Research Supported by NSF & ARO

Why Should I Care About Stochastic Hybrid Systems?

João Hespanha

UC SANTA BARBARA engineering

Why Care Should I Care About SHSs?

Examples

- Feedback over shared networks
- Estimation using remote sensors
- Biology / Gene regulation

Modeling/Analysis tools

- Time-triggered SHSs
- Lyapunov-based analysis
- Moments dynamics

(ex) students: D. Antunes (IST), A. Mesquita (UCSB), Y. Xu (Advertising.com), A. Singh (UCSD)

collaborators: M. Khammash (UCSB), C. Silvestre (IST)

acknowledgements: NSF, Institute for Collaborative bio-technologies (ARO), AFOSR (STTR program)

disclaimer: This is an overview, technical details in papers referenced in bottom right corner… http://www.ece.ucsb.edu/~hespanha

Talk outline

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Examples

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Disclaimer:

Several other important applications/researchers not mentioned in this talk. E.g.,

- air traffic control [Bujorianu, Lygeros, Prandini, Hu, Tomlin,...]
- network traffic modeling [Bohacek, Lee, Yin, ...]
- queuing systems [Cassandras,...]
- economics [Davis, Yin,…]
- biology [Hu, Julius, Lygeros, Pappas,...]

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Deterministic Hybrid Systems

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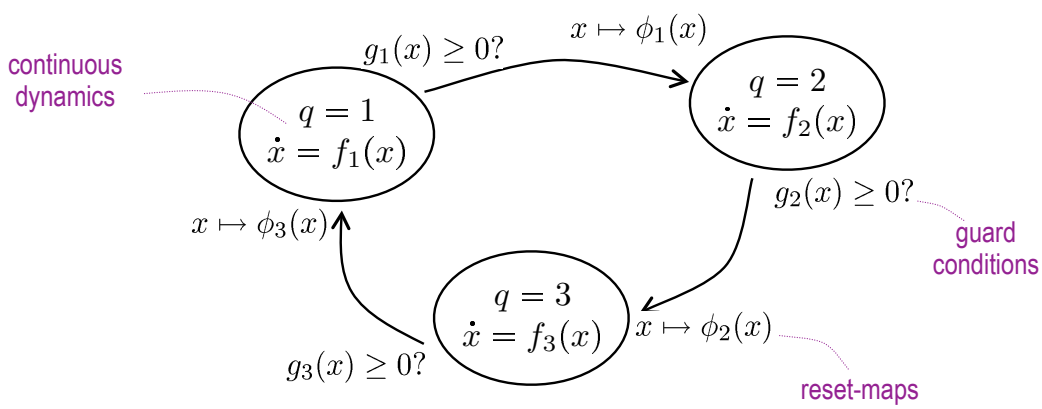
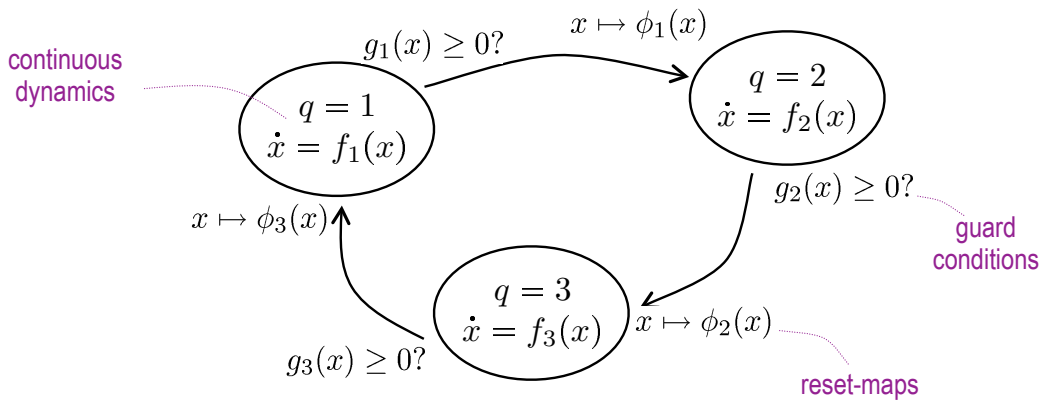


$q(t) \in \mathcal{Q}=\{1,2,\ldots\}$ $\equiv$ discrete state

$x(t) \in \mathbb{R}^n$ $\equiv$ continuous state

Deterministic Hybrid Systems



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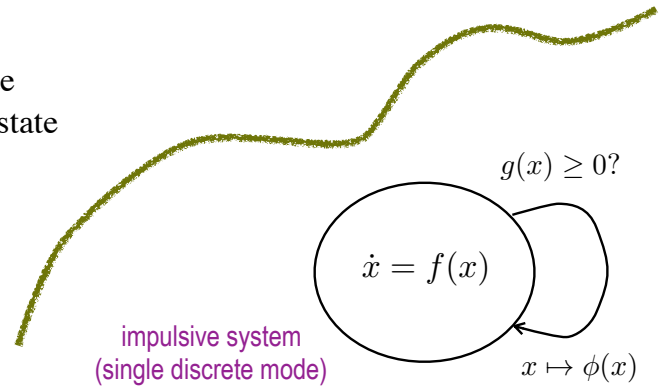
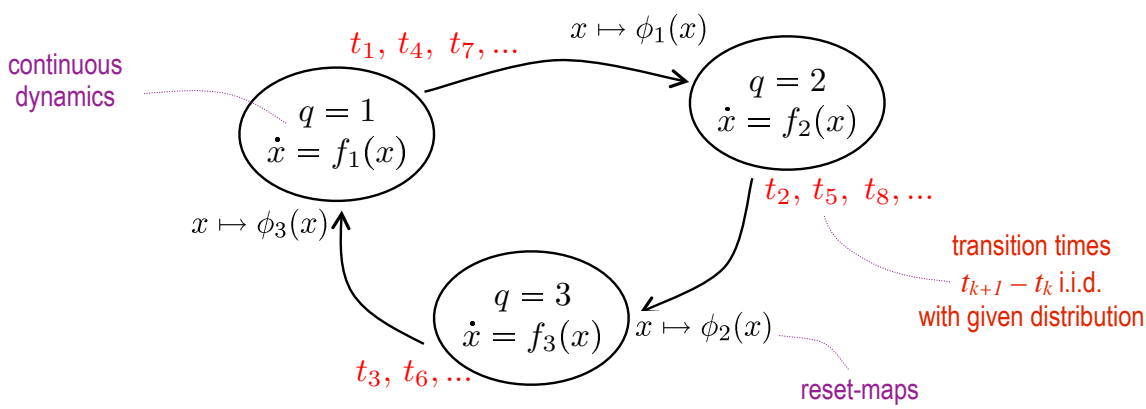


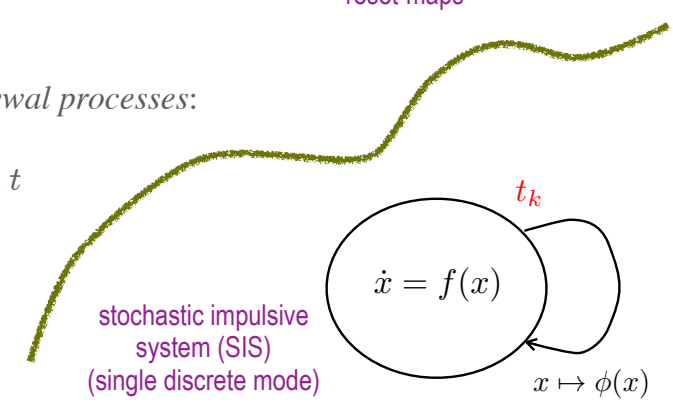
Stochastic Hybrid Systems (time-triggered)



