

# LIMITS OF PERFORMANCE IN REFERENCE-TRACKING AND PATH-FOLLOWING FOR NONLINEAR SYSTEMS

Technical Report

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Abstract: We investigate limits of performance in reference-tracking and path-following and highlight an essential difference between them. For a class of nonlinear systems, we show that in reference-tracking, the smallest achievable  $\mathcal{L}_2$ -norm of the tracking error is equal to the least amount of control energy needed to stabilize the zero-dynamics of the error system. We then show that this fundamental performance limitation does not exist when the control objective is to force the output to follow a geometric *path* without a timing law assigned to it. This is true even when an additional desired speed assignment is required to be satisfied asymptotically or in finite time.

Keywords: Limits of performance, non-minimum phase nonlinear systems, path-following, reference-tracking, cheap-control.

## 1. INTRODUCTION

Obstacles to achieving perfect tracking with linear feedback systems have been quantified with classical Bode integrals and as the limits of cheap optimal control performance [Kwakernaak and Sivan, 1972; Middleton, 1991; Qiu and Davison, 1993; Seron *et al.*, 1999; Chen *et al.*, 2000]. While in the absence of unstable zero dynamics (*non-minimum phase zeros*) perfect tracking of any reference signal is possible, this is not the case with unstable zero dynamics. A formula derived by Qiu and Davison [1993], see also [Su *et al.*, 2003], shows that the tracking error increases as the signal frequencies approach those of the unstable zeros.

Seron *et al.* [1999] re-interpreted the Qiu-Davison formula and generalized it to a class of nonlinear systems. They showed that the best attainable

value of the  $\mathcal{L}_2$ -norm of the tracking error –  $J_T$  – is equal to the lowest control effort, measured by the  $\mathcal{L}_2$ -norm, needed to stabilize the zero dynamics driven by the system output  $y(t)$ . It is its role as a stabilizing control input that prevents the output  $y(t)$  from perfect tracking. Extensions to non-right-invertible systems are given in [Woodyatt *et al.*, 2002; Braslavsky *et al.*, 2002].

In the first part of this paper we present an internal model analog of the results in [Seron *et al.*, 1999; Braslavsky *et al.*, 2002]. In this more general case, the zero dynamics system to be stabilized is driven by the tracking error. As before, the best attainable value of  $J_T$  is the control needed for stabilization.

In the second part of the paper we present our main result for the path-following problem. When, for a class of nonlinear systems, the control objective is to force the output to follow a geometric *path* without a timing law assigned to it, we show that the fundamental performance limitations imposed on reference-tracking by unstable zero dynamics do not apply. Furthermore, the same is

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true for the speed-assigned path-following problem, where an additional desired speed is required to be satisfied asymptotically or in finite time.

Recently, in [Aguiar *et al.*, 2004c] we present conditions under which for non-minimum phase linear systems these path following problems can be solved with arbitrarily small  $\mathcal{L}_2$ -norm of the path following error.

In Section 2 we formulate the reference-tracking and path-following problems. Performance limitations for linear systems for reference-tracking and recent results for path-following are briefly revised in Section 3. Section 4 presents the main results of the paper and concluding remarks are given in Section 5.

## 2. REFERENCE-TRACKING AND PATH-FOLLOWING PROBLEMS

### 2.1 Reference-tracking

For linear systems

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad (1)$$

Davison [1976] and Francis [1977] consider the reference signal  $r(t)$  generated by a known exosystem

$$\dot{w} = Sw, \quad r = Qw, \quad (2)$$

and show that the so-called *servomechanism or regulator problem* is solvable if and only if  $(A, B)$  is stabilizable,  $(C, A)$  is detectable, the number of inputs is at least as large as the number of outputs ( $m \geq q$ ), and the zeros of  $(A, B, C, D)$  do not coincide with the eigenvalues of  $S$ . The internal model approach of [Francis and Wonham, 1976; Francis, 1977] designs the reference-tracking controller

$$u(t) = Kx(t) + (\Gamma - K\Pi)w(t),$$

where  $K$  is such that  $(A + BK)$  is Hurwitz, and  $\Pi$  and  $\Gamma$  satisfy

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma, \\ 0 &= C\Pi + D\Gamma - Q. \end{aligned}$$

Then, the *tracking error*  $e_T(t) := y(t) - r(t)$  converges to zero, and the transients  $\tilde{x} := x - \Pi w$  and  $\tilde{u} := u - \Gamma w$  are governed by  $\dot{\tilde{x}} = (A + BK)\tilde{x}$ ,  $\tilde{u} = K\tilde{x}$ .

For the nonlinear regulator problem

$$\dot{x} = f(x, u), \quad y = h(x, u), \quad (3)$$

$$\dot{w} = s(w), \quad r = q(w), \quad (4)$$

where  $f(0, 0) = 0$ ,  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}^m$  the control,  $y \in \mathbb{R}^q$  the output,  $w \in \mathbb{R}^p$  the internal state of the exosystem, and  $r \in \mathbb{R}^q$  the reference signal, Isidori and Byrnes [1990] shows that it is solvable if and only if there exist smooth mappings  $\Pi(w)$ , and  $c(w)$ , satisfying  $\Pi(0) = 0$ ,  $c(0) = 0$ , and

$$\begin{aligned} \frac{\partial \Pi}{\partial w} s(w) &= f(\Pi(w), c(w)), \\ h(\Pi(w), c(w)) &- q(w) = 0. \end{aligned} \quad (5)$$

Krener [1992] presented necessary and sufficient conditions for local solvability of (5) when either the exosystem has a semisimple pole structure or the plant has a semisimple zero structure.

### 2.2 Path-following

*Path-following problems* are primarily concerned with the design of control laws that drive an object (robot arm, mobile robot, ship, aircraft, etc.) to reach and follow a geometric *path*. A secondary goal is to force the object moving along the path to satisfy some additional dynamic specification. A common approach to the path-following problem is to parameterize the geometric path  $y_d$  by a *path variable*  $\theta$  and then select a *timing law* for  $\theta$ , [Hauser and Hindman, 1995; Al-Hiddabi and McClamroch, 2002; Skjetne *et al.*, 2004; Aguiar *et al.*, 2004a]. Extending the approach of [Hauser and Hindman, 1995], a path-following controller was proposed in [Skjetne *et al.*, 2004] for a class of uncertain strict feedback nonlinear systems. A framework for path-following as a method to avoid some limitations in reference-tracking was described in [Aguiar *et al.*, 2004a]. The key idea is to use  $\theta$  as an additional control input to stabilize the unstable zero-dynamics while the original control variables keep the system on the path.

In path-following, the output  $y(t)$  is required to reach and follow a geometric path  $y_d(\theta)$  generated by the exosystem

$$\begin{aligned} \frac{d}{d\theta} w(\theta) &= s(w(\theta)), \quad w(\theta_0) = w_0 \\ y_d(\theta) &= q(w(\theta)), \end{aligned} \quad (6)$$

where  $\theta$  is the scalar path parameter, and  $w \in \mathbb{R}^p$  is the exogenous state. For a given *timing law*  $\theta(t)$ , the *path-following error* is defined as

$$e_P(t) := y(t) - y_d(\theta(t)). \quad (7)$$

The two path-following problems considered in the paper are:

**Geometric path-following:** For a desired path  $y_d(\theta)$ , design a controller that achieves:

- i) *boundedness*: the state  $x(t)$  is uniformly bounded for all  $t \geq 0$  and every initial condition  $(x(0), w(\theta(0))) = (x_0, w_0)$ , in some neighborhood of  $(0, 0)$ ,
- ii) *error convergence*: the path-following error  $e_P(t)$  converges to zero as  $t \rightarrow \infty$ , and
- iii) *forward motion*:  $\dot{\theta}(t) > c$  for all  $t \geq 0$ , where  $c$  is a positive constant.

**Speed-assigned path-following:** In addition to geometric path-following, given a desired constant speed  $v_d > 0$ , it is required that either  $\dot{\theta}(t) \rightarrow v_d$  as  $t \rightarrow \infty$ , or  $\dot{\theta}(t) = v_d \forall t \geq T$  and some  $T \geq 0$ .

As illustrated by Skjetne *et al.* [2004], these path-following problems provide natural settings for many engineering applications.

