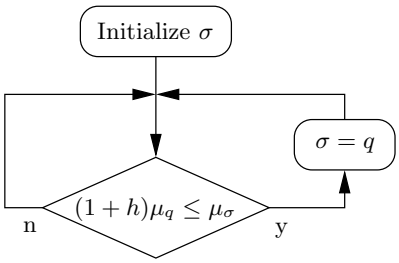

Figure 1: Computer Diagram of $\mathbb{S}_{\mathbb{H}}$.

this value unless and until there is a time $t_1 > t_0$ at which $(1+h)\mu_q \leq \mu_p$ for some $q \in \mathcal{P}$. If this occurs, σ is set equal to q and so on.

The main result of this paper is the Scale-Independent Hysteresis Switching Theorem. This theorem states that under appropriate "open-loop" assumptions, the number of switchings $N_\sigma(t,\tau)$ on any given interval (τ, t) can be bounded in terms of the variation of the monitoring signals on that interval. Using the Scale-Independent Hysteresis Switching Theorem, we will show that, although scale-independent hysteresis does not guarantee the existence of a fixed dwell-time between switchings, it can produce switching that is slow-on-the-average [3]. This will allow us to analyze hysteresis-based supervisory control algorithms in the presence of noise. Scale Independent Hysteresis and Dwell-time Switching are apparently the only two Certainty Equivalence based switching logics proposed thus far for which an analysis has proved possible under circumstances for which switching may not stop.

The switching logic described above is "scale-independent" in that its output σ remains unchanged if its monitoring function/input signal pair $\{\Pi, x\}$ is replaced by another pair $\{\bar{\Pi}, \bar{x}\}$ satisfying

$$\bar{\Pi}(p, \bar{x}, t) = \vartheta(t)\Pi(p, x, t), \qquad \forall p \in \mathcal{P},\ t \geq 0$$

where ϑ is a positive time function. This is because, for any fixed time t, (i) the value of p that minimizes $\Pi(p,x,t)$ also minimizes $\bar{\Pi}(p,\bar{x},t)$ and (ii) $(1+h)\Pi(q,x,t) \leq \Pi(p,x,t)$ is exactly equivalent to $(1+h)\bar{\Pi}(q,\bar{x},t) \leq \bar{\Pi}(p,\bar{x},t)$ for every $p, q \in \mathcal{P}$. Scale-independence simplifies considerably the analysis of estimator-based supervisory control algorithms [6, 7, 11, 12, 13, 14, 15].

This paper is organized as follows. Section 2 contains the statement and proof of the Scale-Independent Hysteresis Switching Theorem. In Section 3 it is illustrated how the scale-independence property can be used to effectively relax the "open-loop" assumptions required by this theorem. In Section 4 we recall some properties of switched systems and Section 5 uses the previous results to analyze a supervisory control algorithm based on the Scale-Independent Hysteresis Switching Logic. Finally, Section 6 contains some concluding remarks and directions for future research.

2 Scale-Independent Hysteresis Switching Theorem

Let $\mathcal{X}_0$ denote a given subset of $\mathcal{X}$, and $\mathcal{S}$ the class of all piecewise-constant functions $s : [0,\infty) \to \mathcal{P}$. In what follows, for each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$, $T_{\{x_0,s\}}$ denotes the length of the maximal interval of existence of solution to the equations

$$\dot{x} = f_{s(t)}(x,t), \qquad x(0) = x_0, \qquad t \geq 0,$$

and $x_{\{x_0,s\}}$ the corresponding solution. The following "open-loop" assumptions are made:

Assumption 1 (Open-Loop). For each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$, the monitoring signals $\mu_p(t) := \Pi(p, x_{\{x_0,s\}}, t)$, $t \in [0, T_{\{x_0,s\}})$, $p \in \mathcal{P}$, are such that:

1. There exists a positive constant ϵ such that $\mu_p(0) \geq \epsilon$, for each $p \in \mathcal{P}$.

2. Each monitoring signal μ_p, $p \in \mathcal{P}$, is monotone nondecreasing on $[0, T_{\{x_0,s\}})$.

First note that, because of the definition of $\mathbb{S}_{\mathbb{H}}$, $\sigma(0)$ must be such that $\mu_{\sigma(0)}(0)$ is strictly smaller than $(1+h)\mu_p(0)$ for every $p \in \mathcal{P}$. Thus, because of the Lipschitz continuity of the f_p and the continuity of Π, there must exist an interval $[0, t_1)$ of maximal length on which $\mu_{\sigma(0)}$ remains strictly smaller than $(1+h)\mu_p$ for every $p \in \mathcal{P}$. This means that σ is constant on $[0, t_1)$. Either this interval is the maximal interval of existence for x or it is not, in which case x is bounded on $[0, t_1)$ {cf. [20]}. If the latter is true, a switch must occur at t_1 and therefore $\mu_{\sigma(t_1)}(t_1) \leq \mu_p(t_1)$ for every $p \in \mathcal{P}$. Again, because of the Lipschitz continuity of the f_p, the continuity of Π, and the assumed boundedness below of each $h\mu_p$ by a positive constant, there must exist an interval $[t_1, t_2)$ of maximal length on which $\mu_{\sigma(t_1)}$ remains strictly smaller than $(1+h)\mu_p$ for every $p \in \mathcal{P}$. σ will then be constant on such an interval. Continuing this reasoning one concludes that there must be an interval $[0, T)$ of maximal length on which there is a unique pair $\{x, \sigma\}$ with x continuous and σ piecewise constant, which satisfies (1) with σ generated by $\mathbb{S}_{\mathbb{H}}$. Moreover, on each strictly proper subinterval $[0, \tau) \subset [0, T)$, σ can switch at most a finite number of times.