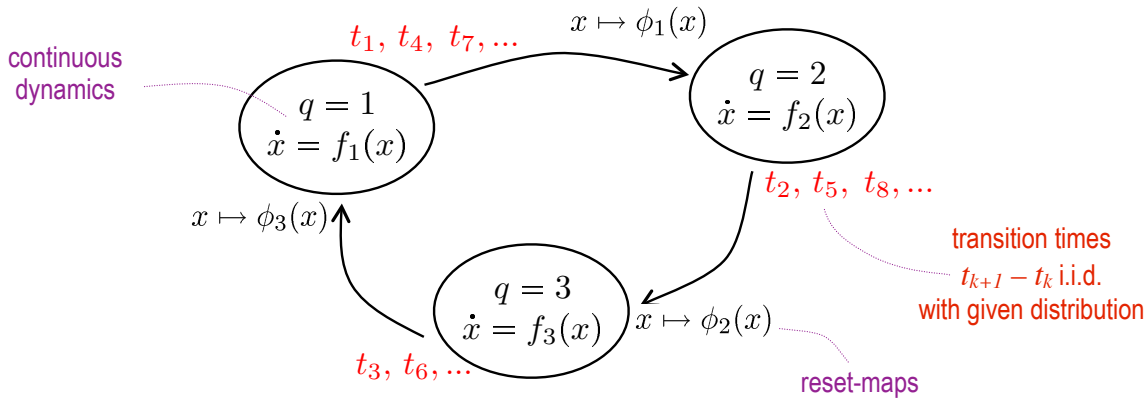
Also known as SHSs *driven by renewal processes*:

$N(t) \equiv$ # of transitions before time t

renewal process
(iid inter-increment times)



Stochastic Hybrid Systems (time-triggered)



Also kno

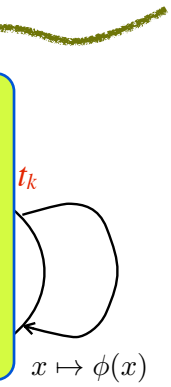
$N(t) \equiv$

re
(iid int

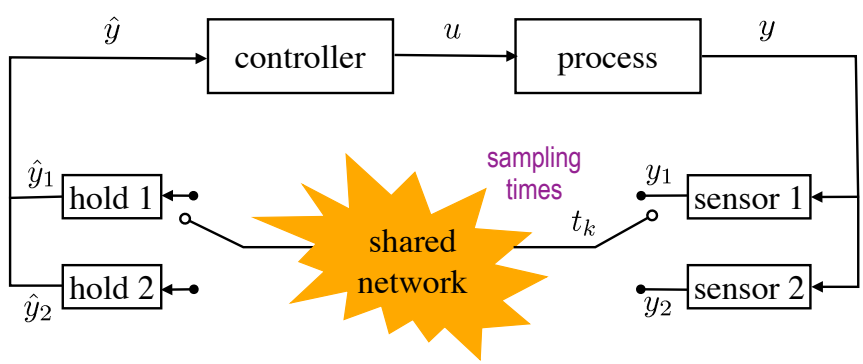
Special case: when $t_{k+1} - t_k$ i.i.d. exponentially distributed

- called Markovian Jump Systems
- in this case $x(t)$ is a Markov Process
- well developed theory (analysis & design)

[Costa, Fragoso, Boukas, Loparo, Lee, Dullerud]



Example #1: Networked Control System



process: $\dot{x}_P = A_P x_P + C_P u$, $y = C_P x_P + D_P u$

controller: $\dot{x}_C = A_C x_C + C_C \hat{y}$, $\hat{y} = C_C x_C + D_C \hat{y}$

round-robin network access:

$$\dot{\hat{y}} = 0$$

hold

$$\begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{cases} \begin{bmatrix} y_1(t_k^-) \\ \hat{y}_2(t_k^-) \end{bmatrix} & k \text{ odd} \\ \begin{bmatrix} \hat{y}_1(t_k^-) \\ y_2(t_k^-) \end{bmatrix} & k \text{ even} \end{cases}$$

Example #1: Networked Control System

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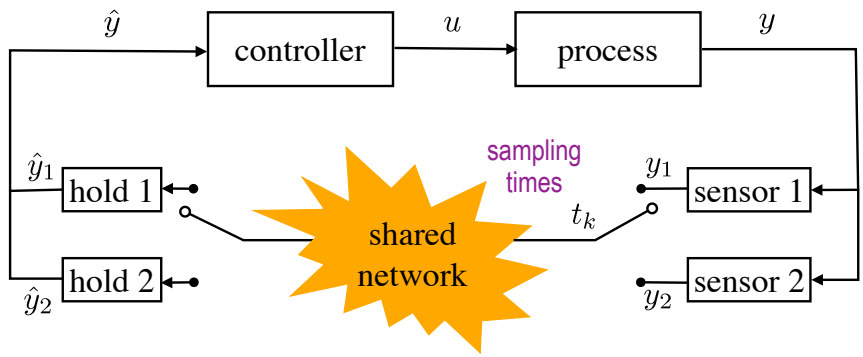

process: $\dot{x}_P = A_P x_P + C_P u$, $y = C_P x_P + D_P u$

controller: $\dot{x}_C = A_C x_C + C_C \hat{y}$, $\hat{y} = C_C x_C + D_C \hat{y}$

What if the network is not available at a sample time t_k ?

1st wait until network becomes available

2nd send (old) data from original sampling of continuous-time output

or

2nd send (latest) data from current sampling of continuous-time output

$\Rightarrow$ intersampling times $t_{k+1} - t_k$ typically become random variables

Example #1: Networked Control System

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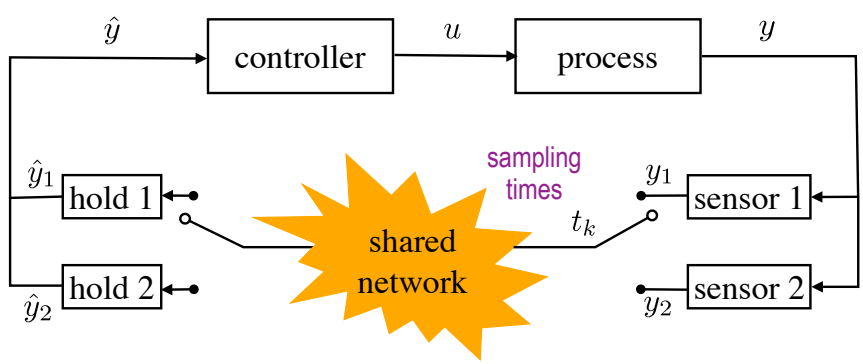
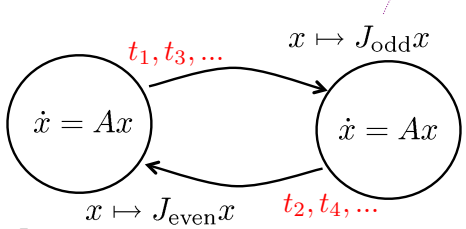


$t_{k+1} - t_k \sim$ time-interval between successive transmissions

$$\begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{bmatrix} y_1(t_k^-) \\ \hat{y}_2(t_k^-) \end{bmatrix}$$



$$x := \begin{bmatrix} x_P \\ x_C \\ \hat{y} \end{bmatrix}$$

$$\begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{bmatrix} \hat{y}_1(t_k^-) \\ y_2(t_k^-) \end{bmatrix}$$

Stability of Linear Time-triggered SIS

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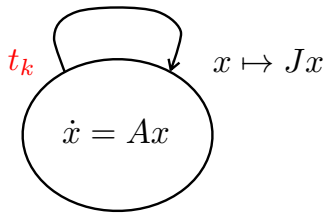

stochastic impulsive system
(single discrete mode)

$t_{k+1} - t_k \sim$ i.i.d., with cumulative distribution function $F(\cdot)$

Defining $x_k := x(t_k)$

state at jump times

$$x_{k+1} = \underbrace{J}_{\text{reset}} \underbrace{e^{A(t_{k+1}-t_k)}}_{\text{continuous evolution}} x_k$$

