

# Stochastic Hybrid Systems: Applications to Communication Networks

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## Talk outline



1. A (simple) model for stochastic hybrid systems (SHSs)
2. SHSs models for network traffic under TCP
3. Analysis tools for SHSs
4. Dynamics of TCP

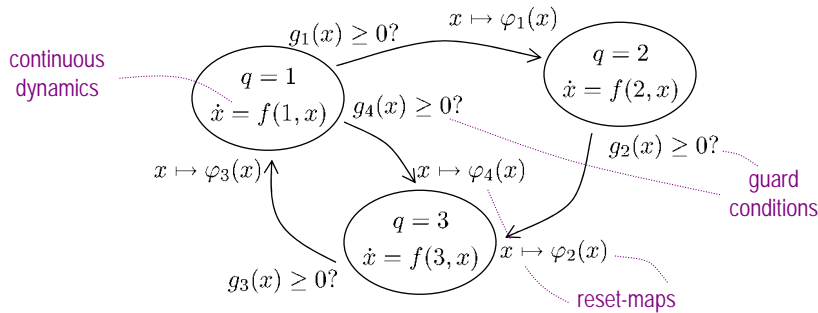
*Collaborators:*

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*Acknowledgements:*

Mustafa Khammash, John Lygeros

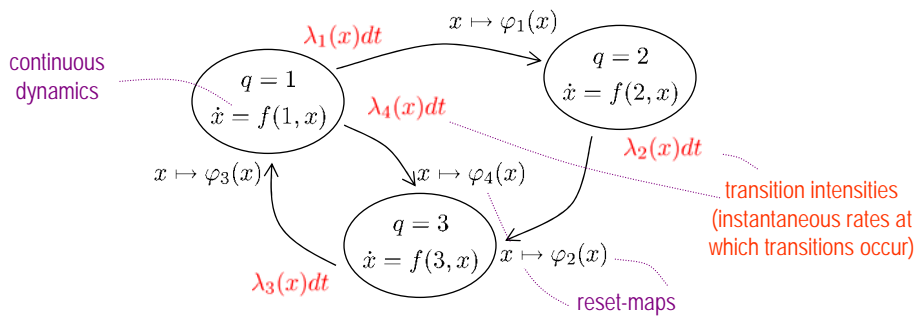
# Deterministic Hybrid Systems



$q(t) \in Q = \{1, 2, \dots\} \equiv$  discrete state  
 $x(t) \in \mathbb{R}^n \equiv$  continuous state
 } right-continuous by convention

we assume here a deterministic system so the invariant sets would be the exact complements of the guards

# Stochastic Hybrid Systems



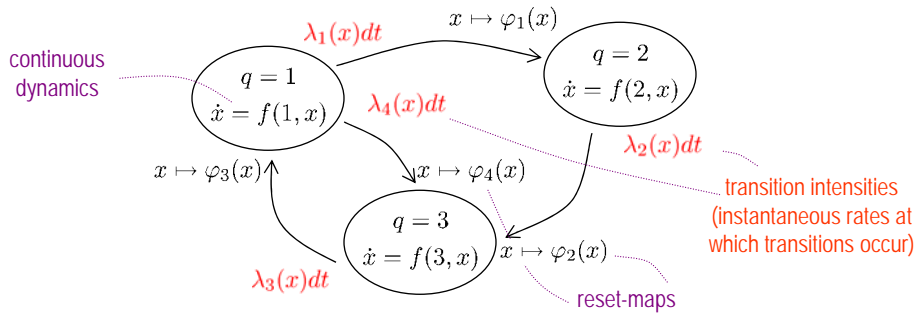
$N_\ell(t) \in \mathbb{N} \equiv$  transition counter, which is incremented by one each time the  $\ell$ th reset-map  $\varphi_\ell(x)$  is "activated"
 } right-continuous by convention

$$P \left( N_\ell(t + dt) > N_\ell(t) \right) \xrightarrow{dt \rightarrow 0} E \left[ \lambda_\ell(x(t)) dt \right]$$

at least one transition on  $(t, t+dt]$

proportional to "elementary" interval length  $dt$  and transition intensity  $\lambda_\ell(x)$

# Stochastic Hybrid Systems



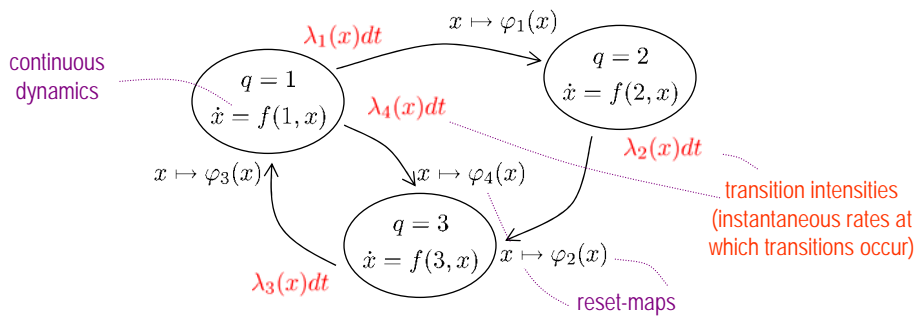
$N_\ell(t) \in \mathbb{N} \equiv$  transition counter, which is incremented by one each time the  $\ell$ th reset-map  $\varphi_\ell(x)$  is “activated” } right-continuous by convention

$$E \left[ N_\ell(t_2) - N_\ell(t_1) \right] = E \left[ \int_{t_1}^{t_2} \lambda_\ell(x(t)) dt \right]$$

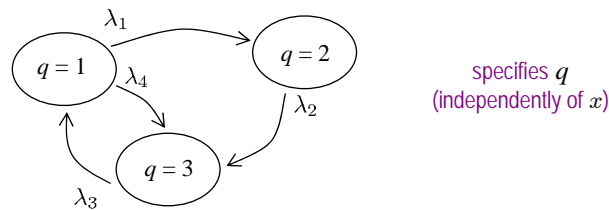
number of transitions on  $(t_1, t_2]$

equal to integral of transition intensity  $\lambda_\ell(x)$  on  $(t_1, t_2]$

# Stochastic Hybrid Systems



Special case: When all  $\lambda_\ell$  are constant, transitions are controlled by a continuous-time Markov process



## Formal model—Summary

State space:  $q(t) \in \mathcal{Q}=\{1,2,\dots\} \equiv$  discrete state  
 $x(t) \in \mathbb{R}^n \equiv$  continuous state

Continuous dynamics:

$$\dot{x} = f(q, x, t) \quad f : \mathcal{Q} \times \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^n$$

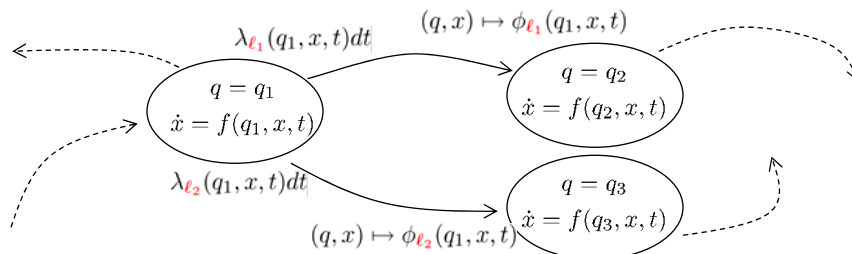
Transition intensities:

$$\lambda_\ell(q, x, t) \quad \lambda_\ell : \mathcal{Q} \times \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty) \quad \ell \in \{1, \dots, m\}$$

Reset-maps (one per transition intensity):

$$(q, x) \mapsto \phi_\ell(q, x, t) \quad \phi_\ell : \mathcal{Q} \times \mathbb{R}^n \times [0, \infty) \rightarrow \mathcal{Q} \times \mathbb{R}^n \quad \ell \in \{1, \dots, m\}$$

# of transitions



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Reset-maps (one per transition intensity):