To establish existence of solution to (1) with σ generated by $\mathbb{S}_{\mathbb{H}}$, only the first Open-Loop Assumption was used. The other assumption enable us to draw conclusions regarding the behavior of σ and the μ_p on $[0, T)$. The following is the main result of this paper.

Theorem 1 (Scale-Independent Hysteresis Switching). *Let $\mathcal{P}$ be a finite set with m elements and assume that the Open-Loop Assumptions hold. For a fixed initial state $\{x_0, \sigma_0\} \in \mathcal{X}_0 \times \mathcal{P}$, let $\{x, \sigma\}$ denote the unique solution to (1) with σ generated by $\mathbb{S}_{\mathbb{H}}$ and let $[0, T)$ denote the largest interval on which this solution is defined. For any $\ell \in \mathcal{P}$,*

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log\Big(\frac{\mu_\ell(t)}{\min_{p\in\mathcal{P}} \mu_p(t_0)}\Big), \qquad 0 \leq t_0 < t < T, \tag{3}$$

and, when the monitoring signals are differentiable, we also have that

$$\int_{t_0}^{t} \dot{\mu}_\sigma(\tau) d\tau \leq m\Big((1+h)\mu_\ell(t) - \min_{p\in\mathcal{P}} \mu_p(t_0)\Big), \qquad 0 \leq t_0 < t < T, \tag{4}$$

where $\dot{\mu}_\sigma$ is defined to be equal to $\dot{\mu}_p$ on intervals where σ is constant and equal to $p \in \mathcal{P}$, and zero at points of discontinuity of σ.

When there is some μ_ℓ, $\ell \in \mathcal{P}$, bounded on $[0, T)$, σ can only have a finite number of discontinuities on $[0, T)$ because of (3). This means that there must be a time $T^* < T$ beyond which σ is constant. Moreover, since $\sigma = \sigma(T^*)$ on $[T^*, T)$,

$$\mu_{\sigma(T^*)}(t) = \mu_{\sigma(T^*)}(T^*) + \mu_{\sigma(T^*)}(t) - \mu_{\sigma(T^*)}(T^*) \leq \mu_{\sigma(T^*)}(T^*) + \int_{T^*}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(\mu_{\sigma(\tau)}(\tau)\Big) d\tau \qquad t \in [T^*, T).$$

Thus $\mu_{\sigma(T^*)}$ must be bounded on $[0, T)$ because of (4). Theorem 1 thus generalizes previous results in [1, 2].

Remark 1. Equation (4) can be generalized for the case when the monitoring signals are not piecewise differentiable. In this case the left-hand-side of (4) must be re-interpreted as a summation of the variation of the corresponding monitoring signals over the intervals on which σ is constant. Equation (4) must then be replaced by

$$\sum_{k=0}^{N_\sigma(t,t_0)} \Big(\mu_{\sigma(t_k)}(t_{k+1}) - \mu_{\sigma(t_k)}(t_k)\Big) \leq m\Big((1+h)\mu_\ell(t) - \min_{p\in\mathcal{P}} \mu_p(t_0)\Big), \qquad 0 \leq t_0 < t < T, \tag{5}$$

where $t_1 < t_2 < \cdots < t_{N_\sigma(t,t_0)}$ denote the discontinuities of σ on (t_0, t) and $t_{N_\sigma(t,t_0)+1} := t$. Since (5) is a generalization of (4), we shall prove (5) instead of (4).

Proof of Theorem 1. For an arbitrary initial state $\{x_0, \sigma_0\} \in \mathcal{X}_0 \times \mathcal{P}$ and times $t_0 < t$, let x, σ, T, ϵ, and the t_k be the signals and constants defined in the statement of the theorem and Remark 1. For each $k \in \{0, 1, \ldots, N_\sigma(t_0, t)\}$, σ remains constant and equal to $\sigma(t_k)$ on $[t_k, t_{k+1})$, thus

$$\mu_{\sigma(t_k)}(t) \leq (1+h)\mu_p(t_k), \qquad \forall t \in [t_k, t_{k+1}], \quad k \in \{0, 1, \ldots, N_\sigma(t_0, t)\}, \quad p \in \mathcal{P} \tag{6}$$

and, since σ switched to $\sigma(t_k)$ at time t_k, $k \in \{1, 2, \ldots, N_\sigma(t_0, t)\}$, we also have

$$\mu_{\sigma(t_k)}(t_k) \leq \mu_p(t_k), \qquad \forall k \in \{1, 2, \ldots, N_\sigma(t_0, t)\}, \quad p \in \mathcal{P}. \tag{7}$$

Moreover, σ switched from $\sigma(t_k)$ to $\sigma(t_{k+1})$ at time t_{k+1}, $k < N_\sigma(t_0, t) - 1$, thus, because of the continuity of $\mu_{\sigma(t_k)}$,

$$\mu_{\sigma(t_k)}(t_{k+1}) = (1+h)\mu_{\sigma(t_{k+1})}(t_{k+1}), \qquad \forall k \in \{0, 1, \ldots, N_\sigma(t_0, t) - 1\}. \tag{8}$$

Inequalities (6)–(8) summarize all the properties of $\mathbb{S}_{\mathbb{H}}$ that are relevant for this proof.