Stability of Linear Time-triggered SIS

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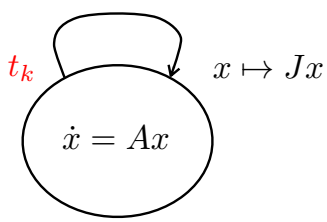

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state at jump times

$$x_{k+1} = \underbrace{J}_{\text{reset}} \underbrace{e^{A(t_{k+1}-t_k)}}_{\text{continuous evolution}} x_k$$

For a given $P = P' > 0$

$$\mathrm{E}[x'_{k+1} P x_{k+1} \mid x_k] = x'_k \, \mathrm{E}_{F(\boldsymbol{\Delta})} \Big[\underbrace{e^{A'\boldsymbol{\Delta}} J' P J e^{A\boldsymbol{\Delta}}} \Big] x_k$$

expectation w.r.t. $\boldsymbol{\Delta} = t_{k+1} - t_k$
(cumulative distribution F)

Stability of Linear Time-triggered SIS

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stochastic impulsive system
(single discrete mode)

t_k

$\dot{x}$

Defining x_k

If there exists

LMI on $P_{n\times n}$

$$P > 0, \quad \mathrm{E}_{F(\boldsymbol{\Delta})}\left[e^{A'\boldsymbol{\Delta}} J' P J e^{A\boldsymbol{\Delta}}\right] < P$$

then

$$\lim_{k\to\infty} \mathrm{E}[\|x_k\|^2] = 0 \quad \text{(exp. fast in index } k\text{)}$$

What about $x(t)$ between jumps?

For a given $P = P' > 0$

$$\mathrm{E}[x'_{k+1} P x_{k+1} \mid x_k] = x'_k \, \mathrm{E}_{F(\boldsymbol{\Delta})}\left[\underbrace{e^{A'\boldsymbol{\Delta}} J' P J e^{A\boldsymbol{\Delta}}}\right] x_k$$

expectation w.r.t. $\boldsymbol{\Delta} = t_{k+1} - t_k$
(cumulative distribution F)

Stability of Linear Time-triggered SIS

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All stability notions require $\lim_{k\to\infty} \|x_k\| = 0$ exponentially fast

the nec. & suff. conditions only differ on the requirements on the tail of distribution

$$1 - F(s) = \mathrm{P}(t_{k+1} - t_k > s)$$

- Mean-square exponential stability, i.e., $\lim_{t\to\infty} \mathrm{E}[\|x(t)\|^2] \overset{\text{exp. fast}}{=} 0$
- Mean-square asymptotic stability, i.e., $\lim_{t\to\infty} \mathrm{E}[\|x(t)\|^2] = 0$
- Mean-square stochastic stability, i.e., $\int_0^\infty \mathrm{E}[\|x(t)\|^2] dt < \infty$

Stability of Linear Time-triggered SIS

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All stability notions require $\lim_{k\to\infty} \|x_k\| = 0$ exponentially fast

the nec. & suff. conditions only differ on the requirements on the tail of distribution

$$1 - F(s) = \mathrm{P}(t_{k+1} - t_k > s)$$

(versions of these results for multiple discrete modes are available)

Theorem:

- Mean-square exponential stability, i.e., $\lim_{t\to\infty} \mathrm{E}[\|x(t)\|^2] \overset{\text{exp. fast}}{=} 0$

$\Leftrightarrow$ $\exists P > 0,\ \mathrm{E}_{F(\boldsymbol{\Delta})}\left[e^{A'\boldsymbol{\Delta}} J' P J e^{A\boldsymbol{\Delta}}\right] < P$ and $\lim_{s\to\infty} e^{A's} e^{As}\big(1 - F(s)\big) \overset{\text{exp. fast}}{=} 0$

- Mean-square asymptotic stability, i.e., $\lim_{t\to\infty} \mathrm{E}[\|x(t)\|^2] = 0$

$\Leftrightarrow$ $\exists P > 0,\ E_{F(\boldsymbol{\Delta})}[e^{A'\boldsymbol{\Delta}} J' P J e^{A\boldsymbol{\Delta}}] < P$ and $\lim_{s\to\infty} e^{A's} e^{As}\big(1 - F(s)\big) = 0$

- Mean-square stochastic stability, i.e., $\int_0^\infty \mathrm{E}[\|x(t)\|^2] dt < \infty$

$\Leftrightarrow$ $\exists P > 0,\ \mathrm{E}_{F(\boldsymbol{\Delta})}\left[e^{A'\boldsymbol{\Delta}} J' P J e^{A\boldsymbol{\Delta}}\right] < P$ and $\int_0^\infty e^{A's} e^{As} F(ds) < \infty$

[Antunes *et al*, 2009]

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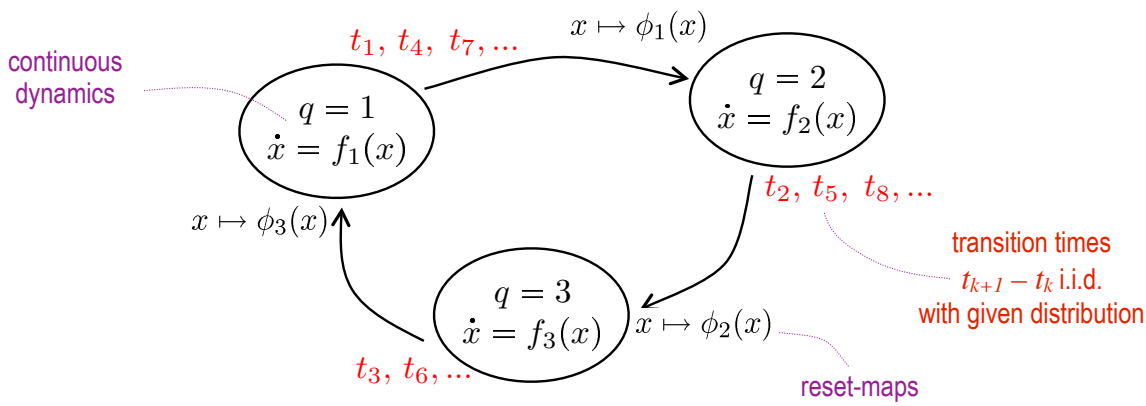
So far time-driven SHSs...



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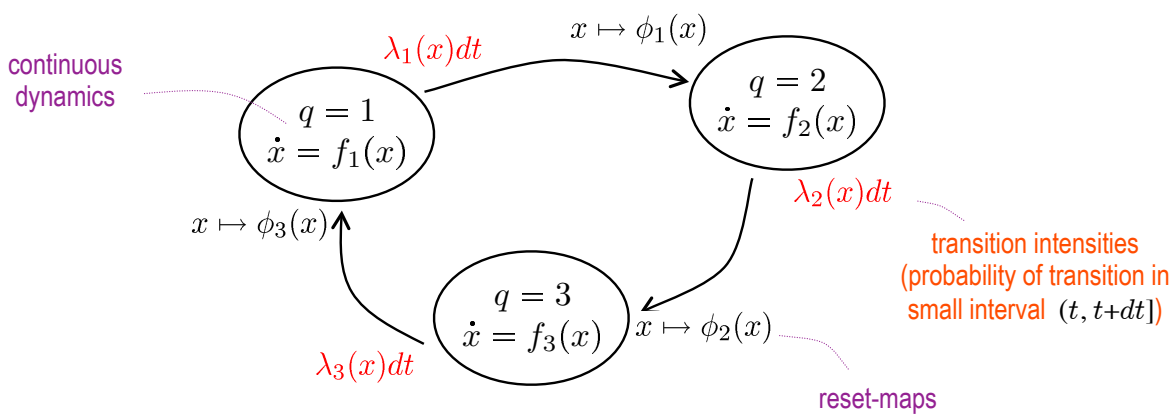
Stochastic Hybrid Systems