From a theoretical standpoint our main interest is to determine whether the freedom to select

a timing law  $\theta(t)$  can be used to achieve an arbitrarily small  $\mathcal{L}_2$ -norm of the path-following error, that is, whether  $\delta^* > 0$  in

$$\int_0^\infty \|e_P(t)\|^2 dt \leq \delta^* \quad (8)$$

can be made arbitrarily small, while keeping the closed-loop stable.

### 3. LIMITS OF PERFORMANCE FOR LINEAR SYSTEMS

#### 3.1 Reference-tracking

An important issue in reference-tracking problems is whether the  $\mathcal{L}_2$ -norm of the tracking error can be made arbitrarily small, that is, whether

$$\int_0^\infty \|e_T(t)\|^2 dt \leq \delta^*, \quad (9)$$

can be satisfied for an arbitrary  $\delta^* > 0$ , while keeping the closed-loop stable. For this to be the case, the zeros of  $(A, B, C, D)$  must lie in the open left half-plane  $\mathbb{C}^-$ .

The non-minimum phase zeros, that is the zeros in  $\mathbb{C}^+$ , impose a fundamental limitation on the attainable tracking performance (9). This is revealed by analyzing the cheap control and showing that the limit as  $\epsilon \rightarrow 0$  of the optimal value of the cost functional

$$J_\epsilon := \min_{\tilde{u}} \int_0^\infty [\|e_T(t)\|^2 + \epsilon^2 \|\tilde{u}(t)\|^2] dt \quad (10)$$

with  $\tilde{u}$  defined in Section 2.1, is strictly positive, [Kwakernaak and Sivan, 1972]. Qiu and Davison [1993] proved the following result:

*Theorem 1.* Let  $x(0) = 0$ ,  $r(t) := \eta_1 \sin \omega t + \eta_2 \cos \omega t$ , and assume that (1) is right-invertible and has no zero at  $j\omega$ . The best attainable performance  $J_T := \lim_{\epsilon \rightarrow 0} J_\epsilon$  is given by  $J_T = \eta' M \eta$ ,  $\eta = \text{col}(\eta_1, \eta_2)$  for some positive semi-definite  $M$  and

$$\text{trace } M = \sum_{i=1}^p \left( \frac{1}{z_i - j\omega} + \frac{1}{z_i + j\omega} \right),$$

where  $z_1, z_2, \dots, z_p$  are the non-minimum phase zeros of  $(A, B, C, D)$ .

For more general reference signals, [Su *et al.*, 2003] give explicit formulas which show the dependence of  $J_T$  on the non-minimum phase zeros and their frequency-dependent directional information.

#### 3.2 Path-following

In contrast to reference-tracking, the attainable performance for path-following is not limited by non-minimum phase zeros, [Aguiar *et al.*, 2004c]. Let

$$y_d(\theta) := \sum_{k=1}^{n_d} [a_k e^{j\omega_k \theta} + a_k^* e^{-j\omega_k \theta}], \quad (11)$$

where the  $\omega_k > 0$  are real numbers and the  $a_k$  are non-zero complex vectors, which can be generated

by an exosystem of the form (6). As in [Qiu and Davison, 1993; Woodyatt *et al.*, 2002; Braslavsky *et al.*, 2002; Su *et al.*, 2003], we assume that initially the system is at rest, i.e.,  $x(0) = 0$ .

*Theorem 2.* For the geometric path-following problem, if  $(A, B)$  is stabilizable, then for any given positive constant  $\delta^*$  there exist constant matrices  $K$  and  $L$ , and a timing law  $\theta(t)$  such that the feedback law

$$u(t) = Kx(t) + Lw(\theta(t)) \quad (12)$$

achieves (8).

*Proof.* See [Aguiar *et al.*, 2004c].  $\square$

We stress that the stabilizability of  $(A, B)$  is the *only* condition (necessary and sufficient) for the solvability of the geometric path-following problem using (12).

Next we show that an arbitrarily small  $\mathcal{L}_2$ -norm of the path-following error is attainable even when the speed assignment  $v_d$  is specified beforehand.

*Theorem 3.* For the speed-assigned path-following problem, let  $v_d$  be specified so that the eigenvalues of  $v_d S$  do not coincide with the zeros of (3), and assume that  $(A, B)$  is stabilizable. Then, (8) can be satisfied for any  $\delta^* > 0$  with a timing law  $\theta(t)$  and a controller of the form (12) but with time-varying piecewise-constant matrices  $K$  and  $L$ .

*Proof.* See [Aguiar *et al.*, 2004c].  $\square$

### 4. LIMITS OF PERFORMANCE FOR NONLINEAR SYSTEMS

To obtain analogous results for nonlinear systems we first extend the results of [Seron *et al.*, 1999; Braslavsky *et al.*, 2002] to the case when the reference signal for reference-tracking is generated by an exosystem (4). We then show that, in contrast to reference-tracking, the path-following problems can be solved with arbitrarily small  $\mathcal{L}_2$ -norm of the path-following error.

#### 4.1 Reference-tracking

We consider the class of nonlinear systems which are locally diffeomorphic to systems in strict-feedback form [Krstić *et al.*, 1995, Appendix G]<sup>3</sup>:

$$\dot{z} = f_0(z) + g_0(z)\xi_1, \quad (13a)$$

$$\dot{\xi}_1 = f_1(z, \xi_1) + g_1(z, \xi_1)\xi_2,$$

$\vdots$

$$\dot{\xi}_r = f_r(z, \xi_1, \dots, \xi_r) + g_r(z, \xi_1, \dots, \xi_r)u, \quad (13b)$$

$$y = \xi_1, \quad (13c)$$

where  $z \in \mathbb{R}^{n_z}$ ,  $\xi := \text{col}(\xi_1, \dots, \xi_r)$ ,  $\xi_i \in \mathbb{R}^m$ ,  $\forall i \in \{1, \dots, r\}$ ,  $u \in \mathbb{R}^m$ , and  $y \in \mathbb{R}^m$ .  $f_i(\cdot)$  and  $g_i(\cdot)$  are  $C^k$  functions of their arguments (for

<sup>3</sup> When convenient we use the compact form (3) for (13). In that case,  $f(\cdot)$  denotes the vector field described by the right-hand-side of (13a)–(13b),  $h(\cdot)$  the output map described by (13c), and  $x = \text{col}(z, \xi_1, \dots, \xi_r)$ .

some large  $k$ ),  $f_i(0, \dots, 0) = 0$ , and the matrices  $g_i(\cdot)$ ,  $i = 0, \dots, r$  are always nonsingular. We assume that initially the system is at rest, i.e.,  $(z, \xi) = (0, 0)$ .

When the reference-tracking problem is solvable, there exist maps  $\Pi = \text{col}(\Pi_0, \dots, \Pi_r)$ ,  $\Pi_0 : \mathbb{R}^p \rightarrow \mathbb{R}^{n_z}$ ,  $\Pi_i : \mathbb{R}^p \rightarrow \mathbb{R}^m$ ,  $\forall i \in \{1, \dots, r\}$ , and  $c : \mathbb{R}^p \rightarrow \mathbb{R}^m$  that satisfies (5). The following locally diffeomorphic change of coordinates

$$\tilde{z} = z - \Pi_0(w), \quad (14)$$

$$\tilde{\xi} := \text{col}(\tilde{\xi}_1, \dots, \tilde{\xi}_r), \quad (15)$$

$$\tilde{\xi}_i = \xi_i - \Pi_i(w), \quad i = 1, \dots, r \quad (16)$$

$$\tilde{u} = u - c(w), \quad (17)$$

transforms the system (13) into the *error system*

$$\dot{\tilde{z}} = \tilde{f}_0(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)e_T, \quad (18a)$$

$$\dot{\tilde{\xi}}_1 = \tilde{f}_1(\tilde{z}, \tilde{\xi}_1, w) + \tilde{g}_1(\tilde{z}, \tilde{\xi}_1, w)\tilde{\xi}_2,$$

$$\vdots \quad (18b)$$

$$\dot{\tilde{\xi}}_r = \tilde{f}_r(\tilde{z}, \tilde{\xi}_1, \dots, \tilde{\xi}_r, w) + \tilde{g}_r(\tilde{z}, \tilde{\xi}_1, \dots, \tilde{\xi}_r, w)\tilde{u},$$

$$e_T = \tilde{\xi}_1, \quad (18c)$$

where

$$\begin{aligned} \tilde{f}_0 &:= f_0(\tilde{z} + \Pi_0(w)) - f_0(\Pi_0(w)) \\ &\quad + \left[ g_0(\tilde{z} + \Pi_0(w)) - g_0(\Pi_0(w)) \right] q(w), \\ \tilde{g}_0 &:= g_0(\tilde{z} + \Pi_0(w)), \end{aligned}$$

$\tilde{f}_0(0, w) = 0$ ,  $\tilde{g}_0(\tilde{z}, 0) = g_0(\tilde{z})$ , and  $\tilde{f}_i(\cdot)$ ,  $\tilde{g}_i(\cdot)$ ,  $i = 1, \dots, r$  are appropriate defined functions that satisfy  $\tilde{f}_i(0, \dots, 0, w) = 0$  and  $\tilde{g}_i(\tilde{z}, \dots, \tilde{\xi}_i, 0) = g_i(\tilde{z}, \dots, \tilde{\xi}_i)$ . For our analysis we consider the following two optimal control problems.