Since there are at most m distinct elements in $\mathcal{P}$, there must be at least one $q \in \mathcal{P}$ such that $\sigma = q$ on, at least[3],

$$\bar{N} := \left\lceil \frac{N_\sigma(t_0, t) - 1}{m} \right\rceil$$

of the intervals

$$[t_1, t_2), [t_2, t_3), \ldots, [t_{N_\sigma(t_0,t)-1}, t_{N_\sigma(t_0,t)}).$$

When $\bar{N} \leq 1$ we must have $N_\sigma(t_0, t) \leq 1 + m$ and therefore (3) automatically holds. Suppose now that $\bar{N} \geq 2$ and let

$$[t_{k_1}, t_{k_1+1}),\ [t_{k_2}, t_{k_2+1}), \cdots, [t_{k_{\bar{N}}}, t_{k_{\bar{N}}+1})$$

be $\bar{N}$ intervals on which $\sigma = q$. Picking any $i \in \{1, 2, \ldots, \bar{N} - 1\}$, in view of (8), the assumed monotonicity of $\mu_{\sigma(t_{k_i+1})}$, and (7), we conclude that

$$\mu_q(t_{k_i+1}) = (1+h)\mu_{\sigma(t_{k_i+1})}(t_{k_i+1}) \geq (1+h)\mu_{\sigma(t_{k_i+1})}(t_{k_i}) \geq (1+h)\mu_q(t_{k_i}). \tag{9}$$

Moreover, because the intervals are non-overlapping, $t_{k_{i+1}} \geq t_{k_i+1}$ and therefore $\mu_q(t_{k_{i+1}}) \geq \mu_q(t_{k_i+1})$. From this and (9), we obtain

$$(1+h)\mu_q(t_{k_i}) \leq \mu_q(t_{k_{i+1}}), \qquad i \in \{1, 2, \ldots, \bar{N} - 1\}.$$

Iterating the above inequality from $i = 1$ until $i = \bar{N} - 1$, yields

$$(1+h)^{\bar{N}-1}\mu_q(t_{k_1}) \leq \mu_q(t_{k_{\bar{N}}}),$$

and therefore

$$(1+h)^{\bar{N}-1}\mu_q(t_{k_1}) \leq \mu_\ell(t_{k_{\bar{N}}}), \qquad \forall \ell \in \mathcal{P},$$

because of (7). But then, using the monotonicity of the μ_p, we conclude that

$$(1+h)^{\bar{N}-1}\mu_q(t_0) \leq \mu_\ell(t), \qquad \forall \ell \in \mathcal{P}.$$

In light of this, we must have

$$(1+h)^{\bar{N}-1} \min_{p \in \mathcal{P}} \mu_p(t_0) \leq \mu_\ell(t), \qquad \forall \ell \in \mathcal{P},$$

[3]Given a scalar a, we denote by $\lceil a \rceil$ the smallest integer larger or equal to a.

from which (3) follows using the fact that $\bar{N} \geq \frac{N_\sigma(t_0,t)-1}{m}$.
It remains to show that (5) holds. Grouping all the terms of the summation in (5) for which $\sigma(t_k)$ is the same, we obtain

$$\sum_{k=0}^{N_\sigma(t_0,t)} \Big(\mu_{\sigma(t_k)}(t_{k+1}) - \mu_{\sigma(t_k)}(t_k)\Big) = \sum_{q\in\mathcal{P}} \sum_{\substack{k=0 \\ \sigma(t_k)=q}}^{N_\sigma(t_0,t)} \Big(\mu_q(t_{k+1}) - \mu_q(t_k)\Big). \tag{10}$$

Take now some value $q \in \mathcal{P}$ that σ takes on $[0,t)$. Since the intervals are non-overlapping, using the monotonicity of μ_q, we conclude that

$$\sum_{\substack{k=0 \\ \sigma(t_k)=q}}^{N_\sigma(t_0,t)} \Big(\mu_q(t_{k+1}) - \mu_q(t_k)\Big) \leq \mu_q(t_{\bar{k}_q+1}) - \mu_q(t_0), \tag{11}$$

where $\bar{k}_q$ denotes the largest index $k \in \{0,1,\ldots,N_\sigma(t_0,t)\}$ for which $\sigma(t_k) = q$. Now, because of (6),

$$\mu_q(t_{\bar{k}_q+1}) \leq (1+h)\mu_\ell(t_{\bar{k}_q+1}), \qquad\qquad \forall \ell \in \mathcal{P}.$$

From this, (11), and the monotonicity of the μ_p, we conclude that

$$\sum_{\substack{k=0 \\ \sigma(t_k)=q}}^{N_\sigma(t_0,t)} \Big(\mu_q(t_{k+1}) - \mu_q(t_k)\Big) \leq (1+h)\mu_\ell(t) - \min_{p\in\mathcal{P}} \mu_p(t_0), \qquad\qquad \forall \ell \in \mathcal{P}.$$

Inequality (5) follows from this, (10), and the fact that $\mathcal{P}$ has m elements. ■

3 Relaxing the Open-Loop Assumptions

The scale-independence property mentioned in Section 1 can be used to somewhat relax the "Open-Loop" Assumptions 1. Suppose that the following assumptions hold.

Assumption 2 (Relaxed Open-Loop). There exists a positive time-function ϑ such that for each pair $\{x_0, s\} \in \mathcal{X}_0 \times \mathcal{S}$, the *scaled monitoring signals* $\bar{\mu}_p(t) := \vartheta(t)\Pi(p, x_{\{x_0,s\}}, t)$, $t \in [0, T_{\{x_0,s\}})$, $p \in \mathcal{P}$, are such that:

1. There exists a positive constant ϵ such that $\bar{\mu}_p(0) \geq \epsilon$, for each $p \in \mathcal{P}$.
2. Each scaled monitoring signal $\bar{\mu}_p$, $p \in \mathcal{P}$, is monotone nondecreasing on $[0, T_{\{x_0,s\}})$.