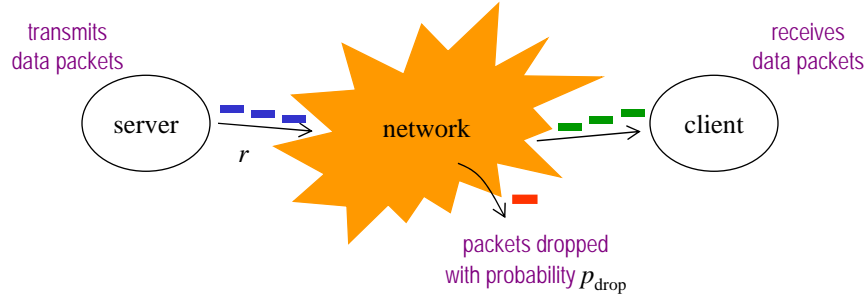
$$(q, x) \mapsto \phi_\ell(q, x, t) \quad \phi_\ell : \mathcal{Q} \times \mathbb{R}^n \times [0, \infty) \rightarrow \mathcal{Q} \times \mathbb{R}^n \quad \ell \in \{1, \dots, m\}$$

# of transitions

### Results:

1. [existence] Under appropriate regularity (Lipschitz) assumptions, there exists a measure “consistent” with the desired SHS behavior
2. [simulation] The procedure used to construct the measure is constructive and allows for efficient generation of *Monte Carlo sample paths*
3. [Markov] The pair  $(q(t), x(t)) \in \mathcal{Q} \times \mathbb{R}^n$  is a (Piecewise-deterministic) Markov Process (in the sense of M. Davis, 1993)

# Transmission Control Protocol

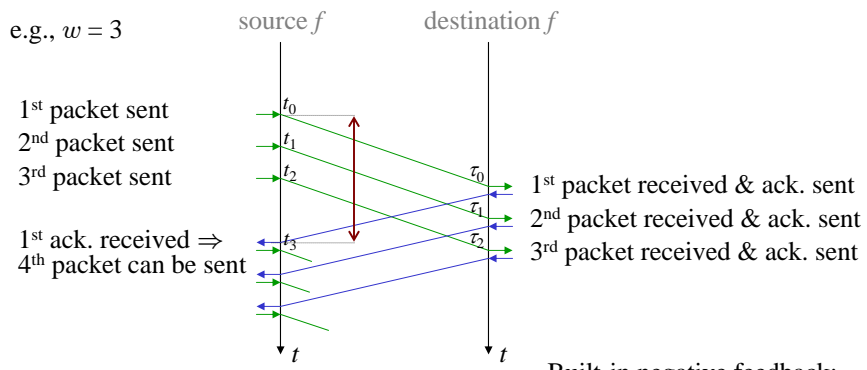


congestion control  $\equiv$  selection of the rate  $r$  at which the server transmits packets  
 feedback mechanism  $\equiv$  packets are dropped by the network to indicate congestion

# TCP window-based control

$w$  (window size)  $\equiv$  number of packets that can remain unacknowledged for by the destination

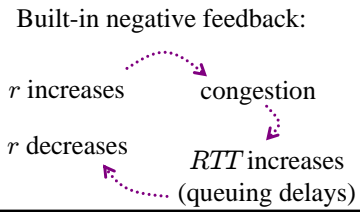
e.g.,  $w = 3$



$w$  effectively determines the sending rate  $r$ :

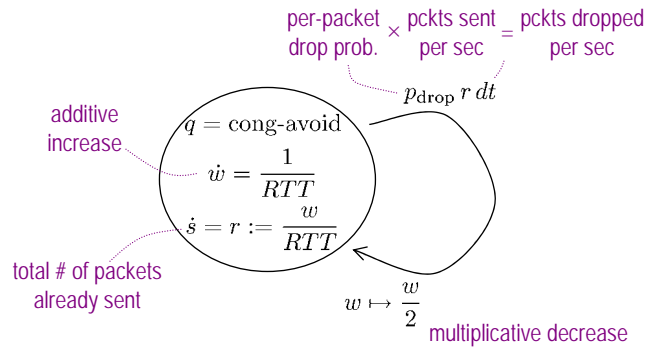
$$r(t) = \frac{w(t)}{RTT(t)}$$

round-trip time



## TCP Reno congestion avoidance

1. While there are no drops, increase  $w$  by 1 on each RTT (additive increase)
2. When a drop occurs, divide  $w$  by 2 (multiplicative decrease)  
(congestion controller constantly probes the network for more bandwidth)

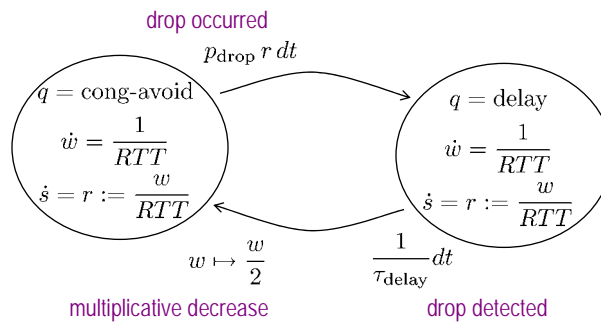


Three feedback mechanisms: As rate  $r$  increases  $\Rightarrow \dots$

- a) congestion  $\Rightarrow RTT$  increases (queuing delays)  $\Rightarrow r$  decreases
- b) congestion  $\Rightarrow p_{\text{drop}}$  increases (active queuing)  $\Rightarrow r$  decreases
- c) multiplicative decrease more likely  $\Rightarrow r$  decreases

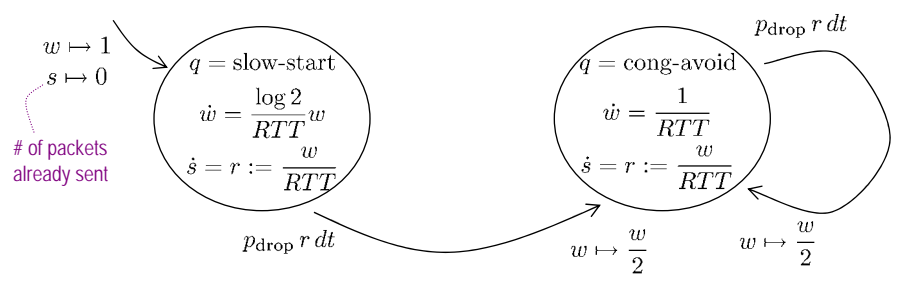
## Delay in drop detection

3. In general there is some delay between drop occurrence and detection  
(assumed exponentially distributed with mean  $\tau_{\text{delay}}$ )



## TCP Reno slow start

- Until a drop occurs double  $w$  on each  $RTT$
- When a drop occurs divide  $w$  by 2 and transition to congestion avoidance (get to full bandwidth utilization fast)



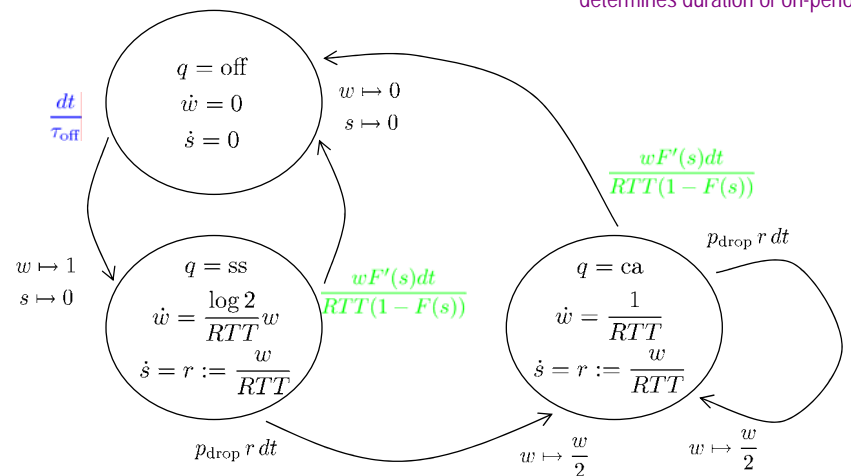
for simplicity this diagram ignores delay