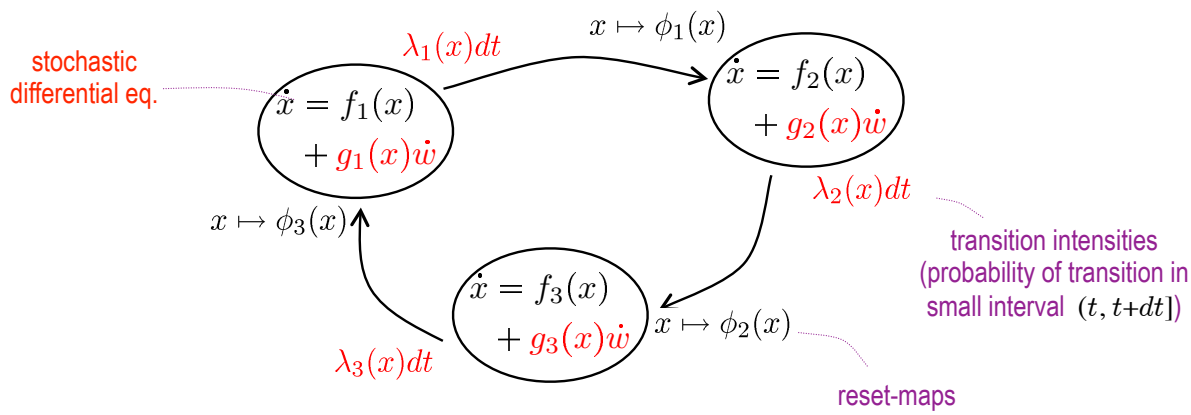


$q(t) \in \mathcal{Q}=\{1,2,\ldots\}$ $\equiv$ discrete state
$x(t) \in \mathbb{R}^n$ $\equiv$ continuous state

Special case: Markovian jump system

When all λ_ℓ are constant $\Rightarrow$ time triggered SIS with exponential $t_{k+1} - t_k$

Stochastic Hybrid Systems with Diffusion



$q(t) \in \mathcal{Q}=\{1,2,\ldots\}$ $\equiv$ discrete state

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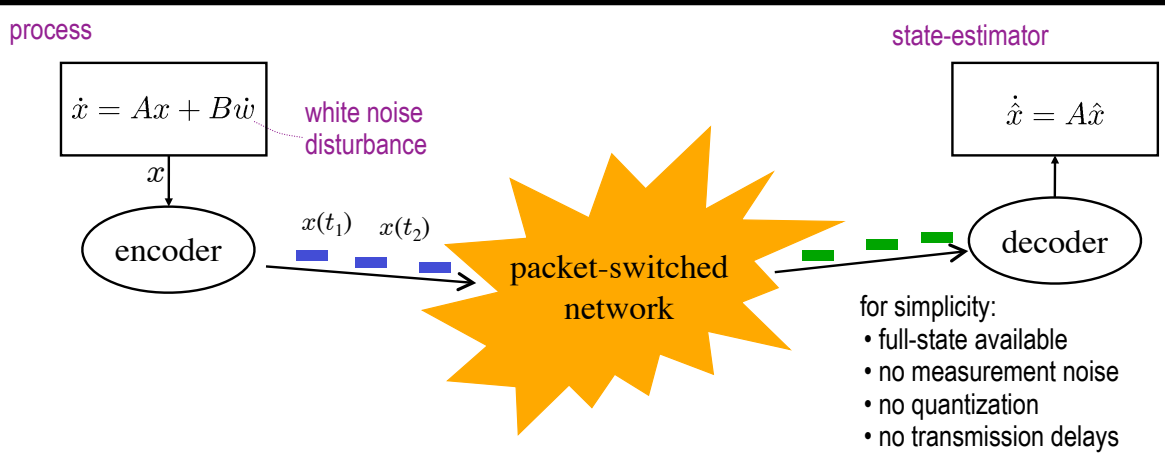
Example #2: Estimation through network



encoder logic $\equiv$ determines *when* to send measurements to the network

decoder logic $\equiv$ determines *how* to incorporate received measurements

Stochastic communication logic

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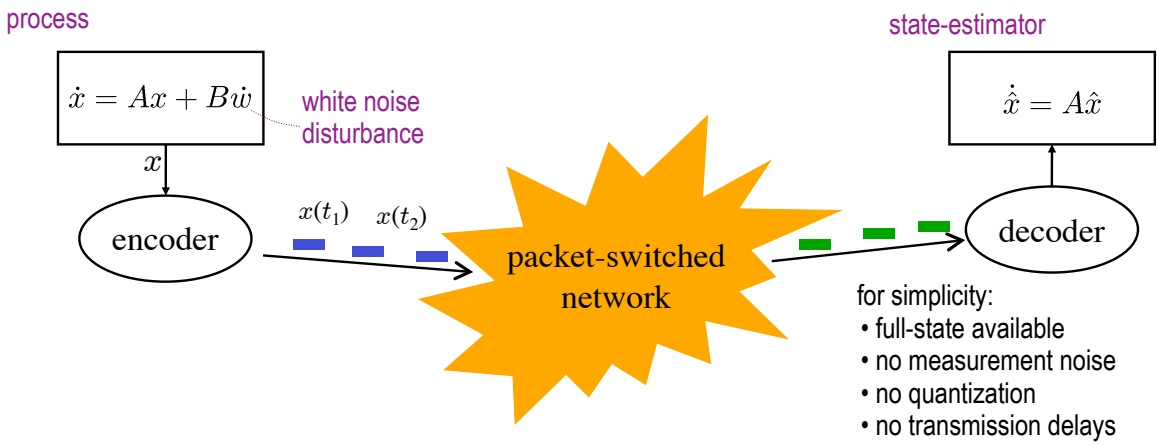

for simplicity:
- full-state available
- no measurement noise
- no quantization
- no transmission delays

decoder logic ≡ determines *how* to incorporate received measurements

1. upon reception of $x(t_k)$, reset $\hat{x}(t_k)$ to $x(t_k)$

encoder logic ≡ determines *when* to send measurements to the network

1. keep track of remote estimate $\hat{x}$
2. send measurements stochastically
3. probability of sending data increases as $\hat{x}$ deviates from x

[related ideas pursued by Astrom, Basar, Hristu, Kumar, Tilbury]

Error Dynamics

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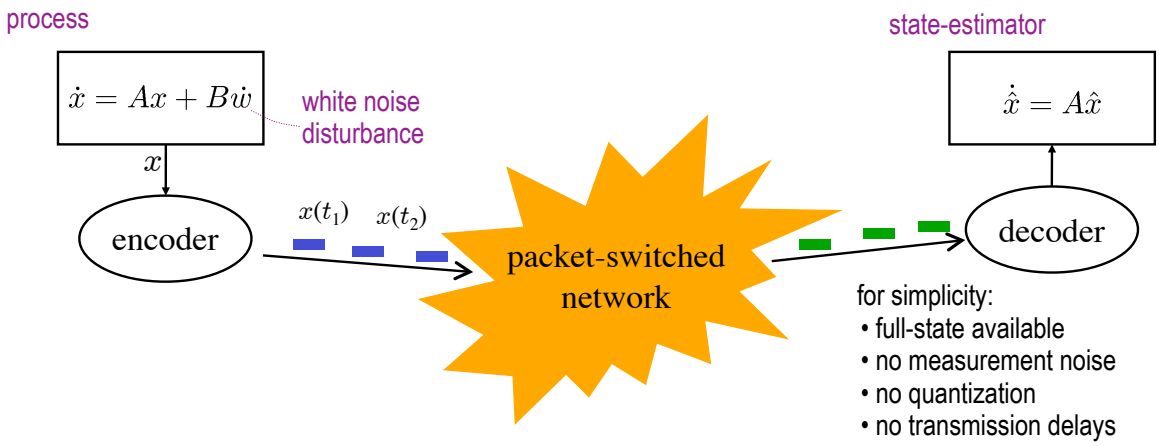
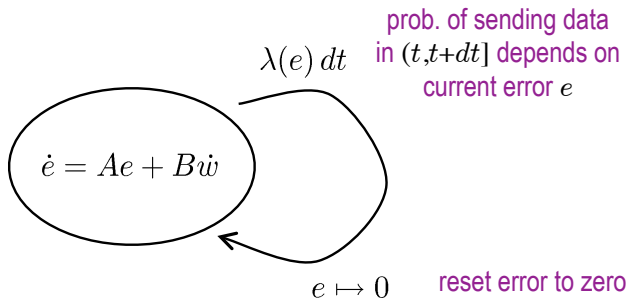


for simplicity:
- full-state available
- no measurement noise
- no quantization
- no transmission delays