In light of the scale independence property, $\mathbb{S}_\mathbb{H}$'s output σ remains unchanged if its monitoring function/input signal pair $\{\Pi, x\}$ is replaced by another pair $\{\bar{\Pi}, x\}$ satisfying

$$\bar{\Pi}(p,x,t) = \vartheta(t)\Pi(p,x,t), \qquad\qquad \forall p \in \mathcal{P},\ t \geq 0.$$

Since the Relaxed Open-Loop Assumptions guarantee that the original Open-Loop Assumptions hold for the pair $\{\bar{\Pi}, x\}$, we obtain the following corollary of Theorem 1.

Corollary 1. *Let $\mathcal{P}$ be a finite set with m elements and assume that the Relaxed Open-Loop Assumptions hold. For a fixed initial state $\{x_0, \sigma_0\} \in \mathcal{X}_0 \times \mathcal{P}$, let $\{x, \sigma\}$ denote the unique solution to (1) with σ generated by $\mathbb{S}_\mathbb{H}$ and let $[0, T)$ denote the largest interval on which this solution is defined. For any $\ell \in \mathcal{P}$,*

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log\Big(\frac{\bar{\mu}_\ell(t)}{\min_{p\in\mathcal{P}} \bar{\mu}_p(t_0)}\Big), \qquad\qquad 0 \leq t_0 < t < T,$$

and, when the scaled monitoring signals are differentiable, we also have that

$$\int_{t_0}^{t} \dot{\bar{\mu}}_\sigma(\tau)d\tau \leq m\Big((1+h)\bar{\mu}_\ell(t) - \min_{p\in\mathcal{P}} \bar{\mu}_p(t_0)\Big), \qquad 0 \leq t_0 < t < T, \tag{12}$$

where $\dot{\bar{\mu}}_\sigma$ is defined to be equal to $\dot{\bar{\mu}}_p$ on intervals where σ is constant and equal to $p \in \mathcal{P}$, and zero at points of discontinuity of σ.

It turns out that in supervisory control the original Open-Loop Assumptions may often be violated, whereas the relaxed ones can be shown to hold (cf. Section 5 and [6, 7, 11, 12, 13, 14, 15]).

Remark 2. Also here, when the scaled monitoring signals are not piecewise differentiable, equation (12) can be generalized to

$$\sum_{k=0}^{N_\sigma(t,t_0)} \Big(\bar{\mu}_{\sigma(t_k)}(t_{k+1}) - \bar{\mu}_{\sigma(t_k)}(t_k)\Big) \leq m\Big((1+h)\bar{\mu}_\ell(t) - \min_{p\in\mathcal{P}} \bar{\mu}_p(t_0)\Big), \qquad 0 \leq t_0 < t < T,$$

where $t_1 < t_2 < \cdots < t_{N_\sigma(t,t_0)}$ denote the discontinuities of σ on (t_0, t) and $t_{N_\sigma(t,t_0)+1} := t$.

4 Stability of Switched Linear Systems

To show how the Scale-Independent Hysteresis Switching Theorem can be used in the analysis of a supervisory control algorithm we need to recall some results from [3]. Consider the time-varying switched system

$$\dot{x} = A_\sigma x, \tag{13}$$

where $\{A_p : p \in \mathcal{P}\}$ is an indexed family of $n \times n$ stability matrices and σ is a piecewise constant switching signal taking values on $\mathcal{P}$. Given a set of piecewise constant switching signals $\mathcal{S}$, we say that (13) is *uniformly exponentially stable over* $\mathcal{S}$ if there exist positive constants c and λ such that[4], for each $\sigma \in \mathcal{S}$,

$$\|\Phi_\sigma(t,\tau)\| \leq ce^{-\lambda(t-\tau)}, \qquad \forall t \geq \tau \geq 0,$$

where $\Phi_\sigma(t,\tau)$ is the state transition matrix of (13). When we want to emphasize the rate of decay in the above bound we add that (13) has *stability margin* λ.

Given a positive constant τ_D, let $\mathcal{S}[\tau_D]$ denote the set of all switching signals with the interval between discontinuities no smaller than τ_D. The constant τ_D is called the *(fixed) dwell-time*. It was shown, e.g., in [6, Lemma 2], that one can pick τ_D sufficiently large so that (13) is uniformly exponentially stable over $\mathcal{S}[\tau_D]$. A similar result holds when $\mathcal{S}[\tau_D]$ is enlarged to contain signals that occasionally have switchings separated by less than τ_D, but for which the average interval between consecutive discontinuities is no less than τ_D. We proceed to formalize this concept of "average dwell-time." For given $N_0, \tau_D > 0$, we denote by $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$ the set of all switching signals for which

$$N_\sigma(t,\tau) \leq N_0 + \frac{t-\tau}{\tau_D}.$$

The constant τ_D is called the *average dwell-time* and N_0 the *chatter bound*. The following is taken from [3]:

Theorem 2. *Given a compact set of $n \times n$ matrices $\{A_p : p \in \mathcal{P}\}$ and a positive constant λ_0 such that $A_p + \lambda_0 I$ is asymptotically stable for each $p \in \mathcal{P}$, then, for any $\lambda \in [0, \lambda_0)$, there is a finite constant τ_D^* such that* (13) *is uniformly exponentially stable over $\mathcal{S}_{\text{ave}}[\tau_D, N_0]$ with stability margin λ, for any average dwell-time $\tau_D \geq \tau_D^*$ and any chatter bound $N_0 > 0$.*

[4]Given a matrix A, we denote by $\|A\|$ the largest singular value of A.

