Why Should I Care About Stochastic Hybrid Systems?

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Why Care Should I Care About SHSs?

Examples
- Feedback over shared networks
- Estimation using remote sensors
- Biology / Gene regulation

Modeling/Analysis tools
- Time-triggered SHSs
- Lyapunov-based analysis
- Moments dynamics

(ex) students: D. Antunes (IST), A. Mesquita (UCSB), Y. Xu (Advertising.com), A. Singh (UCSD)
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Disclaimer:
Several other important applications/researchers not mentioned in this talk. E.g.,
- air traffic control [Bujorianu, Lygeros, Prandini, Hu, Tomlin,...]
- network traffic modeling [Bohacek, Lee, Yin, ...]
- queuing systems [Cassandras,...]
- economics [Davis, Yin,...]
- biology [Hu, Julius, Lygeros, Pappas,...]

Deterministic Hybrid Systems

\[
\begin{align*}
q(t) & \in Q=\{1,2,...\} \quad \equiv \text{discrete state} \\
x(t) & \in \mathbb{R}^n \quad \equiv \text{continuous state}
\end{align*}
\]

\[
\begin{align*}
q = 1 & \quad \hat{x} = f_1(x) \\
q = 2 & \quad \hat{x} = f_2(x) \\
q = 3 & \quad \hat{x} = f_3(x)
\end{align*}
\]

\[
\begin{align*}
g_1(x) & \geq 0? \\
g_2(x) & \geq 0? \\
g_3(x) & \geq 0?
\end{align*}
\]
Deterministic Hybrid Systems

\[ q(t) \in Q = \{1, 2, \ldots\} \quad \equiv \text{discrete state} \]
\[ x(t) \in \mathbb{R}^n \quad \equiv \text{continuous state} \]

\[ \dot{x} = f(x) \]

\[ g_1(x) \geq 0? \quad x \mapsto \phi_1(x) \]
\[ g_2(x) \geq 0? \quad x \mapsto \phi_2(x) \]
\[ g_3(x) \geq 0? \quad x \mapsto \phi_3(x) \]

Stochastic Hybrid Systems (time-triggered)

\[ t_1, t_4, t_7, \ldots \quad \mapsto \phi_1(x) \]
\[ t_2, t_5, t_8, \ldots \quad \mapsto \phi_2(x) \]
\[ t_3, t_6, \ldots \quad \mapsto \phi_3(x) \]

Also known as SHSs driven by renewal processes:

\[ N(t) \equiv \# \text{of transitions before time } t \]

renewal process (iid inter-increment times)

stochastic impulsive system (SIS)
(single discrete mode)
Stochastic Hybrid Systems (time-triggered)

Also known as SHSs driven by renewal processes (iid inter-increment times): continuous dynamics

\[ \dot{x} = f(x) \]

stochastic impulsive system (SIS) (single discrete mode)

Special case: when \( t_{k+1} - t_k \) i.i.d. exponentially distributed

called Markovian Jump Systems

in this case \( x(t) \) is a Markov Process

well developed theory (analysis & design)

[Costa, Fragoso, Boukas, Loparo, Lee, Dullerud]

Example #1: Networked Control System

process: \[ \dot{x} = A_P x_P + C_P u \]
\[ y = C_P x_P + D_P u \]

controller: \[ \dot{x}_C = A_C x_C + C_C \dot{y} \]
\[ \dot{y} = C_C x_C + D_C \dot{y} \]

round-robin network access:

\[ \dot{y} = 0 \]

\[ \begin{bmatrix} \dot{y}_1(t_k) \\ \dot{y}_2(t_k) \end{bmatrix} = \begin{bmatrix} y_1(t_k^-) \\ y_2(t_k^-) \end{bmatrix} \quad k \text{ odd} \]

\[ \begin{bmatrix} \dot{y}_1(t_k^-) \\ \dot{y}_2(t_k^-) \end{bmatrix} \quad k \text{ even} \]
Example #1: Networked Control System

process: \[ \dot{x}_P = A_P x_P + C_P u \quad \text{controller:} \quad \dot{x}_C = A_C x_C + C_C \hat{y} \]
\[ y = C_P x_P + D_P u \quad \quad \quad \quad \quad \quad \dot{y} = C_C x_C + D_C \hat{y} \]

What if the network is not available at a sample time \( t_k \)?