TCP has several other modes that can be modeled by hybrid systems  
 [Bohacek, Hespanha, Lee, Obraczka. A Hybrid Systems Modeling Framework for Fast and Accurate Simulation of Data Communication Networks. In SIGMETRICS'03]

## On-off TCP model

- Between transfers, server remains *off* for an exponentially distributed time with mean  $\tau_{\text{off}}$
- Transfer-size is a random variable with cumulative distribution  $F(s)$

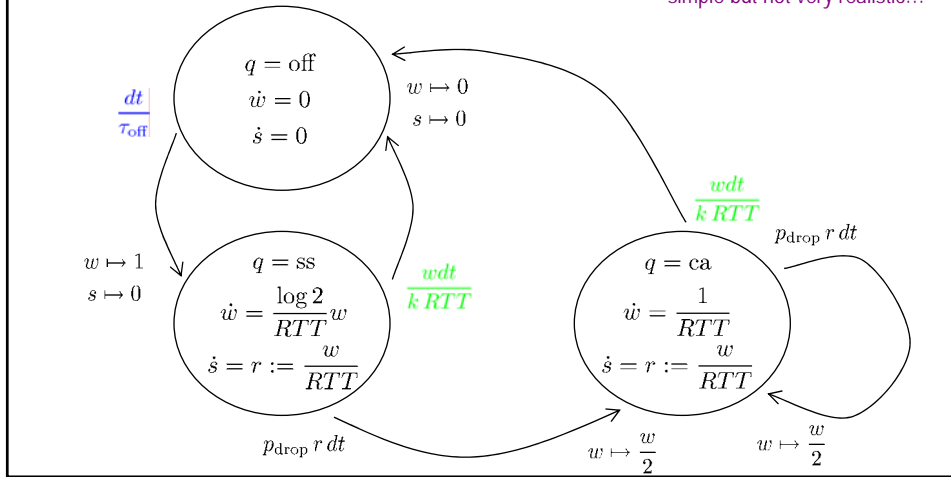
determines duration of on-periods



# On-off TCP model

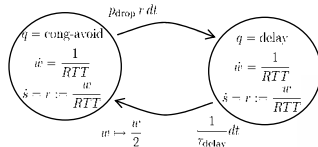
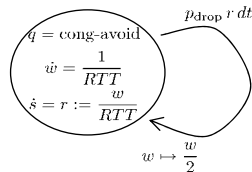
- Between transfers, server remains *off* for an exponentially distributed time with mean  $\tau_{off}$
- Transfer-size is exponentially distributed with average  $k$  (packets)

simple but not very realistic...



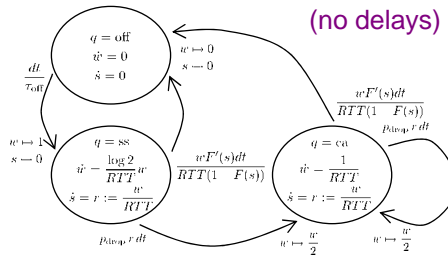
# SHS models for TCP

long-lived TCP flows (no delays)

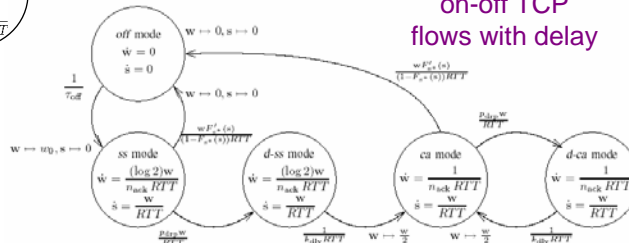


long-lived TCP flows with delay

on-off TCP flows (no delays)



on-off TCP flows with delay



# Analysis—Lie Derivative

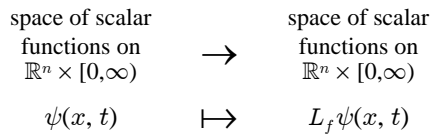
$$\dot{x} = f(x, t) \quad x \in \mathbb{R}^n$$

Given function  $\psi : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$

$$\dot{\psi}(x, t) = \underbrace{\frac{\partial \psi}{\partial x} f(x, t)}_{L_f \psi} + \frac{\partial \psi}{\partial t}$$

derivative along solution to ODE  
Lie derivative of  $\psi$

One can view  $L_f$  as an operator



$L_f$  completely defines the system dynamics

# Analysis—Generator of the SHS

$\dot{x} = f(q, x, t)$	$\lambda_\ell(q, x, t)$	$(q, x) = \phi_\ell(q^-, x^-, t)$
continuous dynamics	transition intensities	reset-maps

Given function  $\psi : \mathcal{Q} \times \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$

$$\frac{d}{dt} E[\psi(q, x, t)] = E[(L\psi)(q, x, t)]$$

generator for the SHS  
 Dynkin's formula (in differential form)

where

$$(L\psi)(q, x, t) := \underbrace{\frac{\partial \psi}{\partial x} f(q, x, t)}_{L_f \psi} + \sum_{\ell=1}^m \underbrace{\left( \psi(\phi_\ell(q, x, t), t) - \psi(q, x, t) \right)}_{\text{reset instantaneous variation}} \lambda_\ell(q, x, t)$$

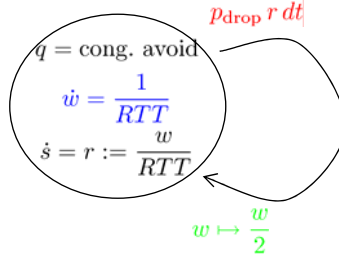
Lie derivative of  $\psi$ 
intensity

$L$  completely defines the SHS dynamics

*Disclaimer:* see following paper for technical assumptions  
 Hespanha. Stochastic Hybrid Systems: Applications to Communication Networks. HSCC'04

## Long-lived TCP flows

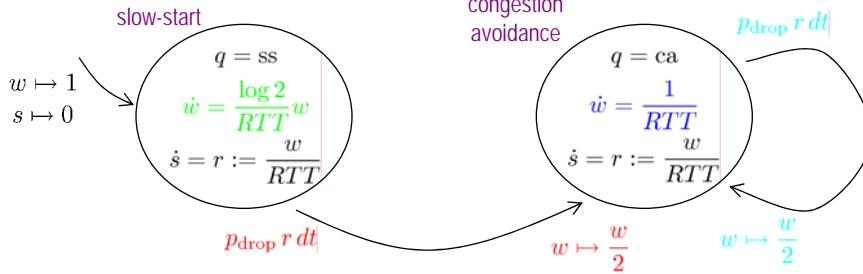
long-lived  
TCP flows



$$(L\psi)(w, t) = \frac{\partial \psi}{\partial w} \frac{1}{RTT} + \frac{\partial \psi}{\partial t} + \left[ \psi\left(\frac{w}{2}, t\right) - \psi(w, t) \right] p_{\text{drop}} \frac{w}{RTT}$$

## Long-lived TCP flows

long-lived TCP flows  
(with slow start)



$$(L\psi)(q, w, t) = \begin{cases} \frac{\partial \psi}{\partial w} \frac{\log 2}{RTT} w + \frac{\partial \psi}{\partial t} + \left[ \psi\left(\text{ca}, \frac{w}{2}\right) - \psi(\text{ss}, w) \right] \frac{p_{\text{drop}} w}{RTT} & q = \text{ss} \\ \frac{\partial \psi}{\partial w} \frac{1}{RTT} + \frac{\partial \psi}{\partial t} + \left[ \psi\left(\text{ca}, \frac{w}{2}\right) - \psi(\text{ss}, w) \right] \frac{p_{\text{drop}} w}{RTT} & q = \text{ca} \end{cases}$$

## Analysis—PDF for SHS state

$\dot{x} = f(q, x, t)$	$\lambda_\ell(q, x, t)$	$(q, x) = \phi_\ell(q^-, x^-, t)$
continuous dynamics	transition intensities	reset-maps

Let  $\rho(\cdot; t)$  be the *probability density function* (PDF) for the state  $(x, q)$  at time  $t$  :

$$E[\psi(q(t), x(t), t)] = \sum_{q \in \mathcal{Q}} \int_{\mathbb{R}^n} \psi(q, x, t) \rho(q, x; t) dx$$