Error dynamics: $e := x - \hat{x}$



ODE – Lie Derivative

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$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

Given scalar-valued function $V : \mathbb{R}^n \to \mathbb{R}$

$$\frac{dV\big(x(t)\big)}{dt} = \underbrace{\frac{\partial V\big(x(t)\big)}{\partial x} f\big(x(t)\big)}_{L_f V}$$

derivative along solution to ODE

$L_f V$ Lie derivative of V

Basis of "Lyapunov" formal arguments to establish boundedness and stability…

E.g., picking $V(x) := \|x\|^2$

$$\frac{dV\big(x(t)\big)}{dt} = \frac{\partial V}{\partial x} f\big(x\big) \leqslant 0 \quad \Rightarrow \quad V\big(x(t)\big) = \|x(t)\|^2 \leqslant \|x(0)\|^2$$

$\|x\|^2$ remains bounded along trajectories !

Generator of a Stochastic Hybrid System

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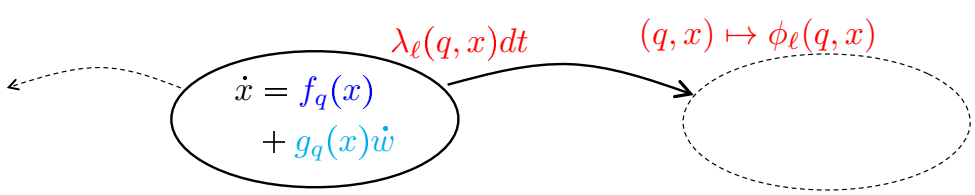

Given scalar-valued function $V : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}$

$$\frac{d}{dt}\mathrm{E}\Big[V\big(q(t),x(t)\big)\Big] = E\Big[(LV)\big(q(t),x(t)\big)\Big]$$

x & q are discontinuous, but the expected value is differentiable

Dynkin's formula (in differential form)

where

$$(LV)(q,x) := \frac{\partial V}{\partial x}(q,x) f_q(x)$$

Lie derivative

(extended) generator of the SHS

$$+ \sum_{\ell=1}^{m} \underbrace{\lambda_\ell(q,x)}_{\text{intensity}} \overbrace{\Big(V\big(\phi_\ell(q,x)\big) - V(q,x)\Big)}^{\text{instantaneous variation}}$$

Reset term

$$+ \frac{1}{2}\,\mathrm{trace}\left(g_q(x)' \frac{\partial^2 V}{\partial x^2} g_q(x)\right)$$

Diffusion term

Lyapunov Analysis – SHSs

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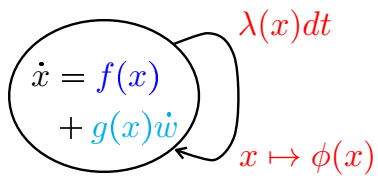

$$\boxed{\frac{d}{dt}\mathrm{E}\left[V\big(x(t)\big)\right] = E\Big[(LV)\big(x(t)\big)\Big]}$$

class-K functions:
(zero at zero & mon. increasing)

probability of $||x(t)||$ exceeding any given bound M, can be made arbitrarily small by making $||x_0||$ small

sample-path notions

$$\begin{cases} \alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \\ LV(x) \le -\alpha_3(\|x\|) \end{cases} \Rightarrow \begin{cases} \mathrm{P}\left(\exists t : \|x(t)\| \ge M\right) \le \frac{\alpha_2(\|x_0\|)}{\alpha_1(M)} \\ \mathrm{P}\left(x(t) \to 0\right) = 1 \end{cases}$$

almost sure (a.s.) asymptotic stability

expected-value notions

$$\begin{cases} V(x) \ge 0 \\ LV(x) \le -W(x) \end{cases} \Rightarrow \int_0^\infty \mathrm{E}\left[W\big(x(t)\big)\right]dt < \infty$$

stochastic stability (mean square when $W(x) = \|x\|^2$)

$$\begin{cases} V(x) \ge W(x) \ge 0 \\ LV(x) \le -\mu V + c \end{cases} \Rightarrow \mathrm{E}\left[W\big(x(t)\big)\right] \le e^{-\mu t}V(x_0) + \frac{c}{\mu}$$

exponential stability (mean square when $W(x) = \|x\|^2$)

Example #2: Remote estimation

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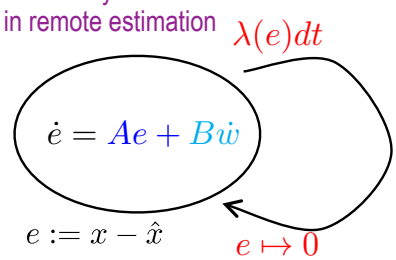
error dynamics in remote estimation



Dynkin's formula

$$\frac{d}{dt}\mathrm{E}\left[V\big(e(t)\big)\right] = E\Big[(LV)\big(e(t)\big)\Big]$$

$$(LV)(e) := \frac{\partial V}{\partial e}Ae + \lambda(e)\Big(V(0) - V(e)\Big) + \frac{1}{2}\,\mathrm{trace}\left(B'\frac{\partial^2 V}{\partial e^2}B\right)$$

For constant rate: $\lambda(e) = \gamma$ (exp. distributed inter-jump times) using $V(e) = e'Pe$

1. $\mathrm{E}[\,e\,] \to 0$ if and only if $\gamma > \Re\big[\lambda_i(A)\big],\ \forall i$
2. $\mathrm{E}[\,\|e\|^m\,]$ bounded if and only if $\gamma > m\,\Re\big[\lambda_i(A)\big],\ \forall i$

getting more moments bounded requires higher comm. rates

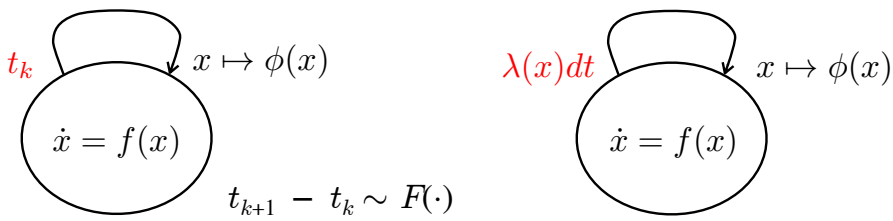
For radially unbounded rate: $\lambda(e)$ (reactive transmissions) using $V(e) = \|e\|^2$

3. $\mathrm{E}[\,e\,] \to 0$ (always)
4. $\mathrm{E}[\,\|e\|^m\,]$ bounded $\forall\, m$

Moreover, one can achieve the same $\mathrm{E}[\,\|e\|^2\,]$ with less communication than with a constant rate or periodic transmissions…

[Xu *et al*, 2006]

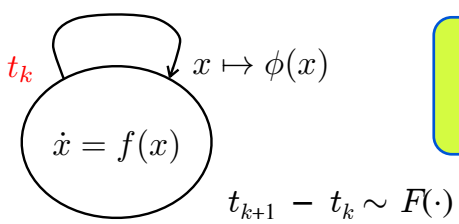
Back to Time-triggered SIS...



Can we pick an intensity $\lambda(\cdot)$ to obtain the desired distribution for the t_k ?