**Cheap control problem:** For the system consisting of the error system (18) and the exosystem (4) with initial condition  $(\tilde{z}(0), \tilde{\xi}(0), w(0)) = (\tilde{z}_0, \tilde{\xi}_0, w_0)$ , find the optimal feedback law  $\tilde{u} = \alpha_{\delta, \epsilon}^{cc}(\tilde{z}, \tilde{\xi}, w)$  that minimizes the cost functional

$$\frac{1}{2} \int_0^\infty (\|e_T(t)\|^2 + \delta \|\tilde{z}(t)\|^2 + \epsilon^{2r} \|\tilde{u}(t)\|^2) dt \quad (19)$$

for some small  $\delta > 0$ ,  $\epsilon > 0$ . We denote by  $J_{\delta, \epsilon}^{cc}(\tilde{z}_0, \tilde{\xi}_0, w_0)$  the corresponding optimal value. The best-attainable cheap control performance for trajectory tracking is then defined by  $J_T := \lim_{(\delta, \epsilon) \rightarrow 0} J_{\delta, \epsilon}^{cc}(\tilde{z}_0, \tilde{\xi}_0, w_0)$ . As shown in [Krener, 2001],  $J_{\delta, \epsilon}^{cc}(\tilde{z}, \tilde{\xi}, w)$  is  $\mathcal{C}^{k-2}$  in some neighborhood of  $(0, 0, 0)$  for every  $\delta > 0$ ,  $\epsilon > 0$  under the following assumption:

*Assumption 1.* The local linearization around  $(z, \xi) = (0, 0)$  of system (13) is stabilizable and detectable, and the local linearization around  $w = 0$  of the exosystem (4) is stable.

**Minimum-energy problem:** For the system

$$\dot{\tilde{z}} = \tilde{f}_0(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)e_T, \quad (20a)$$

$$\dot{w} = s(w), \quad (20b)$$

with  $e_T$  viewed as the input, and initial condition  $(\tilde{z}(0), w(0)) = (\tilde{z}_0, w_0)$ , find the optimal feedback law  $e_T = \alpha_{\delta}^{me}(\tilde{z}, w)$  that minimizes the cost

$$\frac{1}{2} \int_0^\infty (\delta \|\tilde{z}(t)\|^2 + \|e_T(t)\|^2) dt, \quad (21)$$

for some small  $\delta > 0$ . We denote by  $J_{\delta}^{me}(\tilde{z}_0, w_0)$  the corresponding optimal value. Under Assumption 1,  $J_{\delta}^{me}(\tilde{z}, w)$  is  $\mathcal{C}^{k-2}$  in some neighborhood of  $(0, 0)$ .

Our analysis reveals that the best attainable value of  $J_T$  is equal to the least control effort needed to stabilize the corresponding zero dynamics system (20) driven by the tracking error  $e_T$ .

*Theorem 4.* Suppose that Assumption 1 holds and that (5) has a solution in some neighborhood of  $w = 0$ . Then, for any initial condition  $(\tilde{z}(0), \tilde{\xi}(0), w(0)) = (\tilde{z}_0, \tilde{\xi}_0, w_0)$  in some neighborhood of  $(0, 0, 0)$  there exists a solution to the cheap control problem and

$$J_T = \lim_{\delta \rightarrow 0} J_{\delta}^{me} \quad (22)$$

*Proof (outline).* Under Assumption 1 and from the formulations of the cheap control and minimum-energy problems, we conclude that for every  $\delta > 0$ ,  $\epsilon > 0$ , and every initial condition  $(\tilde{z}_0, \tilde{\xi}_0, w_0)$  in some neighborhood of  $(0, 0, 0)$ , the value functions  $J_{\delta}^{me}(\tilde{z}, w)$  and  $J_{\delta, \epsilon}^{cc}(\tilde{z}, \tilde{\xi}, w)$  exist and satisfy

$$J_{\delta}^{me}(\tilde{z}_0, w_0) \leq J_{\delta, \epsilon}^{cc}(\tilde{z}_0, \tilde{\xi}_0, w_0).$$

Therefore,

$$\lim_{\delta \rightarrow 0} J_{\delta}^{me} \leq \lim_{(\epsilon, \delta) \rightarrow 0} J_{\delta, \epsilon}^{cc} := J_T. \quad (23)$$

On the other hand, from Lemma 1 in Appendix we have

$$J_{\delta, \epsilon}^{cc}(\tilde{z}_0, \tilde{\xi}_0, w_0) \leq J_{\delta}^{me}(\tilde{z}_0, w_0) + O(\epsilon). \quad (24)$$

Therefore, the result (22) follows from (23)–(24), and the fact that  $\delta$ ,  $\epsilon$  can be made arbitrarily small.  $\square$

#### 4.2 Path-following

One can also define a cheap control problem for path-following by replacing  $e_T$  by  $e_P$  in (19). The following result shows that, in contrast to reference-tracking, the path-following problem can be solved with arbitrarily small  $\mathcal{L}_2$ -norm for the error.

We assume that the vector field  $s(w)$  and the output map  $q(w)$  of the exosystem (6) are linear, i.e., that  $s(w) = Sw$ ,  $q(w) = Qw$ , and that all eigenvalues of  $S \in \mathbb{R}^{p \times p}$  are non-zero and semisimple.

*Theorem 5.* Assume that (5) has a solution for  $s(w) = v_d Sw$ , for  $v_d$  almost everywhere on  $(0, \infty)$ . There exists a neighborhood around  $w = 0$  such that for all initial conditions  $w(\theta(0)) = w_0$  in that neighborhood and any  $\delta^* > 0$ , we can find a timing law for  $\theta(t)$  and a feedback law

$$u = c(w) + \alpha_{\delta, \epsilon}(z, \xi, w) \quad (25)$$

that solves the geometric path-following and satisfies (8).

*Proof (outline).* Choosing the timing law

$$\dot{\theta}(t) = v_d, \quad \theta(0) = 0, \quad (26)$$

where  $v_d > 0$  is a constant to be selected latter, the path-following problem can be viewed as the reference-tracking of the signal  $r(t)$  generated by

$$\dot{w}(t) = v_d S w(t), \quad r(t) = Q w(t). \quad (27)$$

The substitution in (5) yields

$$\begin{aligned} \frac{\partial \Pi}{\partial w} v_d S w &= f(\Pi(w), c(w)), \\ h(\Pi(w), c(w)) - Q w &= 0. \end{aligned} \quad (28)$$

The feedback law (25) is determined by the function  $c(w)$  that solves (28) and  $\alpha_{\delta, \epsilon}(z, \xi, w)$  which minimizes (19) for the error system (18) together with the exosystem (27) and for some small  $\delta > 0$ ,  $\epsilon > 0$ . From the choice of the timing law (26) and applying Theorem 4, we can conclude that in the limit as  $(\epsilon, \delta) \rightarrow 0$ ,  $J_P = J_T = \lim_{\delta \rightarrow 0} J_\delta^{me}$ .