When  $\rho(\cdot; t)$  is smooth, one can deduce from the generator that

$$\begin{aligned} \frac{\partial \rho(q, x; t)}{\partial t} = & -\frac{\partial f(q, x, t) \rho(q, x; t)}{\partial x} + \sum_{\ell=1}^m \left( -\lambda_\ell(q, x, t) \rho(q, x; t) + \right. \\ & \left. + \sum_{\substack{\bar{q} \in \mathcal{Q}: \\ \phi_\ell^q(\bar{q}, \phi_\ell^{-x}(\bar{q}, x, t)) = q}} \lambda_\ell(\bar{q}, \phi_\ell^{-x}(\bar{q}, x, t), t) \rho(\bar{q}, \phi_\ell^{-x}(\bar{q}, x, t); t) \left| \frac{\partial \phi_\ell^{-x}(\bar{q}, x, t)}{\partial z} \right| \right) \end{aligned}$$

inverse of  
 $x \mapsto \phi_\ell^x(q, x, t)$

functional PDE  
very difficult to solve

## Analysis—Moments for SHS state

$\dot{x} = f(q, x, t)$	$\lambda_\ell(q, x, t)$	$(q, x) = \phi_\ell(q^-, x^-, t)$
continuous dynamics	transition intensities	reset-maps

*often a few low-order moments suffice to study a SHS...*

$\mathbf{z}$  (scalar) random variable with mean  $\mu$  and variance  $\sigma^2$

$$P(\mathbf{z} \geq \epsilon \mid \mathbf{z} \geq 0) \leq \frac{\mu}{\epsilon}$$

Markov inequality

( $\forall \epsilon > 0$ )

$$P(|\mathbf{z} - a| \geq \epsilon) \leq \frac{E[|\mathbf{z} - a|^n]}{\epsilon^n}$$

Bienaymé inequality

( $\forall \epsilon > 0, a \in \mathbb{R}, n \in \mathbb{N}$ )

$$P(|\mathbf{z} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

Tchebychev inequality

## Polynomial SHSs

$$\dot{x} = f(q, x, t) \quad \lambda_\ell(q, x, t) \quad (q, x) = \phi_\ell(q^-, x^-, t)$$

continuous dynamics
transition intensities
reset-maps

Given function  $\psi : \mathcal{Q} \times \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$

$$\frac{d}{dt} \mathbb{E}[\psi(q, x, t)] = \mathbb{E}[(L\psi)(q, x, t)]$$

generator for the SHS  
 Dynkin's formula  
 (in differential form)

where

$$(L\psi)(q, x, t) := \frac{\partial \psi}{\partial x} f(q, x, t) + \frac{\partial \psi}{\partial t} + \sum_{\ell=1}^m \left( \psi(\phi_\ell(q, x, t), t) - \psi(q, x, t) \right) \lambda_\ell(q, x, t)$$

A SHS is called a *polynomial SHS* (pSHS) if its generator maps finite-order polynomial on  $x$  into finite-order polynomials on  $x$   
 Typically, when

$$x \mapsto f(q, x, t) \quad x \mapsto \lambda_\ell(q, x, t) \quad x \mapsto \phi_\ell(q, x, t)$$

are all polynomials  $\forall q, t$

## Moment dynamics for pSHS

$$x(t) \in \mathbb{R}^n \quad q(t) \in \mathcal{Q} = \{1, 2, \dots\}$$

continuous state
discrete state

Test function: Given  $\bar{q} \in \mathcal{Q} \quad m = (m_1, m_2, \dots, m_n) \in \mathbb{N}_{\geq 0}^n$

$$\psi_{\bar{q}}^{(m)}(q, x) := \begin{cases} x_1^{m_1} x_2^{m_2} \dots x_n^{m_n} & q = \bar{q} \\ 0 & q \neq \bar{q} \end{cases} \rightarrow \text{for short } x^{(m)}$$

Uncentered moment:

$$\mu_{\bar{q}}^{(m)}(t) := \mathbb{E}[\psi_{\bar{q}}^{(m)}(q(t), x(t))]$$

E.g,

$$\mu_1^{(0,0,\dots,0)}(t) = \mathbb{P}(q(t) = 1) \quad \mu_1^{(1,1,\dots,0)}(t) = \mathbb{E}[x_1(t)x_2(t)I_{q(t)=1}]$$

$$\mu_1^{(0,1,\dots,0)}(t) = \mathbb{E}[x_2(t)I_{q(t)=1}] \quad \mu_1^{(2,0,\dots,0)}(t) = \mathbb{E}[x_1(t)^2 I_{q(t)=1}]$$

## Moment dynamics for pSHS

$x(t) \in \mathbb{R}^n$  continuous state       $q(t) \in \mathcal{Q} = \{1, 2, \dots\}$  discrete state

Test function: Given  $\bar{q} \in \mathcal{Q}$        $m = (m_1, m_2, \dots, m_n) \in \mathbb{N}_{\geq 0}^n$

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For polynomial SHS...

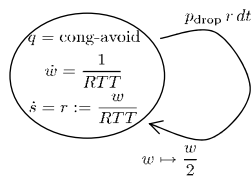
$$\begin{array}{ccc} \psi_{\bar{q}}^{(m)}(q, x) & \Rightarrow & L\psi_{\bar{q}}^{(m)}(q, x) \\ \text{monomial on } x & & \text{polynomial on } x \\ & & \Rightarrow \\ & & L\psi_{\bar{q}}^{(m)}(q, x) \\ & & \text{linear comb. of} \\ & & \text{test functions} \end{array}$$

$$\dot{\mu}_{\bar{q}}^{(m)} = \frac{d}{dt} \mathbb{E} [\psi_{\bar{q}}^{(m)}(q, x)] = \mathbb{E} [L\psi_{\bar{q}}^{(m)}(q, x)] = \sum_{i=1}^k \alpha_i \mu_{q_i}^{(m_i)}$$

linear moment dynamics

## Moment dynamics for TCP

long-lived  
TCP flows



$$\mu^{(k)} := \mathbb{E}[r^k] = \mathbb{E} \left[ \frac{w^k}{RTT^k} \right]$$

$$\dot{\mu}^{(1)} = \frac{1}{RTT^2} - \frac{p}{2} \mu^{(1)}$$

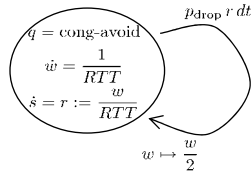
$$\dot{\mu}^{(2)} = \frac{2}{RTT^2} \mu^{(1)} - \frac{3p}{4} \mu^{(3)}$$

$$\dot{\mu}^{(3)} = \frac{3}{RTT^2} \mu^{(2)} - \frac{7p}{8} \mu^{(4)}$$

⋮

# Moment dynamics for TCP

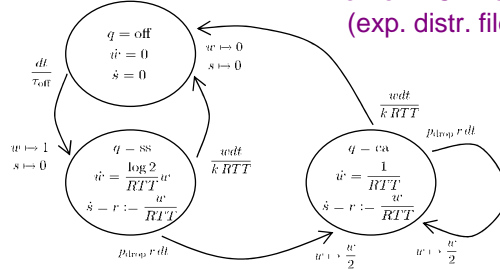
long-lived TCP flows



$$\mu^{(k)} := E[r^k] = E\left[\frac{w^k}{RTT^k}\right]$$

$$\begin{aligned} \dot{\mu}^{(1)} &= \frac{1}{RTT^2} - \frac{p}{2}\mu^{(1)} \\ \dot{\mu}^{(2)} &= \frac{2}{RTT^2}\mu^{(1)} - \frac{3p}{4}\mu^{(3)} \\ \dot{\mu}^{(3)} &= \frac{3}{RTT^2}\mu^{(2)} - \frac{7p}{8}\mu^{(4)} \\ &\vdots \end{aligned}$$

on-off TCP flows (exp. distr. files)