Take now a bounded family of $n \times m$ matrices $\mathcal{B} := \{B_p : p \in \mathcal{P}\}$, also parameterized by the elements of the index set $\mathcal{P}$, and consider the m-input system

$$\dot{z} = A_\sigma z + B_\sigma u, \tag{14}$$

where $u : [0, \infty) \to \mathbb{R}^m$ denotes some piecewise continuous input and σ some switching signal in $\mathcal{S}$. Given a nonnegative constant λ, we say that (14) has *input-to-state $e^{\lambda t}$-weighted, $\mathcal{L}_2$-induced norm uniformly bounded over $\mathcal{S}$* if there exist finite constants $\mathfrak{g}$, $\mathfrak{g}_0$ such that, for each piecewise continuous input u and each $\sigma \in \mathcal{S}$,

$$\left(\int_0^t e^{2\lambda\tau} \|z(\tau)\|^2 \right)^{\frac{1}{2}} \leq \mathfrak{g} \left(\int_0^t e^{2\lambda\tau} \|u(\tau)\|^2 \right)^{\frac{1}{2}} + \mathfrak{g}_0 \|z(0)\|, \quad t \geq 0, \tag{15}$$

along solutions to (14) [5]. If (15) is replaced by

$$e^{\lambda t}\|z(t)\| \leq \mathfrak{g} \sup_{[0,t)} e^{\lambda\tau}\|u(\tau)\| + \mathfrak{g}_0\|z(0)\|, \qquad t \geq 0,$$

we say that (14) has *input-to-state $e^{\lambda t}$-weighted, $\mathcal{L}_\infty$-induced norm uniformly bounded over $\mathcal{S}$*, and if (15) is replaced by

$$e^{\lambda t}\|z(t)\| \leq \mathfrak{g} \left(\int_0^t e^{2\lambda\tau} \|u(\tau)\|^2 \right)^{\frac{1}{2}} + \mathfrak{g}_0\|z(0)\|, \qquad t \geq 0,$$

we say that (14) has *input-to-state $e^{\lambda t}$-weighted, $\mathcal{L}_2$-to-$\mathcal{L}_\infty$-induced norm uniformly bounded over $\mathcal{S}$*. The following Lemma is proved in [21].

Lemma 1. *Given a family $\mathcal{S}$ of piecewise constant switching signals, if* (13) *is uniformly exponentially stable over $\mathcal{S}$, with stability margin λ_0, then, for any $\lambda \in [0, \lambda_0)$,* (14) *has input-to-state $e^{\lambda t}$-weighted, $\mathcal{L}_2$-induced norm uniformly bounded over $\mathcal{S}$. Similarly for the $\mathcal{L}_\infty$ and $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ induced norms.*

The finiteness of these three induced norms is crucial to the analysis of adaptive switching algorithms, especially when noise and unmodeled dynamics are taken into account (cf. Section 5 and [5, 6, 7, 8, 9, 10, 21]).

5 Supervisory Control

In this section we show how the previous results can be used in the context of supervisory control. We follow closely the formulation in [6, 7].

The problem addressed here is the set-point control of an imprecisely modeled process $\mathbb{P}$. In particular, we want to generate the control input u to the process so as to drive its output y to a constant reference r. The process has two other exogenous inputs that cannot be measured: a bounded measurement noise signal $\mathbf{n}$ and a bounded disturbance $\mathbf{d}$. For simplicity the signals u, y, $\mathbf{n}$, and $\mathbf{d}$ are scalar. $\mathbb{P}$ is assumed linear, time-invariant, with a stabilizable (through u) and detectable realization

$$\dot{x}_\mathbb{P} = A_\mathbb{P} x + B_\mathbb{P} u + D_\mathbb{P}\mathbf{d}, \qquad y = C_\mathbb{P} x + \mathbf{n}, \tag{16}$$

but precise values for $A_\mathbb{P}, B_\mathbb{P}, C_\mathbb{P}, D_\mathbb{P}$ are not known. It is known, however, that $\mathbb{P}$'s transfer function τ, from u to y, belongs to a family of transfer functions of the form $\mathcal{N} := \bigcup_{p\in\mathcal{P}} \mathcal{N}_p$, where p is an unknown parameter taking values in some parameter set $\mathcal{P}$ and each $\mathcal{N}_p$ denotes a family of transfer functions centered around a known, *nominal transfer function* ν_p, e.g., $\mathcal{N}_p := \{(1+\delta)\nu_p : \|\delta\|_\infty \leq \epsilon\}$. Here, ϵ denotes some small positive constant and δ a stable transfer function with $\mathcal{H}_\infty$-norm smaller than ϵ. For simplicity, in the sequel we assume that $\epsilon = 0$ and the set $\mathcal{P}$ is finite and equal to $\{1, 2, \ldots, m\}$. In a future paper we shall consider the general case of $\epsilon > 0$ and infinite $\mathcal{P}$.

[5] Given a vector x, we denote by $\|x\|$ the Euclidean norm of x.