Now, we need to show that  $J_\delta^{me}$  can be made arbitrarily small by selecting a sufficiently large  $v_d$ . To this end, we prove in Lemma 2 in Appendix that for the minimum-energy problem and every initial condition in some neighborhood of  $(\tilde{z}, w) = (0, 0)$ , there exist a sufficiently small  $\delta > 0$  in (21) and a feedback law  $e_T = \hat{\alpha}_\delta^{me}(\tilde{z}, w)$  for which  $J_\delta^{me}(\tilde{z}_0, w_0)$  can be bounded by

$$J_\delta^{me}(\tilde{z}_0, w_0) \leq \frac{1}{2} \tilde{z}'_0 P_0 \tilde{z}_0,$$

for some  $P_0 > 0$  that does not depend on  $v_d$ . Observe also that  $\tilde{z}_0 = \Pi_0(w_0)$ , since  $z(0) = 0$ . The proof is completed using Lemma 3 in Appendix.  $\square$

Next we show that an arbitrarily small  $\mathcal{L}_2$ -norm of the path-following error is attainable even when the speed assignment  $v_d$  is specified beforehand.

*Theorem 6.* Let  $v_d$  be specified so that (28) has a solution in some neighborhood of  $w = 0$ . For the speed-assigned path-following problem, (8) can be satisfied for any  $\delta^* > 0$  with a suitable timing law  $\theta(t)$  and a controller of the form (25) with time-varying piecewise-continuous mappings  $c(w)$  and  $\alpha(z, \xi, w)$ .

*Proof (outline).* To construct a path-following controller that satisfies (8) we start with

$$u = c_\sigma(w) + \alpha_\sigma(z, \xi, w), \quad (29a)$$

$$\dot{\theta} = v_\sigma, \quad (29b)$$

where for each positive constant  $v_\ell$ ,  $\ell \in \mathcal{I} := \{0, 1, 2, \dots, N\}$ , the mappings  $\Pi_\ell := \text{col}(\Pi_{\ell_0}, \Pi_{\ell_\xi})$ ,  $\Pi_{\ell_\xi} := \text{col}(\Pi_{\ell_{\xi_1}}, \dots, \Pi_{\ell_{\xi_r}})$ ,  $\Pi_{\ell_0} : \mathbb{R}^p \rightarrow \mathbb{R}^{n_z}$ ,  $\Pi_{\ell_i} : \mathbb{R}^p \rightarrow \mathbb{R}^m$ ,  $i = 1, \dots, r$ , and  $c_\ell : \mathbb{R}^p \rightarrow \mathbb{R}^m$  satisfy

$$\begin{aligned} \frac{\partial \Pi_\ell}{\partial w} v_\ell S w &= f(\Pi_\ell(w), c_\ell(w)), \\ h(\Pi_\ell(w), c_\ell(w)) - Q w &= 0, \end{aligned} \quad (30)$$

and  $\sigma(t) : [t_0 := 0, \infty) \rightarrow \mathcal{I}$ , is the piecewise constant switching signal

$$\sigma(t) = \begin{cases} i, & t_i \leq t < t_{i+1}, \quad i = 0, \dots, N-1 \\ N, & t \geq t_N \end{cases}$$

Each  $\alpha_\ell(z, \xi, w)$  is the optimal feedback-law that minimizes

$$\int_0^\infty (\|e_P\|^2 + \delta \|z - \Pi_{\ell_0}(w)\|^2 + \epsilon^{2r} \|u - c_\ell(w)\|^2) dt,$$

for some  $\delta > 0$ ,  $\epsilon > 0$  small. We observe that (29) is a speed-assignment path-following controller for which  $\dot{\theta}(t)$  converges to  $v_N = v_d$  in finite time.

We now prove that for any given  $\delta^*$ , (8) can be satisfied by appropriate selection of the finite sequence  $t_0, t_1, \dots, t_N$  together with  $(v_0, \Pi_0, \alpha_0, c_0)$ ,  $(v_1, \Pi_1, \alpha_1, c_1), \dots, (v_N, \Pi_N, \alpha_N, c_N)$  used in the feedback controller (29). To this end, we show in Lemma 4 in Appendix that  $J_P$  is bounded by

$$\begin{aligned} J_P &\leq \frac{1}{2} \tilde{z}'_0 P_0 \tilde{z}_0 + \frac{\lambda_{max}(P_0)}{2} \gamma \sum_{\ell=1}^N (v_{\ell-1} - v_\ell)^2 \\ &\quad + \lambda_{max}(P_0) \sum_{\ell=1}^N \tilde{z}_{\ell-1}(t_\ell)' [\tilde{z}_{\ell-1}(t_\ell) - \tilde{z}_\ell(t_\ell)] \\ &\quad + \frac{\lambda_{max}(P_0)}{2} \sum_{\ell=1}^N \|\tilde{z}_{\ell-1}(t_\ell)\|^2, \end{aligned} \quad (31)$$

where  $P_0 > 0$ ,  $\gamma$  is a positive constant,  $\tilde{z}_0 = \tilde{z}(0)$ ,  $\tilde{z}_\ell := \Pi_{\ell_0}(w)$ , and the transient errors  $\tilde{z}_\ell := z - \Pi_{\ell_0}(w)$  converge to zero as  $t \rightarrow \infty$ .

We show that each term of (31) can be upper-bounded by  $\frac{\delta^*}{4}$  so that  $J_P \leq \delta^*$ . Applying the same arguments used to prove Theorem 5, the first term in (31) can be bounded by  $\frac{\delta^*}{4}$  by choosing a sufficiently large  $v_0$ . To prove that the second term in (31) is smaller than  $\frac{\delta^*}{4}$ , we select the parameters  $v_\ell$ ,  $\ell \in \mathcal{I}$  to satisfy

$$v_{\ell-1} - v_\ell = \mu, \quad v_N = v_d, \quad \ell = 1, 2, \dots, N \quad (32)$$

where  $\mu := \frac{\delta^*}{4\gamma(v_0 - v_N)}$ , and  $N := \frac{v_0 - v_N}{\mu}$ . Then, it follows that

$$\gamma \sum_{\ell=1}^N (v_{\ell-1} - v_\ell)^2 \leq \gamma N \mu^2 = \gamma (v_0 - v_N) \mu = \frac{\delta^*}{4}.$$

The above selection for the  $v_\ell$ ,  $\ell \in \mathcal{I}$ , is made under the constraint that the reference-tracking of the signal  $r(t)$  generated by (27) (with  $v_d$  replaced by  $v_\ell$ ) be solvable. This can always be satisfied by appropriately adjusting  $v_0$ . Finally, for a given finite  $N$ , each of the last two terms in (31) can be made smaller than  $\frac{\delta^*}{4}$  by choosing  $t_\ell$ ,  $\ell = 1, 2, \dots, N$  sufficiently large.  $\square$

## 5. CONCLUSIONS

We have first presented an analogy of the results in [Seron *et al.*, 1999; Braslavsky *et al.*, 2002] for reference-tracking using the internal model approach. We have shown that the best attainable value of  $J_T$  is the least amount of energy required to stabilize the corresponding zero-dynamics of

the error system. In main part of the paper we have demonstrated that the performance limitations can be avoided by reformulating the problem as path-following, where the path variable  $\theta$  is treated as an additional control variable. This conceptual result may be of practical significance, because the path-following formulation is convenient for many applications. Design of path-following controllers for non-minimum phase systems is a topic of current research.