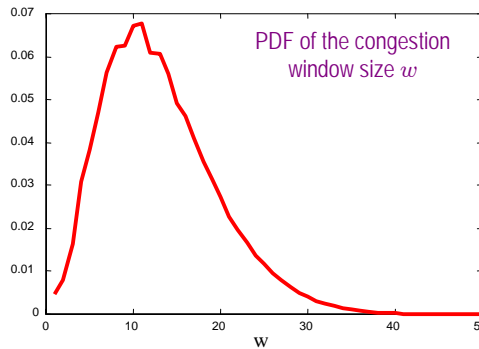


$$\mu_{ca}^{(k)} := E[r^k I_{q=ca}] = E\left[\frac{w^k}{RTT^k} I_{q=ca}\right] \dots$$

$$\begin{bmatrix} \dot{\mu}_{off}^{(0)} \\ \dot{\mu}_{ss}^{(0)} \\ \dot{\mu}_{ca}^{(0)} \\ \dot{\mu}_{ss}^{(1)} \\ \dot{\mu}_{ca}^{(1)} \\ \dot{\mu}_{ss}^{(2)} \\ \dot{\mu}_{ca}^{(2)} \\ \vdots \end{bmatrix} = \begin{bmatrix} -\tau_{off}^{-1}\mu_{off}^{(0)} + \frac{1}{k}\mu_{ss}^{(1)} + \frac{1}{k}\mu_{ca}^{(1)} \\ \tau_{off}^{-1}\mu_{off}^{(0)} - (\frac{1}{k} + p)\mu_{ss}^{(1)} \\ p\mu_{ss}^{(1)} - \frac{1}{k}\mu_{ca}^{(1)} \\ \frac{\tau_{off}^{-1}w_0}{RTT}\mu_{off}^{(0)} + \frac{\log 2}{RTT}\mu_{ss}^{(1)} - (\frac{1}{k} + p)\mu_{ss}^{(2)} \\ \frac{1}{RTT^2}\mu_{ca}^{(0)} + \frac{p}{2}\mu_{ss}^{(2)} - (\frac{1}{k} + \frac{p}{2})\mu_{ca}^{(2)} \\ \frac{\tau_{off}^{-1}w_0}{RTT^2}\mu_{off}^{(0)} + \frac{\log 4}{RTT}\mu_{ss}^{(2)} - (\frac{1}{k} + p)\mu_{ss}^{(3)} \\ \frac{2}{RTT^2}\mu_{ca}^{(1)} + \frac{p}{4}\mu_{ss}^{(3)} - (\frac{1}{k} + \frac{3p}{4})\mu_{ca}^{(3)} \\ \vdots \end{bmatrix}$$

# Truncated moment dynamics

Experimental evidence indicates that the sending rate is well approximated by a Log-Normal distribution

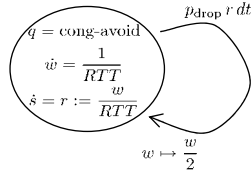


$$\begin{aligned} \mathbf{z} \text{ Log-Normal} &\Rightarrow E[\mathbf{z}^3] = \frac{E[\mathbf{z}^2]^3}{E[\mathbf{z}]^3} \\ \mathbf{r} \text{ Log-Normal (on each mode)} &\Rightarrow \mu_{q,3} = E[\mathbf{r}^3 | \mathbf{q} = q] \mu_{q,0} \approx \frac{E[\mathbf{r}^2 | \mathbf{q} = q]^3}{E[\mathbf{r} | \mathbf{q} = q]^3} \mu_{q,0} = \frac{\mu_{q,2}^3 \mu_{q,0}}{\mu_{q,1}^3} \end{aligned}$$

Data from: Bohacek, A stochastic model for TCP and fair video transmission. INFOCOM'03

# Moment dynamics for TCP

long-lived TCP flows

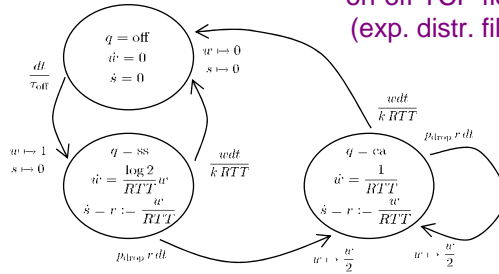


$$\mu^{(k)} := E[r^k] = E \left[ \frac{w^k}{RTT^k} \right]$$

$$\begin{aligned} \dot{\mu}^{(1)} &= \frac{1}{RTT^2} - \frac{p}{2} \mu^{(1)} \\ \dot{\mu}^{(2)} &= \frac{2}{RTT^2} \mu^{(1)} - \frac{3p}{4} \left( \frac{\mu^{(2)}}{\mu^{(1)}} \right)^3 \end{aligned}$$

finite-dimensional nonlinear ODEs

on-off TCP flows (exp. distr. files)

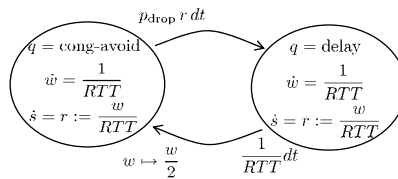


$$\mu_{ca}^{(k)} := E[r^k I_{q=ca}] = E \left[ \frac{w^k}{RTT^k} I_{q=ca} \right] \dots$$

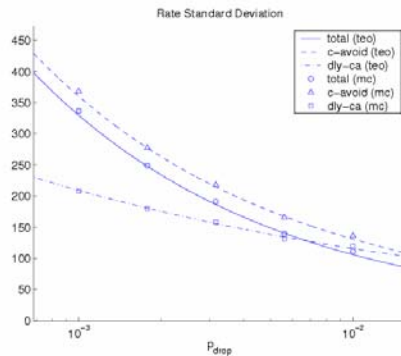
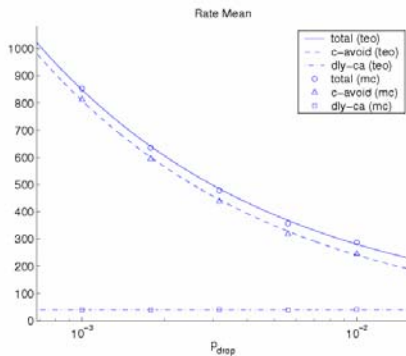
$$\begin{bmatrix} \dot{\mu}_{off}^{(0)} \\ \dot{\mu}_{ss}^{(0)} \\ \dot{\mu}_{ca}^{(0)} \\ \dot{\mu}_{ss}^{(1)} \\ \dot{\mu}_{ca}^{(1)} \\ \dot{\mu}_{ss}^{(2)} \\ \dot{\mu}_{ca}^{(2)} \end{bmatrix} = \begin{bmatrix} -\tau_{off}^{-1} \mu_{off}^{(0)} + \frac{1}{k} \mu_{ss}^{(1)} + \frac{1}{k} \mu_{ca}^{(1)} \\ \tau_{off}^{-1} \mu_{off}^{(0)} - \left( \frac{1}{k} + p \right) \mu_{ss}^{(1)} \\ p \mu_{ss}^{(1)} - \frac{1}{k} \mu_{ca}^{(1)} \\ \frac{\tau_{off}^{-1} w_0}{RTT} \mu_{off}^{(0)} + \frac{\log 2}{RTT} \mu_{ss}^{(1)} - \left( \frac{1}{k} + p \right) \mu_{ss}^{(2)} \\ \frac{1}{RTT^2} \mu_{ca}^{(0)} + \frac{p}{2} \mu_{ss}^{(2)} - \left( \frac{1}{k} + \frac{p}{2} \right) \mu_{ca}^{(2)} \\ \frac{\tau_{off}^{-1} w_0}{RTT^2} \mu_{off}^{(0)} + \frac{\log 4}{RTT} \mu_{ss}^{(2)} - \left( \frac{1}{k} + p \right) \mu_{ss}^{(3)} \left( \frac{\mu_{ss}^{(2)}}{\mu_{ss}^{(1)}} \right)^3 \\ \frac{2}{RTT^2} \mu_{ca}^{(1)} + \frac{p}{4} \mu_{ss}^{(3)} \left( \frac{\mu_{ss}^{(2)}}{\mu_{ss}^{(1)}} \right)^3 - \left( \frac{1}{k} + \frac{3p}{4} \right) \mu_{ca}^{(3)} \left( \frac{\mu_{ca}^{(2)}}{\mu_{ca}^{(1)}} \right)^3 \end{bmatrix}$$