The solution proposed in [6, 7] to solve this problem is based on Certainty Equivalence and starts with the selection of a family of linear, time-invariant *candidate controllers* $\mathcal{C} := \{\kappa_p : p \in \mathcal{P}\}$. Each κ_p would make the feedback closed-loop system in Figure 2 asymptotically stable if the process transfer function τ was known to belong to $\mathcal{N}_p$. To avoid pole-zero cancellations it is assumed that $\mathbb{P}$ does not have transmission zeros at the origin.

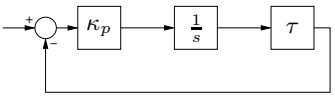

Figure 2: Feedback configuration.

In case we knew to which set $\mathcal{N}_p$ the actual process transfer function τ belonged, stability of the closed loop could be achieved with a nonadaptive, linear, time-invariant controller with transfer function equal to $\frac{1}{s}\kappa_p$. Since the process transfer function is not known in advance we build a "multi-controller" $\mathbb{C}$ that effectively allows switching between all the controller transfer functions in $\mathcal{C}$. If $\{(\bar{A}_p, \bar{B}_p, \bar{C}_p, \bar{D}_p) : p \in \mathcal{P}\}$ is a family of n-dimensional, stabilizable and detectable realizations for the transfer functions in $\mathcal{C}$, the multi-controller $\mathbb{C}$ can be defined by

$$\dot{x}_{\mathbb{C}} = \bar{A}_\sigma x_{\mathbb{C}} + \bar{B}_\sigma \mathbf{e_T}, \qquad v = \bar{C}_\sigma x_{\mathbb{C}} + \bar{D}_\sigma \mathbf{e_T}, \qquad \dot{u} = v, \tag{17}$$

where $\mathbf{e_T} := r - y$ and $\sigma : [0, \infty) \to \infty$ denotes a "switching signal" that, at each instant of time, determines which candidate controller is put into the feedback loop. The system that generates the switching signal σ is called a supervisor. Here we are interested in estimator-based supervisors like the one in Figure 3. An

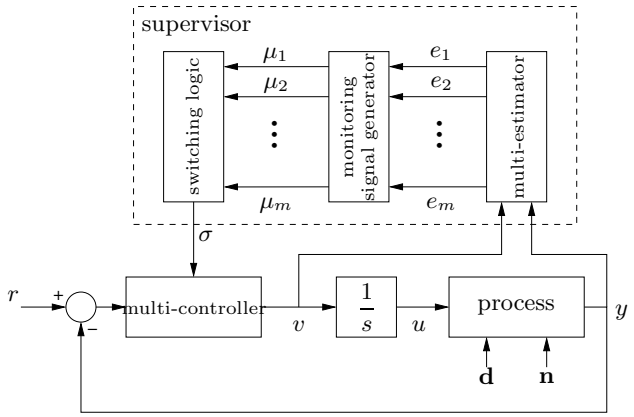

Figure 3: Supervisory control architecture.

estimator-based supervisor consists of three blocks: a multi-estimator, a monitoring signal generator, and a switching logic.

The *multi-estimator* $\mathbb{E}$ is a linear, time-invariant system whose inputs are the outputs of the process and multi-controller and whose outputs are the *output estimation errors* e_p, $p \in \mathcal{P}$. Each e_p is a signal that would converge to zero if the process transfer function τ was equal to the nominal transfer function ν_p. The reader is referred to [6, 7] for the precise structure of $\mathbb{E}$. Denoting by x the combined state of the multi-estimator and multi-controller (excluding the integrator), the evolution of x is determined by

$$\dot{x} = A_\sigma x + d_\sigma e_\sigma, \tag{18}$$

$$\mathbf{e_T} = c_{p^*} x + e_{p^*}, \tag{19}$$

where A_p, d_p, c_p, $p \in \mathcal{P}$ are appropriately defined matrices, and p^* is the element of $\mathcal{P}$ for which $\tau \in \mathcal{N}_{p^*}$. Equation (18) is obtained from equation (23) in [6] with $l = \sigma$ and (19) is obtained from equation (26) in [6] with $l = p^*$. It is also known that there exists a positive constant λ_0 for which each $\lambda_0 I + A_p$ is asymptotically stable (cf. Remark 4 in [6]). Moreover, from equation (28) in [6], one concludes that e_{p^*} is bounded and

$$\int_0^t e^{2\lambda\tau} e_{p^*}(\tau)^2 d\tau \leq c_n e^{2\lambda t} + c_0, \qquad t \geq 0, \tag{20}$$

$$\|e_{p^*}(t)\| \leq d_n + d_0 e^{-\lambda t}, \qquad t \geq 0, \tag{21}$$

where λ is any constant in $(0, \lambda_0)$, c_0, d_0 are positive constants that depends only on initial conditions, and c_n, d_n are positive constants that depends only on upper bounds on the norms of $\mathbf{n}$ and $\mathbf{d}$.

The *monitoring signal generator* $\mathbb{G}$ takes as inputs the output estimation errors e_p, $p \in \mathcal{P}$, and produces the *monitoring signals* μ_p, $p \in \mathcal{P}$ defined by

$$\dot{\tilde{\mu}}_p = -2\lambda\tilde{\mu}_p + e_p^2, \qquad \mu_p = \tilde{\mu}_p + \epsilon_\mu, \qquad p \in \mathcal{P}, \tag{22}$$

with $\lambda \in (0, \lambda_0)$ and $\epsilon_\mu > 0$ constant. $\mathbb{G}$ is initialized so that $\tilde{\mu}_p(0) \geq 0$, $p \in \mathcal{P}$.

The *switching logic* $\mathbb{S}$ generates the switching signal σ based on the values of the monitoring signals μ_p, $p \in \mathcal{P}$. The logic used here is the Scale-Independent Hysteresis Switching Logic defined in Section 1.