## APPENDIX

*Lemma 1.* Suppose that Assumption 1 holds. For every  $\delta > 0$  and every initial condition  $(\tilde{z}_0, \tilde{\xi}_0, w_0)$  for the error system (18) and the exosystem (4) in some neighborhood of  $(0, 0, 0)$ , there exist a sufficiently small  $\epsilon > 0$  and feedback law  $\tilde{u} = \hat{\alpha}_{\delta, \epsilon}^{cc}(\tilde{z}, \tilde{\xi}, w)$  for which (19) is smaller than or equal to

$$J_{\delta}^{me}(\tilde{z}_0, w_0) + O(\epsilon). \quad (33)$$

*Proof.* This proof is inspired from [Braslavsky *et al.*, 2002]. Let  $\alpha_{\delta}^{me}(\tilde{z}, w)$  be the optimal control law for the minimum-energy problem, that is,

$$\alpha_{\delta}^{me}(\tilde{z}, w) = -\tilde{g}'_0 \frac{\partial J_{\delta}^{me}}{\partial \tilde{z}}, \quad (34)$$

which exists under Assumption 1. Consider the change of coordinates

$$\begin{aligned} \eta_1 &= \tilde{\xi}_1 - \alpha_{\delta}^{me}, & \eta_2 &= \dot{\tilde{\xi}}_1 - \dot{\alpha}_{\delta}^{me}, \dots \\ \eta_r &= \tilde{\xi}_1^{(r-1)} - \alpha_{\delta}^{(r-1)} \end{aligned} \quad (35)$$

that takes (18) to the form

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)\alpha_{\delta}^{me}(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)C\eta, \\ \dot{\eta} &= A\eta + B\tilde{f}_{\eta}(\tilde{z}, \eta, w) + Bv, \\ \eta_1 &= C\eta, \end{aligned} \quad (36)$$

where  $\eta := \text{col}(\eta_1, \eta_2, \dots, \eta_r)$ , the input  $v := \tilde{g}\tilde{u}$ ,  $\tilde{g} := \Pi_{i=1}^r \tilde{g}_i$ ,  $\tilde{f}_{\eta}(\cdot)$  is a suitable defined function, and

$$\begin{aligned} A &:= \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, & B &:= \begin{bmatrix} 0 \\ 0 \\ \dots \\ I \end{bmatrix}, \\ C &:= [I \ 0 \ 0 \ \dots \ 0]. \end{aligned} \quad (37)$$

The proof is organized in two parts: we first prove that for any  $\delta > 0$  and any initial condition  $(\tilde{z}_0, \eta_0, w_0)$  in some neighborhood of  $(0, 0, 0)$  for system (36) together with the exosystem (4), there exist a sufficient small  $\epsilon > 0$  and a feedback law  $v = \hat{\alpha}_{\delta, \epsilon}(\tilde{z}, \eta, w)$  for which

$$\begin{aligned} V_{\delta, \epsilon}(\tilde{z}_0, \eta_0, w_0) &:= \frac{1}{2} \int_0^{\infty} (\|\eta_1 + \alpha_{\delta}^{me}(\tilde{z}, w)\|^2 \\ &\quad + \delta\|\tilde{z}\|^2 + \epsilon^{2r}\|v\|^2) dt \end{aligned}$$

is smaller than or equal to (33) along trajectories to (4), (36). Then, in the second part, we derive a feedback law for  $\tilde{u}$  that is obtained from  $\hat{\alpha}_{\delta, \epsilon}$  and show that (19) is smaller than or equal to (33).

Let  $P \in \mathbb{R}^{mr \times mr}$  be the unique positive-definite solution of the algebraic Riccati equation

$$PA + A'P - PBB'P = -C'C \quad (38)$$

where  $A, B, C$  are given by (37), and consider the feedback law

$$v = -\frac{1}{\epsilon^r} B'PE\eta, \quad (39)$$

where

$$E = \begin{bmatrix} I & 0 & \dots & 0 \\ 0 & \epsilon I & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \epsilon^{r-1} I \end{bmatrix}.$$

Substituting (39) into (36), we obtain the closed-loop system

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)\alpha_{\delta}^{me}(\tilde{z}, w) \\ &\quad + \tilde{g}_0(\tilde{z}, w)CE\eta, \\ \epsilon E\dot{\eta} &= (A - BB'P)E\eta + \epsilon^r B\tilde{f}_{\eta}(\tilde{z}, \eta, w). \end{aligned} \quad (40)$$

We now show that (39) achieves asymptotic stability of the closed-loop system (40). Consider the candidate Lyapunov function

$$W = J_{\delta}^{me}(\tilde{z}, w) + \frac{\epsilon}{2} \eta'EQE\eta,$$

where  $Q \in \mathbb{R}^{mr \times mr}$  is the unique positive definite solution of the algebraic Riccati equation

$$Q(A - BB'P) + (A' - PBB')Q = -C'C - I$$

which exists because  $(A - BB'P)$  is Hurwitz. Computing  $\dot{W}$  along (40), we obtain

$$\begin{aligned} \dot{W} &= \frac{\partial J_{\delta}^{me}}{\partial \tilde{z}}(\tilde{f}_0 + \tilde{g}_0\alpha_{\delta}^{me} + \tilde{g}_0CE\eta) + \frac{\partial J_{\delta}^{me}}{\partial w}s(w) \\ &\quad + \frac{1}{2} \eta'E[Q(A - BB'P) + (A' - PBB')Q]E\eta \\ &\quad + \epsilon^r \eta'EQB\tilde{f}_{\eta} \\ &= -\frac{1}{2} \delta \|\tilde{z}\|^2 - \frac{1}{2} \|\alpha_{\delta}^{me}\|^2 - \alpha_{\delta}^{me'} C\eta - \frac{1}{2} \|CE\eta\|^2 \\ &\quad - \frac{1}{2} \|E\eta\|^2 + \epsilon^r \eta'EQB\tilde{f}_{\eta}, \end{aligned}$$

where we have used the fact that  $J_{\delta}^{me}(\tilde{z}, w)$  satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} \frac{\partial J_{\delta}^{me}}{\partial \tilde{z}} \tilde{f}_0(\tilde{z}, w) + \frac{\partial J_{\delta}^{me}}{\partial w} s(w) + \frac{1}{2} \delta \|\tilde{z}\|^2 \\ - \frac{1}{2} \left\| \tilde{g}'_0 \frac{\partial J_{\delta}^{me}}{\partial \tilde{z}} \right\|^2 = 0. \end{aligned} \quad (41)$$

By noting that  $\tilde{f}_{\eta}(\cdot)$  satisfies

$$\|\tilde{f}_{\eta}(\tilde{z}, \eta, w)\| \leq \gamma_{\tilde{z}} \|\tilde{z}\| + \gamma_{\eta} \|\eta\|,$$

for any positive constants  $\rho_{\tilde{z}}, \rho_{\eta}, \rho_w$  so that  $\|\tilde{z}\| \leq \rho_{\tilde{z}}, \|\eta\| \leq \rho_{\eta}, \|w\| \leq \rho_w$ , and sufficient large positive constants  $\gamma_{\tilde{z}}, \gamma_{\eta}$ , it follows that

$$\begin{aligned} \dot{W} &\leq -\frac{1}{2} \delta \|\tilde{z}\|^2 - \frac{1}{2} \|C\eta + \alpha_{\delta}^{me}\|^2 - \frac{1}{2} \|E\eta\|^2 \\ &\quad + \epsilon^r (\gamma_1 \|\tilde{z}\| + \gamma_2 \|\eta\|) \|E\eta\| \\ &\leq -\frac{1}{2} (\delta - \gamma_1 \epsilon^r) \|\tilde{z}\|^2 - \frac{1}{2} (1 - \gamma_1 \epsilon^r - 2\gamma_2 \epsilon) \|E\eta\|^2 \end{aligned}$$

where  $\gamma_1 := \gamma_{\tilde{z}} \|QB\|, \gamma_2 := \gamma_{\eta} \|QB\|$ , and we have used the fact that  $\epsilon^r \|\eta\| \leq \epsilon \|E\eta\|$  for  $\epsilon < 1$ .

Consequently, for any  $\delta > 0$ , we can always pick a sufficiently small  $\epsilon > 0$  such that  $\dot{W} \leq 0$ . Asymptotic stability of  $(\tilde{z}, \eta) = (0, 0)$  thus follows from LaSalle's theorem. Note also that  $W$  converges to zero as  $t \rightarrow \infty$  because  $J_\delta^{me}(0, w) = 0$ .