YES

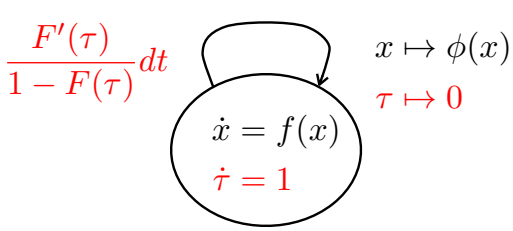
Back to Time-triggered SIS...

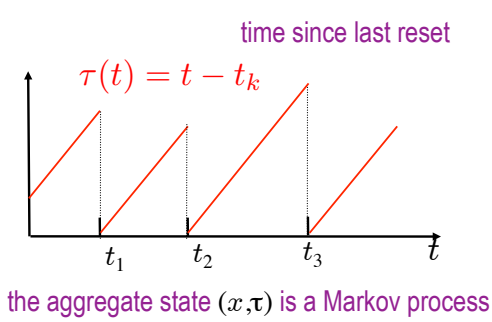


This representation allows one to *combine in the same SHS* time- and event-triggered transitions!

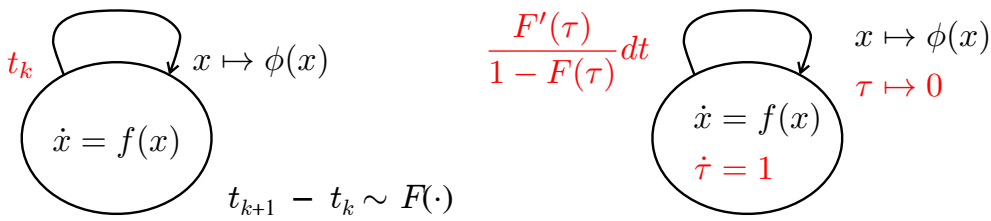
Can we pick an intensity $\lambda(\cdot)$ to obtain the desired distribution for the t_k ?

YES





Converse Lyapunov Stability



Theorem:

System is mean-square exponentially stable, i.e., $\lim_{t\to\infty} \mathrm{E}[\|x(t)\|^2] \overset{\text{exp. fast}}{=} 0$

$$\Updownarrow$$

$\exists P(\tau)$ such that defining $V(x,\tau) = x' P(\tau) x$

Lyapunov-like function
quadratic on x for fixed τ

$$\begin{cases} c_1 I \le P(\tau) \le c_2 I & \Rightarrow \quad V \text{ is positive definite} \\ (LV)(x,\tau) \le -\epsilon V(x,\tau) & \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}[V(x,\tau)] \le -\epsilon\, \mathrm{E}[V(x,\tau)] \end{cases}$$

(motivates choices for Lyapunov function for nonlinear systems)

Talk outline

Examples

- Feedback over shared networks
- Estimation using remote sensors
- Biology / Gene regulation

Modeling/Analysis tools

- Time-triggered SHSs
- Lyapunov-based analysis
- Moments dynamics

(ex) students: D. Antunes (IST), A. Mesquita (UCSB), Y. Xu (Advertising.com), A. Singh (UCSD)

collaborators: M. Khammash (UCSB), C. Silvestre (IST)

acknowledgements: NSF, Institute for Collaborative bio-technologies (ARO), AFOSR (STTR program)

disclaimer: This is an overview, technical details in papers referenced in bottom right corner… http://www.ece.ucsb.edu/~hespanha

Example III:(Unregulated) Gene Expression

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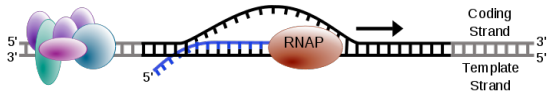
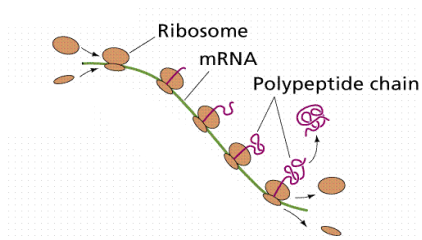
Gene expression ≡ process by which a gene (encoded in the DNA) produces proteins:

http://en.wikipedia.org





$* \longrightarrow \text{mRNA}$ transcription (constant rate)

$\text{mRNA} \longrightarrow \text{X} + \text{mRNA}$ translation

$\text{mRNA} \longrightarrow *$ mRNA decay

$\text{X} \longrightarrow *$ protein decay

Example III:(Unregulated) Gene Expression

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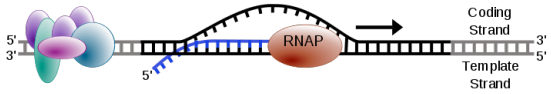
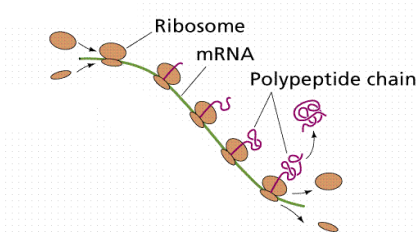




$* \longrightarrow \text{mRNA}$ transcription (constant rate)

$\text{mRNA} \longrightarrow \text{X} + \text{mRNA}$ translation

$\text{mRNA} \longrightarrow *$ mRNA decay

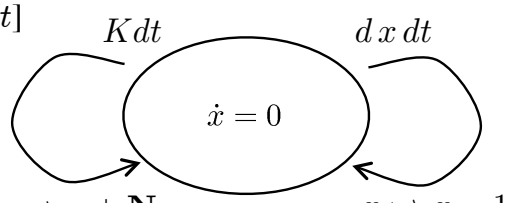
one transcription event

$\text{X} \longrightarrow *$ protein decay

$x(t) \equiv$ number of proteins at time t

prob. of one transcription event in $(t, t{+}dt]$

prob. of one decay event in $(t, t{+}dt]$



# of proteins produced per transcription event

equivalent to Gillespie's stochastic simulation algorithm (SSA)

Example III:(Unregulated) Gene Expression

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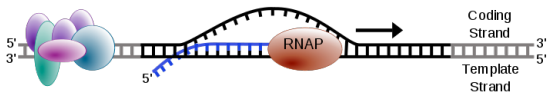
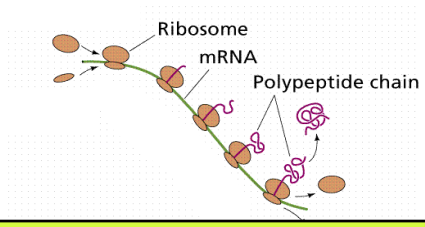




$* \longrightarrow \text{mRNA}$ transcription (constant rate)

$\text{mRNA} \longrightarrow \text{X} + \text{mRNA}$ translation

$\text{mRNA} \longrightarrow *$ mRNA decay

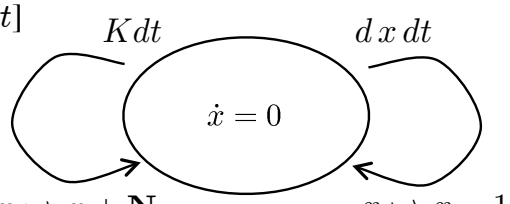
one transcription event

$\text{X} \longrightarrow *$ protein decay

How to go beyond stability/bounds and study the dynamics of means, variances, co-variances, etc.?

prob. of one transcription event in $(t, t{+}dt]$

prob. of one decay event in $(t, t{+}dt]$



# of proteins produced per transcription event

equivalent to Gillespie's stochastic simulation algorithm (SSA)