Suppose now that we define *scaled monitoring signals* $\bar{\mu}_p := \vartheta\mu_p$, with $\vartheta(t) := e^{2\lambda t}$, $t \geq 0$. From (22) one concludes that, for each $t \geq t_0 \geq 0$,

$$\bar{\mu}_p(t) = \tilde{\mu}_p(t_0) + e^{2\lambda t}\epsilon_\mu + \int_{t_0}^t e^{2\lambda\tau} e_p(\tau)^2 d\tau, \tag{23}$$

and therefore each $\bar{\mu}_p$ is always monotone increasing and never smaller than ϵ_μ. By the Scale-Independent Hysteresis Switching Theorem (or more precisely Corollary 1), we can then conclude that, for any $\ell \in \mathcal{P}$ and $0 \leq t_0 \leq t < T$,

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m}{\log(1+h)} \log\Big(\frac{\bar{\mu}_\ell(t)}{\inf_{p\in\mathcal{P}} \bar{\mu}_p(t_0)}\Big) \tag{24}$$

and

$$\int_{t_0}^t \left(2\lambda e^{2\lambda\tau}\epsilon_\mu + e^{2\lambda\tau} e_\sigma^2\right) d\tau \leq m\Big((1+h)\bar{\mu}_\ell(t) - \inf_{p\in\mathcal{P}} \bar{\mu}_p(t_0)\Big). \tag{25}$$

In (25) we used the fact that $\frac{\mathrm{d}}{\mathrm{d}\tau}\big(\bar{\mu}_{\sigma(\tau)}(\tau)\big) = 2\lambda e^{2\lambda\tau}\epsilon_\mu + e^{2\lambda\tau} e_\sigma^2$, wherever the derivative exists. Now, from (20) and (23) we obtain $\bar{\mu}_{p^*}(t) \leq e^{2\lambda t}(\epsilon_\mu + c_n) + \bar{c}_0$, $t \geq 0$, where $\bar{c}_0 := c_0 + \tilde{\mu}_{p^*}(0)$. From this, (24)–(25) with $\ell := p^*$, and the fact that $\bar{\mu}_p(t_0) \geq e^{2\lambda t_0}\epsilon_\mu$, $p \in \mathcal{P}$, we conclude that

$$N_\sigma(t, t_0) \leq 1 + m + \frac{m \log\left(e^{2\lambda(t-t_0)}\left(1 + \frac{c_n}{\epsilon_\mu}\right) + \frac{\bar{c}_0}{\epsilon_\mu}\right)}{\log(1+h)},$$

$$\int_{t_0}^t e^{2\lambda\tau} e_\sigma^2 d\tau \leq m\Big((1+h)\left(e^{2\lambda t}(\epsilon_\mu + c_n) + \bar{c}_0\right) - e^{2\lambda t_0}\epsilon_\mu\Big) - \left(e^{2\lambda t} - e^{2\lambda t_0}\right)\epsilon_\mu,$$

for every $t \geq t_0 \geq 0$. Since for $a, b > 0$, $\log(a+b) \leq \log(2a) + \log(2b)$, we also conclude that

$$N_\sigma(t, t_0) \leq N_0 + \frac{t - t_0}{\bar{\tau}_D}, \tag{26}$$

$$\int_{t_0}^t e^{2\lambda\tau} e_\sigma^2 d\tau \leq \bar{m} e^{2\lambda t} c_n + \bar{m}\bar{c}_0 - (m-1)e^{2\lambda t_0}\epsilon_\mu + (\bar{m} - 1)e^{2\lambda t}\epsilon_\mu, \tag{27}$$

with $\tau_D := \frac{\log(1+h)}{2\lambda m}$, $\bar{m} = m(1+h)$, and

$$N_0 := 1 + m + \frac{m}{\log(1+h)} \log\Big(\frac{4\bar{c}_0}{\epsilon_\mu}\Big(1 + \frac{c_n}{\epsilon_\mu}\Big)\Big).$$

Now, because of Lemma 1 and Theorem 2, there is a finite constant τ_D^* such that (18) has input-to-state $e^{\lambda t}$-weighted $\mathcal{L}_2$-to-$\mathcal{L}_\infty$ norm uniformly bounded over $\mathcal{S}_{\text{ave}}[\tau_D^*, N_0]$. If we then choose λ and h so that $\frac{\lambda}{\log(1+h)} \leq \gamma := \frac{1}{2m\tau_D^*}$, we get $\tau_D \geq \tau_D^*$ and the output σ of the switching logic is guaranteed to be in $\mathcal{S}_{\text{ave}}[\tau_D^*, N_0]$. From this and (27) one concludes that x is bounded and, because of (17) and (19), $\mathbf{e_T}$ and v are also bounded. The boundedness of u and the internal state of the process follows from the detectability of the cascade formed by the integrator in (17) and the process (16). The following can then be stated.