To prove that

$$V_{\delta, \epsilon}(\tilde{z}_0, \eta_0, w_0) \leq J_\delta^{me}(\tilde{z}_0, w_0) + O(\epsilon), \quad (42)$$

we define the positive-definite function

$$\hat{V} := J_\delta^{me}(\tilde{z}, w) + \epsilon \hat{V}_\epsilon(\eta),$$

where

$$\hat{V}_\epsilon := \frac{1}{2} \eta' E P E \eta.$$

The time-derivatives of  $J_\delta^{me}$  and  $\hat{V}_\epsilon$  along (40) satisfy

$$\dot{J}_\delta^{me} = -\frac{1}{2} \delta \|\tilde{z}\|^2 - \frac{1}{2} \|\alpha_\delta^{me}\|^2 - \alpha_\delta^{me'} C \eta, \quad (43)$$

$$\begin{aligned} \dot{\hat{V}}_\epsilon &= -\frac{1}{2\epsilon} \|C E \eta\|^2 - \frac{1}{2\epsilon} \|B' P E \eta\|^2 \\ &\quad + \epsilon^{r-1} \eta' E P B \tilde{f}_\eta. \end{aligned} \quad (44)$$

and

$$\begin{aligned} \dot{\hat{V}} &= -\frac{1}{2} \delta \|\tilde{z}\|^2 - \frac{1}{2} \|C \eta + \alpha_\eta\|^2 \\ &\quad - \frac{1}{2} \|B' P E \eta\|^2 + \epsilon^r \eta' E P B \tilde{f}_\eta. \end{aligned} \quad (45)$$

From the definitions of  $\hat{V}$  and  $W$ , and the fact that  $W$  converges to zero as  $t \rightarrow \infty$ , we can conclude that  $\hat{V}$  also converges to zero as  $t \rightarrow \infty$ . Thus, integrating (45) and by noticing that

$$V_{\delta, \epsilon} = \frac{1}{2} \int_0^\infty (\|C \eta + \alpha_\eta\|^2 + \delta \|\tilde{z}\|^2 + \|B' P E \eta\|^2) dt,$$

we get

$$V_{\delta, \epsilon} = \hat{V}(\tilde{z}_0, \eta_0, w_0) + \int_0^\infty \epsilon^r \eta' E P B \tilde{f}_\eta dt.$$

Integrating (44), the term  $\int_0^\infty \epsilon^{r-1} \eta' E P B \tilde{f}_\eta dt$  can be upper-bounded by  $\|\int_0^\infty \epsilon^{r-1} \eta' E P B \tilde{f}_\eta dt\| \leq V_\epsilon(\eta_0)$ , and therefore we conclude (42).

The second part of the proof consists of showing that for every  $\delta > 0$  and every initial condition in some neighborhood of  $(z, \xi, w) = (0, 0, 0)$ , there exists a  $\epsilon^* > 0$  for which (19) with the feedback law

$$\tilde{u} = \tilde{g}^{-1} \hat{\alpha}_{\delta, \epsilon^*}(\tilde{z}, \eta, w) = -\frac{1}{\epsilon^{*r}} \tilde{g}^{-1} B' P E \eta$$

is smaller than or equal to (33).

Let  $\gamma^r$  be the lowest value of the smallest singular value of  $\tilde{g}$  in a compact set containing the trajectories generated by  $u$ . Note that  $\gamma > 0$  because  $\tilde{g}_i^{-1}$ ,  $i = 1, \dots, r$  are nonsingular. Let  $\epsilon^* = \frac{\epsilon}{\gamma}$ , then,

$$\begin{aligned} \frac{1}{2} \int_0^\infty (\|e_T\|^2 + \delta \|\tilde{z}\|^2 + \epsilon^{2r} \|\tilde{u}\|^2) dt &\leq \\ \frac{1}{2} \int_0^\infty (\|e_T\|^2 + \delta \|\tilde{z}\|^2 + \frac{\epsilon^{2r}}{\gamma^{2r}} \|\tilde{g} \tilde{u}\|^2) dt & \\ = V_{\delta, \epsilon^*} \leq J_\delta^{me}(\tilde{z}_0, w_0) + O(\epsilon^*). & \end{aligned}$$

□

*Lemma 2.* Consider the minimum-energy problem formulated in Section 4. For every initial condition  $(\tilde{z}(0), w(0)) = (\tilde{z}_0, w_0)$  for (20) in some neighborhood of  $(0, 0)$ , there exist a sufficiently small  $\delta > 0$  and a feedback law  $e_T = \hat{\alpha}_\delta^{me}(\tilde{z}, w)$  for which (21) is smaller than or equal to

$$\frac{1}{2} \tilde{z}'_0 P_0 \tilde{z}_0.$$

for some  $P_0 > 0$  that does not depend on  $v_d$ .

*Proof.* Let

$$\dot{z} = F_0 z + G_0 \xi_1$$

be the local linearization around  $z = 0$  of (13a). The  $\tilde{z}$  dynamics (18a) can be written as

$$\dot{\tilde{z}} = F_0 \tilde{z} + G_0 e + h_{\tilde{f}_0}(\tilde{z}, w) + h_{\tilde{g}_0}(\tilde{z}, w) e_T, \quad (46)$$

where  $h_{\tilde{f}_0}(\tilde{z}, w) := \tilde{f}'_0(\tilde{z}, w) - F_0 \tilde{z}$ ,  $h_{\tilde{g}_0}(\tilde{z}, w) := \tilde{g}'_0(\tilde{z}, w) - G_0$  and satisfy for all  $\|\tilde{z}\| \leq \rho_{\tilde{z}}$ ,  $\|w\| \leq \rho_w$

$$\|h_{\tilde{f}_0}(\tilde{z}, w)\| \leq \gamma_1 \|\tilde{z}\|^2 + \gamma_2 \|\tilde{z}\| \|\Pi_0(w)\|,$$

$$\|h_{\tilde{g}_0}(\tilde{z}, w)\| \leq \gamma_3 \|\tilde{z}\|^2 + \gamma_4 \|\Pi_0(w)\|,$$

for some positive constants  $\gamma_i$ ,  $i = 1, \dots, 4$ . Consider (46) in closed-loop with the feedback law

$$e_T = \hat{\alpha}_\delta^{me}(\tilde{z}, w) := -G'_0 P_0 \tilde{z}, \quad (47)$$

where  $P_0 > 0$  satisfies

$$F'_0 P_0 + P_0 F_0 + I - P_0 G_0 G'_0 P_0 = 0. \quad (48)$$

Let

$$\hat{V}_\delta^{me} := \frac{1}{2} \tilde{z}' P_0 \tilde{z}.$$

Computing its time-derivative along the trajectories of (46), (47), yields

$$\begin{aligned} \dot{\hat{V}}_\delta^{me} &= \frac{1}{2} \tilde{z}' [P_0 F_0 + F'_0 P_0 - 2P_0 G_0 G'_0 P_0] \tilde{z} \\ &\quad + \frac{1}{2} \tilde{z}' [P h_{\tilde{f}_0} + h'_{\tilde{f}_0} P - 2P_0 h_{\tilde{g}_0} G'_0 P_0] \tilde{z} \\ &\leq -\frac{1}{2} \|e_T\|^2 - \frac{1}{2} \left[ 1 - 2\lambda_{\max}(P_0) (\gamma_1 \|\tilde{z}\|^2 \right. \\ &\quad \left. + \gamma_2 \|\tilde{z}\| \|\Pi_0(w)\| - \lambda_{\max}(P_0) (\gamma_3 \|\tilde{z}\| \right. \\ &\quad \left. + \gamma_4 \|\Pi_0(w)\|) \right] \|\tilde{z}\|^2. \end{aligned}$$

Therefore, there exist sufficiently small positive constants  $\rho_{\tilde{z}}$ ,  $\rho_w$  and a  $\delta > 0$  such that for all  $\|\tilde{z}\| \leq \rho_{\tilde{z}}$ ,  $\|w\| \leq \rho_w$ ,  $\hat{V}_\delta^{me}$  satisfies

$$\dot{\hat{V}}_\delta^{me} \leq -\frac{1}{2} \|e_T\|^2 - \frac{\delta}{2} \|\tilde{z}\|^2. \quad (49)$$

Thus,  $\hat{V}_\delta^{me}$  converges to zero as  $t \rightarrow \infty$  and integrating (49) we obtain

$$\hat{V}_\delta^{me}(\tilde{z}_0) \geq \frac{1}{2} \int_0^\infty (\|e_T\|^2 + \delta \|\tilde{z}\|^2) dt. \quad \square$$

*Lemma 3.* Consider the reference-tracking problem for the nonlinear system (13) with the vector field  $s(w)$  and the output map  $q(w)$  of the exosystem (4) given by  $s(w) = v_d S w$ ,  $q(w) = Q w$ .