Moment Dynamics

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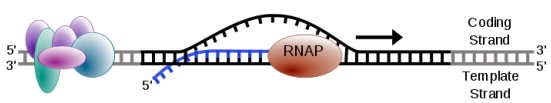
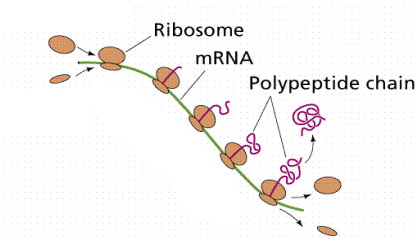
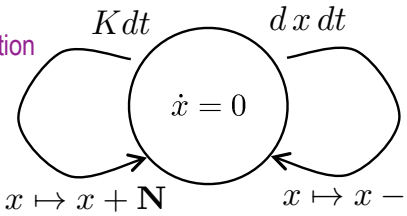




transcription event



decay event

$$\frac{d}{dt}\,\mathrm{E}\left[V(x)\right] = E\left[(LV)(x)\right]$$

$$(LV)(x) = K\Big(V(x+\mathbf{N}) - V(x)\Big) + d\,x\Big(V(x-1) - V(x)\Big)$$

$$\frac{dE[x]}{dt} =$$

$$\frac{dE[x^2]}{dt} =$$

Moment Dynamics

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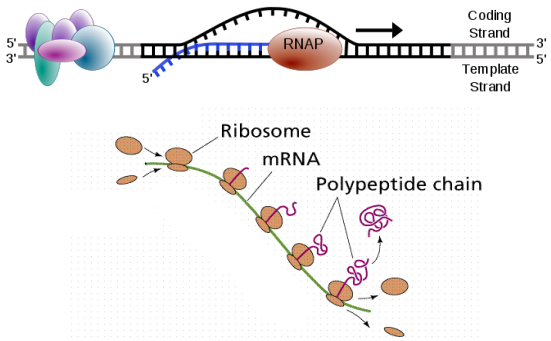

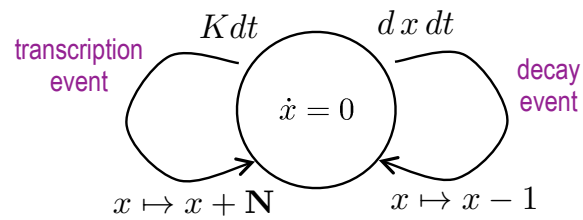

$$\frac{d}{dt}\,\mathrm{E}\left[V(x)\right] = E\left[(LV)(x)\right]$$

$$(LV)(x) = K\Big(V(x+\mathbf{N}) - V(x)\Big) + d\,x\Big(V(x-1) - V(x)\Big)$$

$$\frac{dE[x]}{dt} = K\,\mathrm{E}[\mathbf{N}] - d\,\mathrm{E}[x]$$

$$\frac{dE[x^2]}{dt} = K\,\mathrm{E}[\mathbf{N}^2] + (2K\,\mathrm{E}[\mathbf{N}] + d)\mathrm{E}[x] - 2d\,\mathrm{E}[x^2]$$

One can show that $\mathrm{E}[\mathbf{N}] = N := \dfrac{\text{mRNA translation rate}}{\text{mRNA decay rate}}$ $\qquad \mathrm{Var}[\mathbf{N}] = N^2 - N$

(Unregulated) Gene Expression

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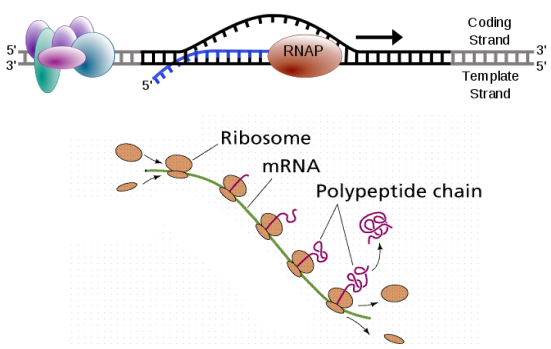
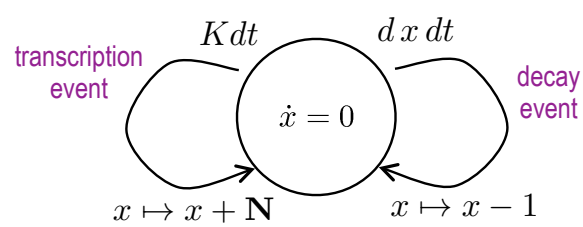
http://en.wikipedia.org





Thus, at steady-state,

$$\mathrm{E}[x] = \frac{KN}{d}$$

$$\mathrm{CV}[x] = \frac{\mathrm{StdDev}[x]}{\mathrm{E}[x]} = \sqrt{\frac{d}{K}} = \sqrt{\frac{N}{\mathrm{E}[x]}}$$

- measure of stochastic fluctuations in protein level x (normalized by mean population)
- **intrinsic noise** (solely due to random protein expression/degradation)

$$N := \frac{\text{mRNA translation rate}}{\text{mRNA decay rate}}$$

Auto-Regulated Gene Expression

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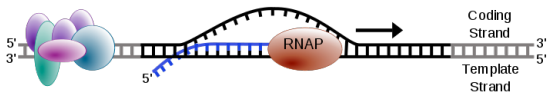
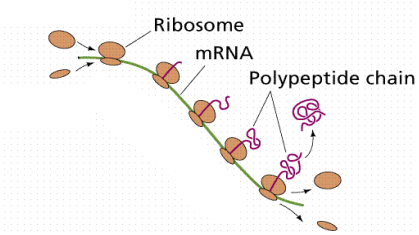




$* \longrightarrow \text{mRNA}$ transcription (**non-constant** rate)

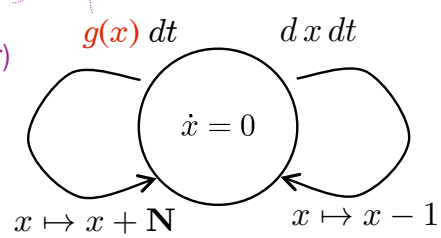
$\text{mRNA} \longrightarrow \text{X} + \text{mRNA}$ translation

$\text{mRNA} \longrightarrow *$ mRNA decay

$\text{X} \longrightarrow *$ protein decay

Protein production rate is a function of the current protein molecule count through *transcription regulation*:

transcriptional response (stochastic rate at which transcription events occur)



- Altering the RNA polymerase specificity for a given promoter or set of promoters
- Binding to non-coding sequences on the DNA to impede RNA polymerase's progress

Auto-Regulatory Negative Feedback

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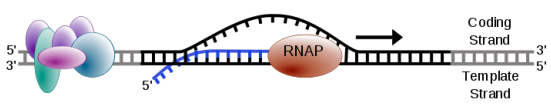
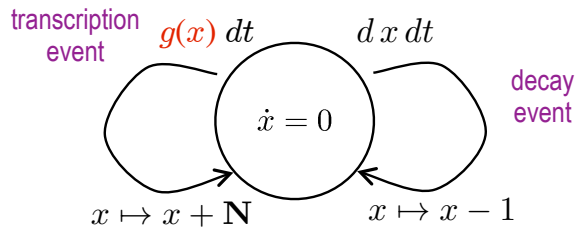
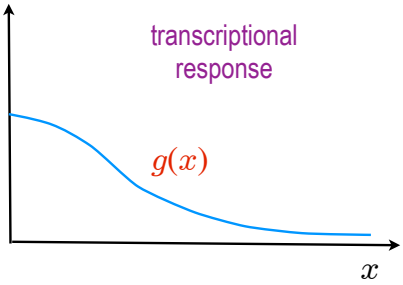




negative feedback ≡ protein production rate is a **decreasing** function of the protein molecule count



- Common form of auto regulation (e.g., half of the repressors in *E. Coli*)
- Experimentally shown to exhibit noise reduction ability

Moment Dynamics

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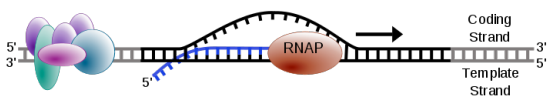