# Long-lived TCP flows

long-lived TCP flows with delay (one RTT average)

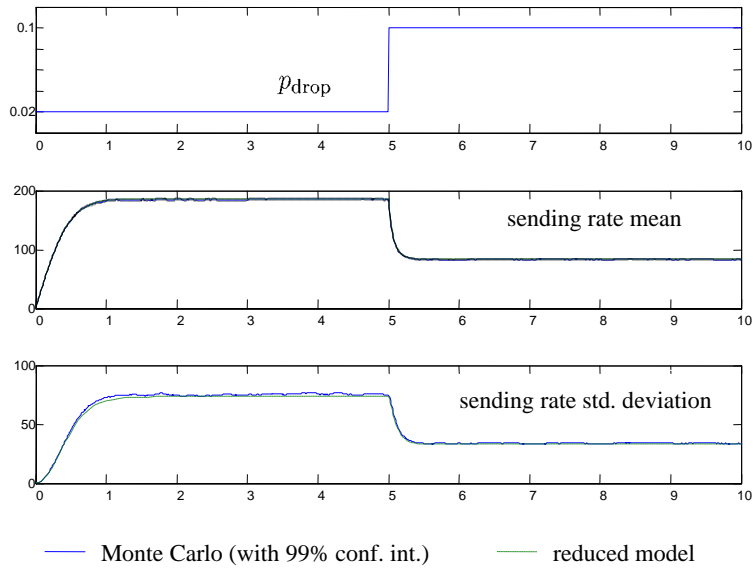


Monte-Carlo & theoretical steady-state distribution (vs. drop probability)

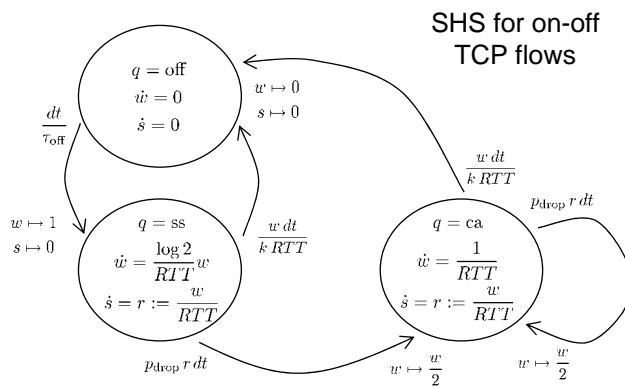


## Long-lived TCP flows

moment dynamics in response to abrupt change in drop probability

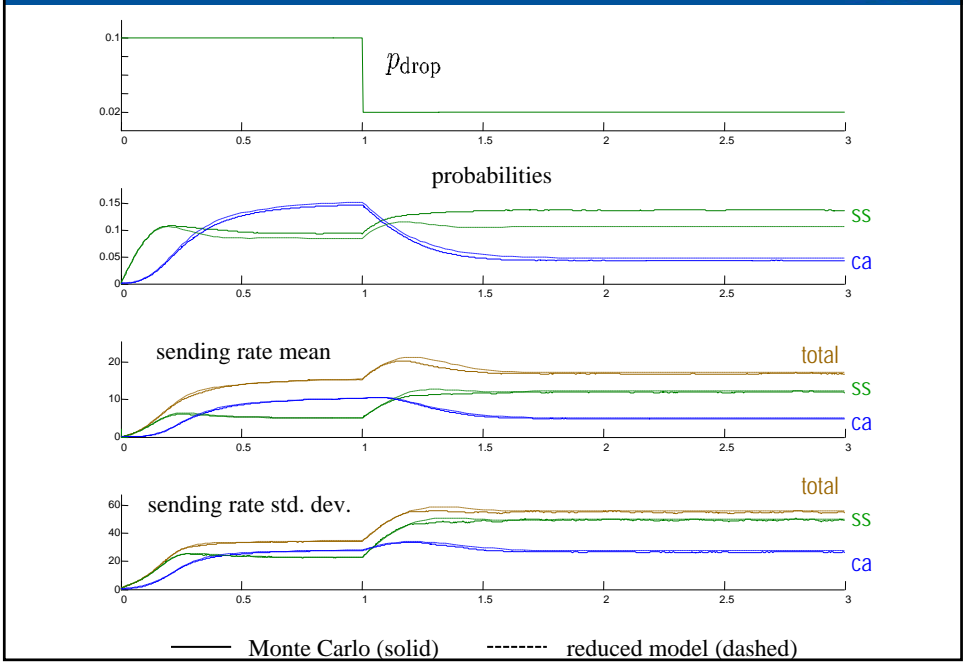


## On-off TCP flows

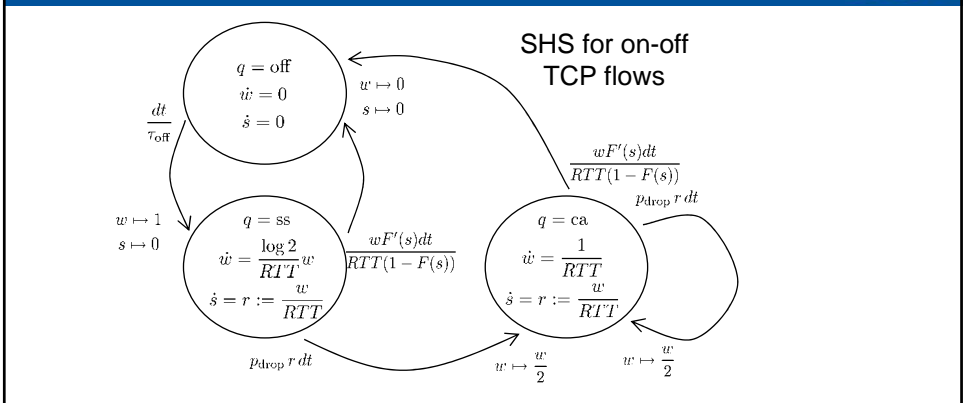


Transfer-sizes exponentially distributed with mean  $k = 30.6\text{KB}$  (Unix'93 files)  
 Off-time exponentially distributed with mean  $\tau_{\text{off}} = 1$  sec  
 Round-trip-time  $RTT = 50\text{msec}$

# On-off TCP flows



# On-off TCP flows



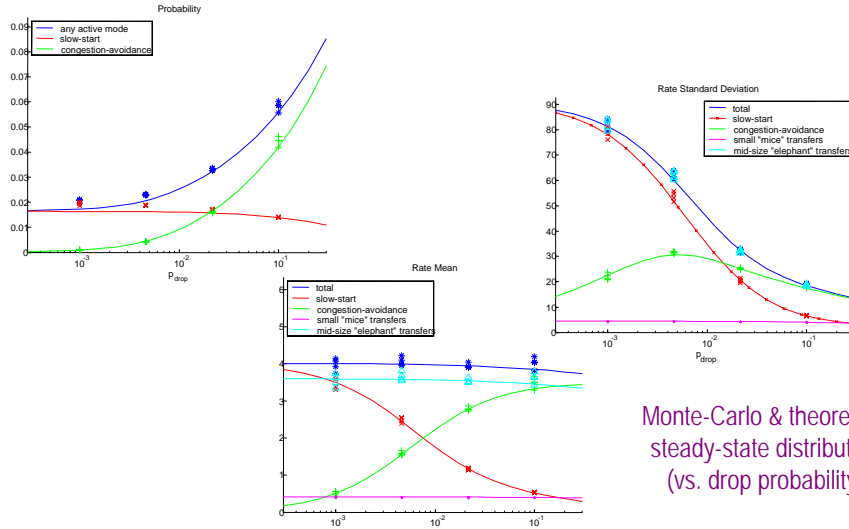
Mixture of exponential distribution for transfer-sizes:

$m$  exp. distr. with mean  $\{ k_1, k_2, \dots, k_m \}$        $m$  probabilities  $\{ p_1, p_2, \dots, p_m \}$        $\sum_{i=1}^m p_i = 1$   
 transfer-sizes are extracted from an exponential distribution with mean  $k_i$  w.p.  $p_i$   
*can approximate well distributions found in the literature ...*

## On-off TCP flows: Case I

Transfer-size approximating the distribution of file sizes in the UNIX file system

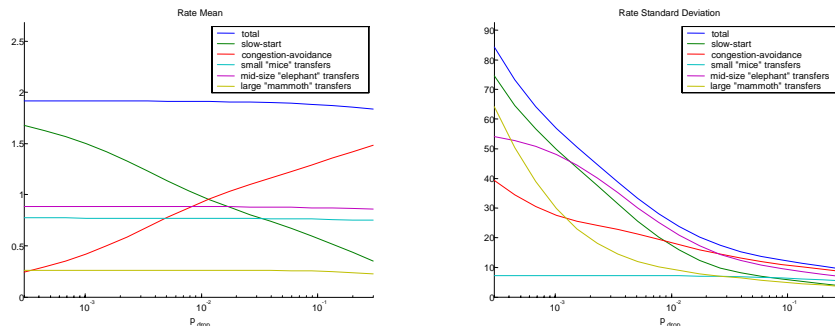
$p_1 = 88.87\%$      $k_1 = 3.5\text{KB}$     (“mice” files)  
 $p_2 = 11.13\%$      $k_2 = 246\text{KB}$     (“elephant” files, but not heavy tail)



## On-off TCP flows: Case II

Transfer-size distribution at an ISP's www proxy [Arlitt *et al.*]

$p_1 = 98\%$      $k_1 = 6\text{KB}$     (“mice” files)  
 $p_2 = 1.7\%$      $k_2 = 400\text{KB}$     (“elephant” files)  
 $p_3 = .02\%$      $k_3 = 10\text{MB}$     (“mammoth” files)



Theoretical steady-state distribution (vs. drop probability)

## Truncated moment dynamics (revisited)

For polynomial SHS...

$$\dot{\mu}_q^{(m)} = \frac{d}{dt} \mathbb{E}[\psi_q^{(m)}(q, x)] = \mathbb{E}[(L\psi_q^{(m)})(q, x)] = \sum_{i=1}^k \alpha_i \mu_{q_i}^{(m_i)}$$

linear moment dynamics

Stacking all moments into an (infinite) vector  $\mu_\infty$

$$\dot{\mu}_\infty = A_\infty \mu_\infty$$

infinite-dimensional linear ODE

In TCP analysis...

$$\mu_\infty = \left. \begin{array}{c} \mu_{ss}^{(0)} \\ \mu_{ca}^{(0)} \\ \mu_{ss}^{(1)} \\ \vdots \\ \mu_{ss}^{(3)} \\ \mu_{ca}^{(3)} \\ \vdots \end{array} \right\} \begin{array}{l} \mu \\ \bar{\mu} \end{array}$$

lower order moments of interest

moments of interest that affect  $\mu$  dynamics

$$\dot{\mu} = A\mu + B\bar{\mu}$$

approximated by nonlinear function of  $\mu$

## Truncation by derivative matching

$$\dot{\mu}_\infty = A_\infty \mu_\infty$$

infinite-dimensional linear ODE

$\dot{\mu} = A\mu + B\bar{\mu}$ <p>truncated linear ODE (nonautonomous, not nec. stable)</p>	$\dot{\nu} = A\nu + B\varphi(\nu)$ <p>nonlinear approximate moment dynamics</p>
--	---

**Assumption:** 1)  $\mu$  and  $\nu$  remain bounded along solutions to

$$\dot{\mu}_\infty = A_\infty \mu_\infty \quad \text{and} \quad \dot{\nu} = A\nu + B\varphi(\nu)$$

2)  $\dot{\mu}_\infty = A_\infty \mu_\infty$  is asymptotically stable

**Theorem:**  $\forall \delta > 0 \exists N$  s.t. if  $\frac{d^k \mu}{dt^k} = \frac{d^k \nu}{dt^k}, \quad \forall k \in \{1, \dots, N\}$

then

$$\|\mu(t) - \nu(t)\| \leq \beta(\|\mu(t_0) - \nu(t_0)\|, t - t_0) + \delta, \quad \forall t \geq t_0 \geq 0$$

class  $\mathcal{KL}$  function

## Truncation by derivative matching

infinite-dimensional linear ODE

$$\dot{\mu}_\infty = A_\infty \mu_\infty$$

$$\dot{\mu} = A\mu + B\bar{\mu}$$

truncated linear ODE  
(nonautonomous, not nec. stable)

$$\dot{\nu} = A\nu + B\varphi(\nu)$$

nonlinear approximate  
moment dynamics

**Assumption:** 1)  $\mu$  and  $\nu$  remain bounded along solutions to

$$\dot{\mu}_\infty = A_\infty \mu_\infty \quad \text{and} \quad \dot{\nu} = A\nu + B\varphi(\nu)$$

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then

$$\|\mu(t) - \nu(t)\| \leq \beta(\|\mu(t_0) - \nu(t_0)\|, t - t_0) + \delta, \quad \forall t \geq t_0 \geq 0$$

Proof idea:

- 1)  $N$  derivative matches  $\Rightarrow \mu$  &  $\nu$  match on compact interval of length  $T$
- 2) stability of  $A_\infty \Rightarrow$  matching can be extended to  $[0, \infty)$

## Truncation by derivative matching

infinite-dimensional linear ODE

- ☹ Given  $\delta$  finding  $N$  is very difficult
- ☺ In practice, small values of  $N$  (e.g.,  $N=2$ ) already yield good results
- ☺ Can use  $\frac{d^k \mu}{dt^k} = \frac{d^k \nu}{dt^k}, \quad \forall k \in \{1, \dots, N\}$

**Assumption:** to determine  $\varphi(\cdot)$ :  $k=1 \rightarrow$  boundary condition on  $\varphi$   
 $k=2 \rightarrow$  PDE on  $\varphi$

2)  $\dot{\mu}_\infty = A_\infty \mu_\infty$  is asymptotically stable

**Theorem:**  $\forall \delta > 0 \exists N$  s.t. if  $\frac{d^k \mu}{dt^k} = \frac{d^k \nu}{dt^k}, \quad \forall k \in \{1, \dots, N\}$

then

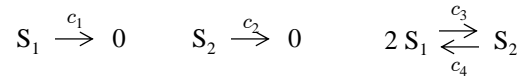
$$\|\mu(t) - \nu(t)\| \leq \beta(\|\mu(t_0) - \nu(t_0)\|, t - t_0) + \delta, \quad \forall t \geq t_0 \geq 0$$

Proof idea:

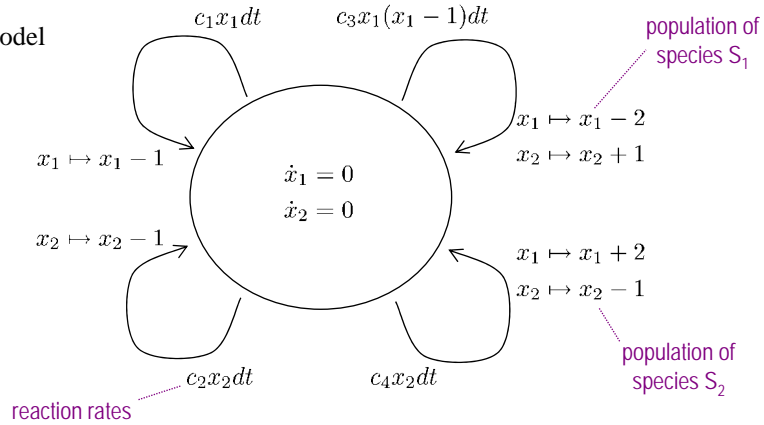
- 1)  $N$  derivative matches  $\Rightarrow \mu$  &  $\nu$  match on compact interval of length  $T$
- 2) stability of  $A_\infty \Rightarrow$  matching can be extended to  $[0, \infty)$