Theorem 3. *There exists a positive constant γ such that, whenever $\frac{\lambda}{\log(1+h)} \leq \gamma$, all signals remain bounded, for any bounded $\mathbf{n}$ and $\mathbf{d}$, and any initialization of $\mathbb{P}$, $\mathbb{E}$, $\mathbb{C}$, $\mathbb{G}$, $\mathbb{S}$, with $\tilde{\mu}_p(0) \geq 0$, $p \in \mathcal{P}$.*

Boundedness of all signals is certainly the primary concern for any control algorithms, but it is also desirable to have convergence to zero in the absence of disturbances and measurement noise. The previous result does not guarantee that will happen. In fact, without too much effort one concludes that, when $\mathbf{d}$ and $\mathbf{n}$ are identically zero, the bound for x is of the form

$$\|x(t)\| \leq \alpha_1 e^{-\lambda(t-t_0)} + \alpha_2 \epsilon_\mu, \qquad\qquad t \geq 0,$$

where α_1 and α_2 are appropriately defined positive constants. However, we believe that it is actually possible to prove that all signals converge to zero in the absence of disturbances and measurement noise. This is supported by computer simulations of the closed-loop system.

6 Conclusions

We showed that, under suitable "open-loop" assumptions, one can establish an upper bound on the number of switchings produced by the Scale-Independent Hysteresis Switching Logic on a given interval. This bound comes as a function of the variation of the inputs to the logic on that interval. Computing upper bounds on the number of switchings produced by hysteresis-based logics has been the main difficulty in applying them to adaptive control when noise and exogenous disturbances are present. With the properties derived here we were able to show that, although this logic does not guarantee the existence of a fixed dwell-time between switchings, it can produce switching that is slow-on-the-average. This allowed us to analyze hysteresis-based supervisory control algorithms in the presence of noise. To the best of our knowledge, this is the first time that an algorithm of this type is analyzed without relying on switching stopping. The results presented here are confined to the case of a finite parameter set $\mathcal{P}$ and the absence of unmodeled dynamics. In a future paper we shall remove these assumptions.

References

[1] J. P. Hespanha, *Logic-Based Switching Algorithms in Control*. PhD thesis, Yale University, New Haven, CT, 1998.

[2] J. P. Hespanha and A. S. Morse, "Scale-independent hysteresis switching," in *Hybrid Systems: Computation and Control* (F. W. Vaandrager and J. H. van Schuppen, eds.), vol. 1569 of *Lecture Notes in Computer Science*, pp. 117–122, Berlin: Springer-Verlag, Mar. 1999.

[3] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. of the 38th Conf. on Decision and Contr.*, pp. 2655–2660, Dec. 1999.

[4] A. S. Morse, "Control using logic-based switching," in *Trends in Control: A European Perspective* (A. Isidori, ed.), pp. 69–113, London: Springer-Verlag, 1995.

[5] M. Chang and E. J. Davison, "Switching control of a family of plants," in *Proc. of the 14th Amer. Contr. Conf.*, vol. 1, pp. 1015–1020, June 1995.

[6] A. S. Morse, "Supervisory control of families of linear set-point controllers—part 1: exact matching," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1413–1431, Oct. 1996.

[7] A. S. Morse, "Supervisory control of families of linear set-point controllers—part 2: robustness," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1500–1515, Nov. 1997.

[8] K. S. Narendra and J. Balakrishnan, "Adaptive control using multiple models," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 171–187, Feb. 1997.

[9] F. M. Pait and F. Kassab, Jr., "Parallel algorithms for adaptive control: Robust stability," in *Control Using Logic-Based Switching* (A. S. Morse, ed.), no. 222 in Lecture Notes in Control and Information Sciences, pp. 262–276, London: Springer-Verlag, 1997.

[10] J. Hockerman-Frommer, S. R. Kulkarni, and P. J. Ramadge, "Controller switching based on output prediction errors," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 596–607, May 1998.

[11] G. Chang, J. P. Hespanha, A. S. Morse, M. Netto, and R. Ortega, "Supervisory field-oriented control of induction motors with uncertain rotor resistance," in *Proc. of the 1998 IFAC Workshop on Adaptive Control and Signal Processing, Glasgow, Scotland*, Aug. 1998.

[12] S. Fujii, J. P. Hespanha, and A. S. Morse, "Supervisory control of families of noise suppressing controllers," in *Proc. of the 37th Conf. on Decision and Contr.*, pp. 1641–1646, Dec. 1998.

[13] J. P. Hespanha and A. S. Morse, "Certainty equivalence implies detectability," *Syst. & Contr. Lett.*, vol. 36, pp. 1–13, Jan. 1999.

[14] J. P. Hespanha and A. S. Morse, "Supervisory control of integral-input-to-state stabilizing controllers," in *Proc. of the 1999 European Contr. Conf.*, Aug. 1999.

[15] J. P. Hespanha, D. Liberzon, and A. S. Morse, "Logic-based switching control of a nonholonomic system with parametric modeling uncertainty," *Syst. & Contr. Lett.* Special Issue on Hybrid Systems, vol. 38, pp. 167–177, Nov. 1999.

[16] J. P. Hespanha and A. S. Morse, "Towards the supervisory control of families of nonlinear regulators," in *Proc. of the 30th Annual Conf. on Information Sciences and Syst.*, vol. 2, pp. 765–768, Mar. 1996.

[17] R. H. Middleton, G. C. Goodwin, D. J. Hill, and D. Q. Mayne, "Design issues in adaptive control," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 50–58, Jan. 1988.

[18] A. S. Morse, D. Q. Mayne, and G. C. Goodwin, "Applications of hysteresis switching in parameter adaptive control," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1343–1354, Sept. 1992.

[19] S. R. Kulkarni and P. J. Ramadge, "Model and controller selection policies based on output prediction errors," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1594–1604, Nov. 1996.

[20] J. K. Hale, *Ordinary Differential Equations*. Wiley-Interscience, 1969.

[21] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," tech. rep., EE–Systems, University of Southern California / Lab. for Control Science & Eng., Yale University, Jan. 1999.