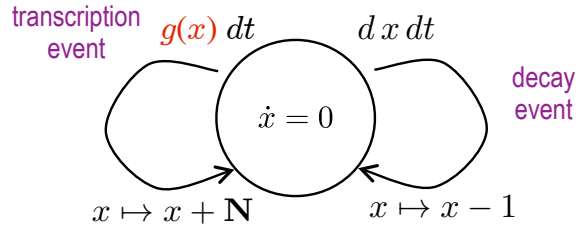

$$\frac{d}{dt}\,\mathrm{E}\left[V(x)\right] = E\left[(LV)(x)\right]$$

$$(LV)(x) = g(x)\Big(V(x+\mathbf{N}) - V(x)\Big) + d\,x\Big(V(x-1) - V(x)\Big)$$

$$\frac{dE[x]}{dt} = \mathrm{E}[\mathbf{N}]\mathrm{E}[g(x)] - d\,\mathrm{E}[x]$$

$$\frac{dE[x^2]}{dt} = \mathrm{E}[\mathbf{N}^2]\mathrm{E}[g(x)] + 2\,\mathrm{E}[\mathbf{N}]\mathrm{E}[g(x)x] + d\mathrm{E}[x] - 2d\,\mathrm{E}[x^2]$$

- When $g(x)$ is an affine function we still get a finite system of linear equations
- When $g(x)$ is a polynomial, we get a closed but infinite system of linear equation (general property of polynomial SHSs)
- For other $g(x)$, one generally does not get a closed system of equations

Auto-Regulated Gene Expression

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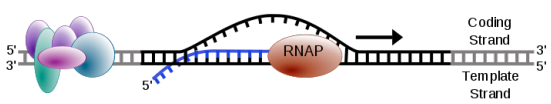

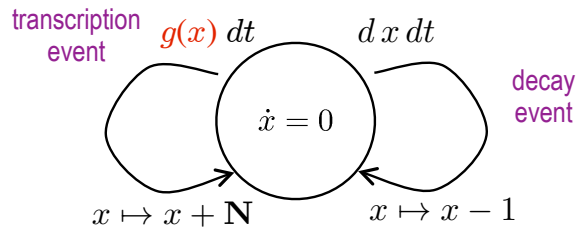

Approximate Analysis Methods

- Distribution-based: assume a specific type of distribution (Normal, LogNormal, Poisson, etc.) and force dynamics to be compatible with this type of distribution
- Large numbers/large volume: take the limit as volume → ∞ and assume concentrations do not → 0
- Derivative matching: force solutions of approximate dynamics to match exact equation locally in time
- Linearization: Linearize transcriptional response around steady-state value of the mean

Auto-Regulatory Negative Feedback

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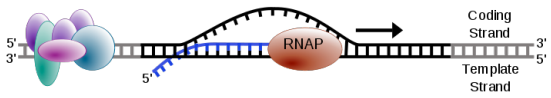
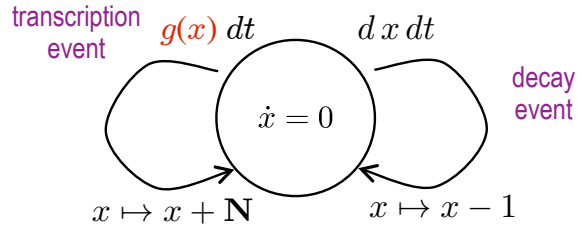




For a transcriptional response approximately linear around steady-state mean

$$g(x) \approx g(x^*) + g'(x^*)(x - x^*)$$

steady-state population mean

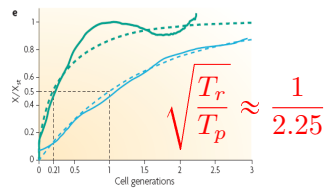
protein's response-time (with feedback)

$$CV[x] = \frac{\mathrm{StdDev}[x]}{\mathrm{E}[x]} = \sqrt{\frac{T_r}{T_p}\frac{N}{\mathrm{E}[x]}}$$

"unregulated" intrinsic noise

protein's half-live (response time without feedback)

Negative feedback reduces T_r with respect to $T_p \Rightarrow$ decreases noise



experimental study using a synthetic gene circuit in which the repressor TetR fused to GFP represses its own promoter

Rosenfeld et al, J. Molecular Biology, 2002

Exogenous Noise

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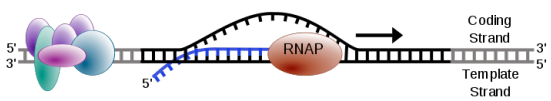

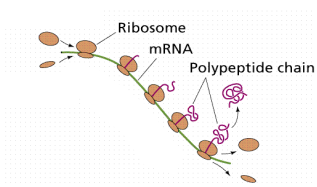

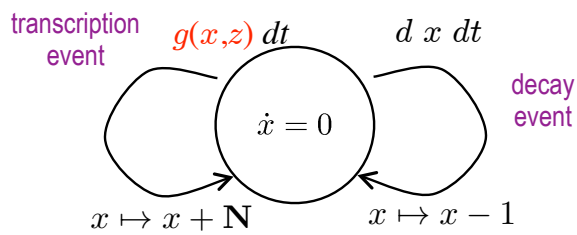

In practice, transcription rate also depends on exogenous species (e.g., RNA polymerase and other enzymes)

$g(x, z) \equiv$ transcriptional response (stochastic rate at which transcription events occur)

exogenous species (with stochastic fluctuations)

Exogenous Noise

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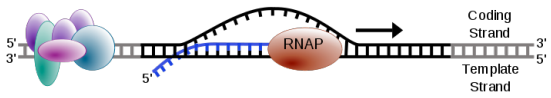
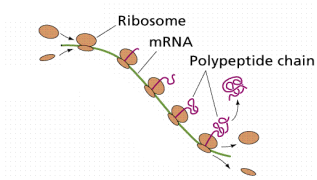
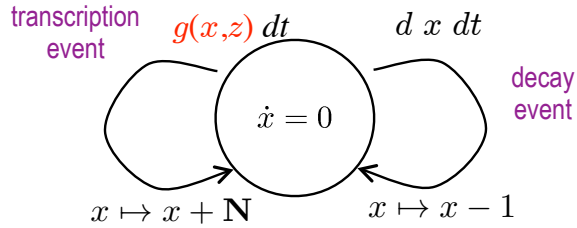






$$CV[x]^2 \approx \underbrace{\frac{T_r}{T_p}\frac{N}{\mathrm{E}[x]}}_{\text{intrinsic noise (as before)}} + \underbrace{\left(\frac{T_r}{T_p}\right)^2 CV[z]^2}_{\text{extrinsic noise}}$$

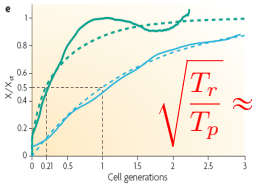
CV of extrinsic species

$T_r \equiv$ protein's response-time (with feedback)

$T_p \equiv$ protein's half-life (response time without feedback)

Negative feedback reduces T_r with respect to T_p

- attenuates both intrinsic and extrinsic noise
- more efficient at reducing extrinsic noise
- surprisingly good matches with experimental results…
- offers a new technique to discover sources of extrinsic noise (solve for $CV[z]$!)



$$\sqrt{\frac{T_r}{T_p}} \approx \frac{1}{2.25}$$

[Singh *et al*, 2009; related results by Paulsson 2004]

Why Should I Care About SHS ?

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1. SHS models that find use in several areas
   (network traffic modeling, networked control systems, distributed estimation, biochemistry, population dynamics in ecosystems)
2. The analysis of SHSs is challenging but there are tools available
   (stability conditions for linear time-triggered SHS, Lyapunov methods, moment dynamics, linearization, truncations)
3. Lots of work to be done:
   - theory
     - stability/robustness/performance of SHSs
   - networked control systems
     - protocol design to optimize performance & minimize communication resources
   - biology
     - quantitative study of common motifs (modules) in systems biology
     - study of spatial processes
   - other applications…