Suppose that all eigenvalues of  $S \in \mathbb{R}^{p \times p}$  are non-zero and semisimple, and that (28) has a solution in some neighborhood of  $w = 0$  for some  $v_d > 0$ . Then, for any  $\rho > 0$ , there exists a sufficiently large  $v_d > 0$  such that the reference-tracking problem is solvable. Moreover, the mapping  $\Pi_0 : \mathbb{R}^p \rightarrow \mathbb{R}^{n_z}$  that satisfies

$$\frac{\partial \Pi_0(w)}{\partial w} S w = \mu [f_0(\Pi_0(w)) + g_0(\Pi_0(w)) Q w] \quad (50)$$

$\mu := \frac{1}{v_d}$ , is bounded by

$$\|\Pi_0(w)\| \leq \rho. \quad (51)$$

*Proof.* Without loss of generality, we assume that  $S$  is a diagonal matrix. For a given neighborhood  $\{w \in \mathbb{C}^p : \|w\| \leq \epsilon\}$  for which the solution exists and a given  $\rho > 0$ , we expand  $\Pi_0(w)$  in Taylor series

$$\begin{aligned} \Pi_0(w) &= \Pi_0^{[1]}(w) + \frac{1}{2} \Pi_0^{[2]}(w) + \dots \\ &\quad + \frac{1}{k!} \Pi_0^{[k]}(w) + O(w)^{k+1} \end{aligned} \quad (52)$$

where the superscript  $[d]$  denotes terms composed of homogeneous polynomials of degree  $d$ , i.e.,

$$\Pi_0^{[k]}(w) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq p} \Pi_{0_{i_1 \dots i_k}} w_{i_1} \dots w_{i_k}, \quad (53)$$

$\Pi_{0_{i_1 \dots i_k}} \in \mathbb{C}^{n_z}$ , and pick a sufficiently large  $N$  such that

$$\|\Pi_0(w) - \sum_{k=1}^N \frac{1}{k!} \Pi_0^{[k]}(w)\| \leq \frac{\rho}{2}, \quad \forall \|w\| \leq \epsilon. \quad (54)$$

We prove (51) if there exists  $\mu > 0$  for which all terms  $k = 1, 2, \dots, N$  of  $\Pi_{0_{i_1 \dots i_k}}$  can be bounded by

$$\|\Pi_{0_{i_1 \dots i_k}}\| \leq \delta, \quad \delta := \frac{\rho}{2(e^{p\epsilon} - 1)}. \quad (55)$$

Note that

$$\begin{aligned} \|\Pi_0(w)\| &\leq \|\Pi_0(w) - \sum_{k=1}^N \frac{1}{k!} \Pi_0^{[k]}(w)\| \\ &\quad + \|\sum_{k=1}^N \frac{1}{k!} \Pi_0^{[k]}(w)\| \\ &\leq \frac{\rho}{2} + \sum_{k=1}^{\infty} \frac{1}{k!} \|\Pi_0^{[k]}(w)\|. \end{aligned}$$

Since  $\|\Pi_0^{[k]}(w)\| \leq \delta p^k \|w\|^k$ , it follows that

$$\|\Pi_0(w)\| \leq \frac{\rho}{2} + \delta \sum_{k=1}^{\infty} \frac{1}{k!} (p\epsilon)^k = \frac{\rho}{2} + \delta(e^{p\epsilon} - 1) = \rho.$$

Expanding  $f_0(z)$ ,  $g_0(z)$  in Taylor series

$$f_0(z) = F_0 z + \frac{1}{2} f_0^{[2]}(z) + \dots + \frac{1}{k!} f_0^{[k]}(z) + O(z)^{k+1},$$

$$g_0(z) = G_0 + g_0^{[1]}(z) + \dots + \frac{1}{k!} g_0^{[k-1]}(z) + O(z)^k,$$

and inserting these series expansions together with the expansion of  $\Pi_0(w)$  into (50) and matching the first order terms, we obtain

$$\frac{\partial \Pi_0^{[1]}(w)}{\partial w} S w = \mu [F_0 \Pi_0^{[1]}(w) + G_0 Q w]. \quad (56)$$

Substituting  $\Pi_0^{[1]}(w) = \sum_{i=1}^p \Pi_i w_i$ ,  $\Pi_i \in \mathbb{R}^{n_z}$  in (56) and matching the term in  $w_i$ , yields

$$\lambda_i \Pi_i w_i = \mu [F_0 \Pi_i + G_0 Q e_i] w_i, \quad \forall_i \quad (57)$$

where the  $\lambda_i$  denote the eigenvalues of  $S$ , and  $e_i$  the  $i^{\text{th}}$  unit vector. Taking norms we get

$$\|\Pi_i\| \leq \mu \frac{\bar{f}_0 \|\Pi_i\| + a_{1_i}}{|\lambda_i|},$$

where  $\bar{f}_0 := \|F_0\|$  and  $a_{1_i} := \|G_0 Q e_i\|$ . Therefore, we can conclude that  $\|\Pi_i\| \leq \delta$  provided that

$$\mu \leq \min_{1 \leq i \leq p} \frac{\delta |\lambda_i|}{\bar{f}_0 \delta + a_{1_i}} \quad \text{and} \quad \mu < \frac{|\lambda_i|}{\bar{f}_0}.$$

To guarantee that (57) has a non trivial solution for all  $\Pi_i$ , (suppose for example that  $\Pi_i$  is an eigenvector of  $F_0$ ), the constant  $\mu$  should also satisfies the non-resonance constraint [Krener, 1992]

$$\mu z_i \neq \lambda_j, \quad i = 1, \dots, n_z; \quad j = 1, \dots, p$$

where  $z_i$  are the eigenvalues of  $F_0$ , which turns out to be the zeros of the local linearization around the origin of (13).

Consider now the degree two part of (50) that satisfies

$$\begin{aligned} \frac{\partial \Pi_0^{[2]}(w)}{\partial w} S w &= \mu [F_0 \Pi_0^{[2]}(w) \\ &\quad + f_0^{[2]}(\Pi_0^{[1]}(w)) + \frac{1}{2} g_0^{[1]}(\Pi_0^{[1]}(w)) Q w]. \end{aligned} \quad (58)$$

We can expand  $\Pi_0^{[2]}$ ,  $f_0^{[2]}$ , and  $g_0^{[1]} Q w$  in terms of the  $w_i$  as

$$\begin{aligned} \Pi_0^{[2]}(w) &= \sum \Pi_{0_{ij}} w_i w_j, \\ f_0^{[2]}(\Pi_0^{[1]}(w)) &= \sum f_{0_{ij}} w_i w_j, \\ g_0^{[1]}(\Pi_0^{[1]}(w)) Q w &= \sum g_{0_{ij}} w_i w_j, \end{aligned}$$

where  $\Pi_{0_{ij}} \in \mathbb{C}^{n_z}$ ,  $f_{0_{ij}} \in \mathbb{C}^{n_z}$ ,  $g_{0_{ij}} \in \mathbb{C}^{n_z}$ , and the sums range over  $1 \leq i \leq j \leq p$ . Substituting these equations into (58), matching the terms  $w_i w_j$  and noticing that

$$\frac{\partial \Pi_0^{[2]}(w)}{\partial w} S w = \sum \Pi_{0_{ij}} (\lambda_i + \lambda_j) w_i w_j, \quad \forall_{i,j}$$

we obtain

$$(\lambda_i + \lambda_j) \Pi_{0_{ij}} w_i w_j = \mu [F_0 \Pi_{0_{ij}} + f_{0_{ij}} + \frac{1}{2} g_{0_{ij}}] w_i w_j.$$

Taking norms, yields

$$\|\Pi_{0_{ij}}\| \leq \mu \frac{\bar{f}_0 \|\Pi_{0_{ij}}\| + a_{2_{ij}}}{|\lambda_i + \lambda_j|},$$

where  $a_{2_{ij}} := \|f_{0_{ij}} + \frac{1}{2} g_{0_{ij}}\|$ . Therefore, we can conclude that  $\|\Pi_{0_{ij}}\| \leq \delta$  provided that

$$\mu \leq \min_{1 \leq i \leq j \leq p} \frac{\delta |\lambda_i + \lambda_j|}{\bar{f}_0 \delta + a_{2_{ij}}} \quad \text{and} \quad \mu < \frac{|\lambda_i + \lambda_j|}{\bar{f}_0}.$$