## and now something completely different... UCSB

Decaying-dimerizing molecular reactions (DDR):

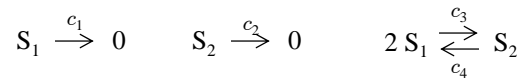


pSHS model



## Moment dynamics for DDR UCSB

Decaying-dimerizing molecular reactions (DDR):

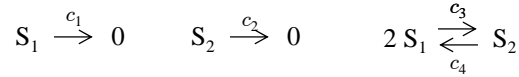


$$\begin{bmatrix} \dot{\mu}^{(1,0)} \\ \dot{\mu}^{(0,1)} \\ \dot{\mu}^{(2,0)} \\ \dot{\mu}^{(0,2)} \\ \dot{\mu}^{(1,1)} \end{bmatrix} = \begin{bmatrix} -c_1 + c_3 & 2c_4 & -c_3 & 0 & 0 \\ -\frac{c_3}{2} & -c_4 - c_2 & \frac{c_3}{2} & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 \\ c_1 - 2c_3 & 4c_4 & -2c_1 + 4c_3 & 0 & 4c_4 \\ -\frac{c_3}{2} & c_4 + c_2 & \frac{c_3}{2} & -2c_4 - 2c_2 & -c_3 \\ c_3 & -2c_4 & -\frac{3c_3}{2} & 2c_4 & -c_1 + c_3 - c_4 - c_2 \end{bmatrix} \begin{bmatrix} \mu^{(1,0)} \\ \mu^{(0,1)} \\ \mu^{(2,0)} \\ \mu^{(0,2)} \\ \mu^{(1,1)} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2c_3 & 0 \\ 0 & c_3 \\ \frac{c_3}{2} & -c_3 \end{bmatrix} \begin{bmatrix} \mu^{(3,0)} \\ \mu^{(2,1)} \end{bmatrix}$$

$$\begin{aligned} \mu^{(1,0)} &:= E[x_1] & \mu^{(2,0)} &:= E[x_1^2] & \mu^{(3,0)} &:= E[x_1^3] \\ \mu^{(0,1)} &:= E[x_2] & \mu^{(0,2)} &:= E[x_2^2] & \mu^{(2,1)} &:= E[x_1^2 x_2] \\ \mu^{(1,1)} &:= E[x_1 x_2] \end{aligned}$$

# Truncated DDR model

Decaying-dimerizing molecular reactions (DDR):

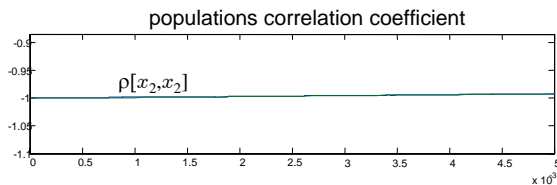
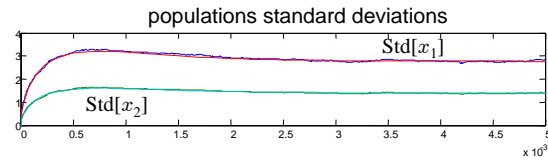
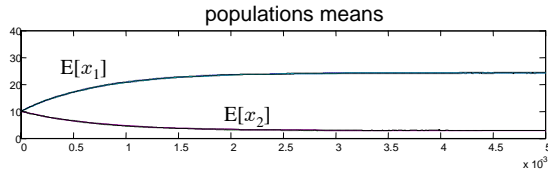


$$\begin{bmatrix} \dot{\mu}^{(1,0)} \\ \dot{\mu}^{(0,1)} \\ \dot{\mu}^{(2,0)} \\ \dot{\mu}^{(0,2)} \\ \dot{\mu}^{(1,1)} \end{bmatrix} \approx \begin{bmatrix} -c_1 + c_3 & 2c_4 & -c_3 & 0 & 0 \\ -\frac{c_3}{2} & -c_4 - c_2 & \frac{c_3}{2} & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 \\ c_1 - 2c_3 & 4c_4 & -2c_1 + 4c_3 & 0 & 4c_4 \\ -\frac{c_3}{2} & c_4 + c_2 & \frac{c_3}{2} & -2c_4 - 2c_2 & -c_3 \\ c_3 & -2c_4 & -\frac{3c_3}{2} & 2c_4 & -c_1 + c_3 - c_4 - c_2 \end{bmatrix} \begin{bmatrix} \mu^{(1,0)} \\ \mu^{(0,1)} \\ \mu^{(2,0)} \\ \mu^{(0,2)} \\ \mu^{(1,1)} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2c_3 & 0 \\ 0 & c_3 \\ \frac{c_3}{2} & -c_3 \end{bmatrix} \begin{bmatrix} \left(\frac{\mu^{(2,0)}}{\mu^{(1,0)}}\right)^3 \\ \left(\frac{\mu^{(2,0)}}{\mu^{(0,1)}}\right) \left(\frac{\mu^{(1,1)}}{\mu^{(1,0)}}\right)^2 \end{bmatrix}$$

by matching  
 $\frac{d^k \mu}{dt^k} = \frac{d^k \nu}{dt^k}, \quad \forall k \in \{1, 2\}$   
 for deterministic distributions

Hespanha. Polynomial Stochastic Hybrid Systems. HSCC'05

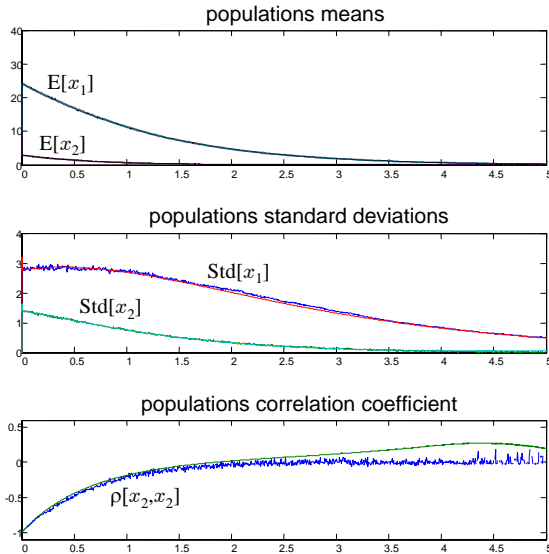
# Monte Carlo vs. truncated model



Fast time-scale transient

(lines essentially undistinguishable at this scale)

## Monte Carlo vs. truncated model



Slow time-scale evolution

only noticeable error when populations become very small (a couple of molecules)

## Conclusions

1. A simple SHS model (inspired by piecewise deterministic Markov Processes) can go a long way in modeling network traffic
2. The analysis of SHSs is generally difficult but there are tools available (generator, Dynkin's equation, moment dynamics, truncations)
3. This type of SHSs (and tools) finds use in several areas (traffic modeling, networked control systems, molecular biology)

- M. Davis. Markov Models & Optimization. Chapman & Hall/CRC, 1993
- S. Bohacek, J. Hespanha, J. Lee, K. Obraczka. Analysis of a TCP hybrid model. In *Proc. of the 39th Annual Allerton Conf. on Comm., Contr., and Computing*, Oct. 2001.
- S. Bohacek, J. Hespanha, J. Lee, K. Obraczka. A Hybrid Systems Modeling Framework for Fast and Accurate Simulation of Data Communication Networks. In *Proc. of the ACM Int. Conf. on Measurements and Modeling of Computer Systems (SIGMETRICS)*, June 2003.
- J. Hespanha. Stochastic Hybrid Systems: Applications to Communication Networks. In Rajeev Alur, George J. Pappas, *Hybrid Systems: Computation and Control*, number 2993 in Lect. Notes in Comput. Science, pages 387-401, Mar. 2004.
- Hespanha. Polynomial Stochastic Hybrid Systems. To be presented at the HSCC'05.
- J. Hespanha, A. Singh. Stochastic Models for Chemically Reacting Systems Using Polynomial Stochastic Hybrid Systems. Submitted to the *Int. J. on Robust Control Special Issue on Control at Small Scales*, Nov. 2004.

All papers (and some ppt presentations) available at  
<http://www.ece.ucsb.edu/~hespanha>