Also, for all  $z_i$ ,  $\mu$  should satisfies the second non-resonance constraint

$$\mu z_i \neq \lambda_{j_1} + \lambda_{j_2}, \quad 1 \leq j_1 \leq j_2 \leq p.$$

Generalizing to an arbitrary degree  $k$ , we have that the degree  $k$  equations of (50) satisfies

$$\begin{aligned} \frac{\partial \Pi_0^{[k]}}{\partial w}(w)Sw &= \mu [F_0 \Pi_0^{[k]}(w) \\ &+ f_0^{[k]}(w) + \frac{1}{k} g_0^{[k-1]}(w)Qw], \end{aligned} \quad (59)$$

where  $f_0^{[k]}(w)$  and  $g_0^{[k-1]}(w)$  are the degree  $k$  and  $k-1$  parts of the composition of  $f_0(z)$  and  $g_0(z)$  with the expansion of  $\Pi_0(w)$  up to degree  $k-1$ . As before, expanding  $\Pi_0^{[k]}$ ,  $f_0^{[k]}$ , and  $g_0^{[k-1]}Qw$  in terms of  $w_{i_1}w_{i_2}\dots w_{i_k}$  with  $i_1 \leq i_2 \leq \dots \leq i_k \leq p$ , and substituting these expansions in (59) yields

$$\begin{aligned} &(\lambda_{i_1} + \dots + \lambda_{i_k})\Pi_{0_{i_1\dots i_k}} w_{i_1}\dots w_{i_k} \\ &= \mu [F_0 \Pi_{0_{i_1\dots i_k}} + f_{0_{i_1\dots i_k}} + \frac{1}{k} g_{0_{i_1\dots i_k}}] w_{i_1}\dots w_{i_k}. \end{aligned}$$

Taking norms, we get

$$\|\Pi_{0_{i_1\dots i_k}}\| \leq \mu \frac{\bar{f}_0 \|\Pi_{0_{i_1\dots i_k}}\| + a_{k_{i_1\dots i_k}}}{|\lambda_{i_1} + \dots + \lambda_{i_k}|},$$

where  $a_{k_{i_1\dots i_k}} := \|f_{0_{i_1\dots i_k}} + \frac{1}{k} g_{0_{i_1\dots i_k}}\|$ . Thus, it follows that  $\|\Pi_{0_{i_1\dots i_k}}\| \leq \delta$  provided that

$$\mu \leq \min_{1 \leq i_1 \leq \dots \leq i_k \leq p} \frac{\delta |\lambda_{i_1} + \dots + \lambda_{i_k}|}{f_0 \delta + a_{k_{i_1\dots i_k}}},$$

and

$$\mu < \frac{|\lambda_{i_1} + \dots + \lambda_{i_k}|}{f_0}.$$

Once more, the selection of  $\mu$  should satisfies all the non-resonance constraints. The  $k^{th}$  non-resonance constraint is given by

$$\mu z_i \neq \lambda_{j_1} + \dots + \lambda_{j_k}, \quad 1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq p.$$

□

*Lemma 4.* Under the conditions of Theorem 6, there exists a sufficiently large constant  $\gamma > 0$  such that  $J_P$  satisfies (31).

*Proof.* We first compute

$$J_\ell := \int_{t_\ell}^{\infty} \|e_P(t)\|^2 dt, \quad \ell \in \mathcal{I}$$

with  $\sigma(t) = \ell$  for all  $t \geq t_\ell$  and note that

$$J_P \leq \sum_{\ell=0}^N J_\ell. \quad (60)$$

As in the derivation of proof of Theorem 5, we get

$$J_\ell \leq \frac{1}{2} \tilde{z}_\ell(t_\ell)' P_0 \tilde{z}_\ell(t_\ell), \quad (61)$$

where  $P_0 > 0$  satisfies (48),  $\tilde{z}_\ell := z - \bar{z}_\ell$  and  $\bar{z}_\ell = \Pi_{\ell_0}(w)$ , that is,  $\bar{z}_\ell$  is the steady-state of  $z$  when  $\sigma(t) = \ell$  for all  $t \geq t_\ell$ . For  $\ell = 1, 2, \dots, N$

we substitute  $z(t_\ell) = \tilde{z}_{\ell-1}(t_\ell) + \bar{z}_{\ell-1}(t_\ell)$  in (61) and get

$$\begin{aligned} J_\ell &\leq \frac{1}{2} (\tilde{z}_{\ell-1}(t_\ell) + \bar{z}_{\ell-1}(t_\ell) - \bar{z}_\ell(t_\ell))' P_0 \\ &\quad (\tilde{z}_{\ell-1}(t_\ell) + \bar{z}_{\ell-1}(t_\ell) - \bar{z}_\ell(t_\ell)) \\ &\leq \frac{\lambda_{max}(P_0)}{2} \left( \|\bar{z}_{\ell-1}(t_\ell) - \bar{z}_\ell(t_\ell)\|^2 + 2\tilde{z}'_{\ell-1}(t_\ell) \right. \\ &\quad \left. [\bar{z}_{\ell-1}(t_\ell) - \bar{z}_\ell(t_\ell)] + \|\tilde{z}_{\ell-1}(t_\ell)\|^2 \right). \end{aligned} \quad (62)$$

We now prove that  $\tilde{\Pi}_{\ell_0}(w) := \bar{z}_{\ell-1} - \bar{z}_\ell = \Pi_{\ell-1_0}(w) - \Pi_{\ell_0}(w)$  can be written as

$$\tilde{\Pi}_{\ell_0}(w) = \alpha_\ell(w) \tilde{\mu}_\ell, \quad (63)$$

where  $\tilde{\mu}_\ell := \mu_{\ell-1} - \mu_\ell$  and  $\alpha_\ell(w)$  is a continuous functions that satisfies

$$\begin{aligned} \frac{\partial \alpha}{\partial w}(w)Sw &= f_0(\Pi_{\ell-1_0}(w)) + g_0(\Pi_{\ell-1_0}(w))Qw \\ &+ \mu_\ell [\beta_f(\Pi_{\ell-1_0}(w), \Pi_{\ell_0}(w))\alpha(w) \\ &+ \beta_g(\Pi_{\ell-1_0}(w), \Pi_{\ell_0}(w))\alpha(w)Qw]. \end{aligned} \quad (64)$$

To show (63), we consider (50) for each  $\Pi_{\ell-1_0}(w)$ ,  $\Pi_{\ell_0}(w)$ , that is,

$$\begin{aligned} \frac{\partial \Pi_{\ell-1_0}(w)}{\partial w}Sw &= \mu_{\ell-1} [f_0(\Pi_{\ell-1_0}(w)) \\ &+ g_0(\Pi_{\ell-1_0}(w))Qw] \end{aligned}$$

$$\begin{aligned} \frac{\partial \Pi_{\ell_0}(w)}{\partial w}Sw &= \mu_\ell [f_0(\Pi_{\ell_0}(w)) \\ &+ g_0(\Pi_{\ell_0}(w))Qw] \end{aligned}$$

Subtracting the equations, yields

$$\begin{aligned} \frac{\partial \tilde{\Pi}_{\ell_0}(w)}{\partial w}Sw &= \tilde{\mu}_\ell [f_0(\Pi_{\ell-1_0}(w)) \\ &+ g_0(\Pi_{\ell-1_0}(w))Qw] + \mu_\ell [f_0(\Pi_{\ell-1_0}(w)) \\ &- f_0(\Pi_{\ell_0}(w)) + (g_0(\Pi_{\ell-1_0}(w)) \\ &- g_0(\Pi_{\ell_0}(w)))Qw]. \end{aligned} \quad (65)$$

Applying the mean value theorem and replacing (63) in (65), it follows (64).

Assuming that  $\Pi_{\ell-1_0}(w)$  and  $\Pi_{\ell_0}(w)$  exist, then  $\alpha(w)$  is bounded and there exists a sufficiently large constant  $\gamma > 0$  such that  $\|\tilde{\Pi}_{\ell_0}(w)\|^2 \leq \gamma \tilde{\mu}_\ell^2$ . This fact, together with (60) and (62) follows (31).

□

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