1\(^{st} \) wait until network becomes available
2\(^{nd} \) send (old) data from original sampling of continuous-time output
or
2\(^{nd} \) send (latest) data from current sampling of continuous-time output
\[ \Rightarrow \text{intersampling times} \ t_{k+1} - t_k \text{ typically become random variables} \]

Example #1: Networked Control System

\[ t_{k+1} - t_k \sim \text{time-interval between successive transmissions} \]

\[ \begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{bmatrix} y_1(t_{k^-}) \\ y_2(t_{k^-}) \end{bmatrix} \]

\[ x \mapsto J_{\text{odd}} x \]

\[ \begin{bmatrix} x_P \\ x_C \\ \hat{y} \end{bmatrix} \]
\[ \dot{x} = Ax \]

Defining \( x_k := x(t_k) \)

For a given \( P = P' > 0 \)

\[
E[x'_{k+1} P x_{k+1} \mid x_k] = x'_k E_{F(\Delta)} \left[ e^{A'\Delta} J' P J e^{A\Delta} \right] x_k
\]

expectation w.r.t. \( \Delta = t_{k+1} - t_k \) (cumulative distribution \( F(\cdot) \))
Stability of Linear Time-triggered SIS

If there exists
\[ P > 0, \quad E_{F(\Delta)} \left[ e^{A'\Delta} J' P J e^{A\Delta} \right] < P \]
then
\[ \lim_{k \to \infty} E[\|x_k\|^2] = 0 \quad \text{(exp. fast in index } k) \]

What about \( x(t) \) between jumps?

For a given \( P = P' > 0 \)
\[ E[x_{k+1}' P x_{k+1} | x_k] = x_k' E_{F(\Delta)} \left[ e^{A'\Delta} J' P J e^{A\Delta} \right] x_k \]
expectation w.r.t. \( \Delta = t_{k+1} - t_k \)
(cumulative distribution \( F \))

All stability notions require \( \lim_{k \to \infty} \|x_k\| = 0 \) exponentially fast

the nec. & suff. conditions only differ on the requirements on the tail of distribution
\[ 1 - F(s) = P(t_{k+1} - t_k > s) \]

- Mean-square **exponential stability**, i.e., \( \lim_{t \to \infty} E[\|x(t)\|^2] \text{ exp. fast} = 0 \)

- Mean-square **asymptotic stability**, i.e., \( \lim_{t \to \infty} E[\|x(t)\|^2] = 0 \)

- Mean-square **stochastic stability**, i.e., \( \int_0^\infty E[\|x(t)\|^2] dt < \infty \)
All stability notions require $\lim_{k \to \infty} \|x_k\| = 0$ exponentially fast

the nec. & suff. conditions only differ on the requirements on the tail of distribution

$1 - F(s) = P(t_{k+1} - t_k > s)$

(versions of these results for multiple discrete modes are available)

Theorem:

- **Mean-square exponential stability**, i.e., $\lim_{t \to \infty} E[\|x(t)\|^2] \exp. \text{fast} = 0$
  \[\Leftrightarrow \exists P > 0, \ E_F(\Delta) [e^{A' \Delta} J' P J e^{A \Delta}] < P \text{ and } \lim_{s \to \infty} e^{A's} e^{As} (1 - F(s)) \exp. \text{fast} = 0\]

- **Mean-square asymptotic stability**, i.e., $\lim_{t \to \infty} E[\|x(t)\|^2] = 0$
  \[\Leftrightarrow \exists P > 0, \ E_F(\Delta) [e^{A' \Delta} J' P J e^{A \Delta}] < P \text{ and } \lim_{s \to \infty} e^{A's} e^{As} (1 - F(s)) = 0\]

- **Mean-square stochastic stability**, i.e., $\int_0^\infty E[\|x(t)\|^2] dt < \infty$
  \[\Leftrightarrow \exists P > 0, \ E_F(\Delta) [e^{A' \Delta} J' P J e^{A \Delta}] < P \text{ and } \int_0^\infty e^{A's} e^{As} F(ds) < \infty\]

[Antunes et al, 2009]

**Talk outline**

**Examples**

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- Moments dynamics

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So far time-driven SHSs...

\[ q(t) \in Q = \{1, 2, \ldots \} \equiv \text{discrete state} \]
\[ x(t) \in \mathbb{R}^n \equiv \text{continuous state} \]

Special case:

When all \( \lambda \) are constant \( \Rightarrow \) time triggered SIS with exponential \( t_{k+1} - t_k \)
Stochastic Hybrid Systems with Diffusion

\[ \dot{x} = f_1(x) + g_1(x)w \]
\[ \dot{x} = f_2(x) + g_2(x)w \]
\[ \dot{x} = f_3(x) + g_3(x)w \]

\( q(t) \in Q = \{1, 2, \ldots \} \equiv \text{discrete state} \)
\( x(t) \in \mathbb{R}^n \equiv \text{continuous state} \)

Example #2: Estimation through network

\[ \dot{x} = Ax + Bw \]
white noise disturbance

encoder

\( x(t_1) \) \( x(t_2) \)

packet-switched network

decoder

\[ \dot{x} = A\dot{x} \]

state-estimator

for simplicity:
- full-state available
- no measurement noise
- no quantization
- no transmission delays

encoder logic \( \equiv \) determines \textit{when} to send measurements to the network
decoder logic \( \equiv \) determines \textit{how} to incorporate received measurements
decoder logic \(\equiv\) determines *how* to incorporate received measurements

1. upon reception of \(x(t_k)\), reset \(\hat{x}(t_k)\) to \(x(t_k)\)

encoder logic \(\equiv\) determines *when* to send measurements to the network

1. keep track of remote estimate \(\hat{x}\)
2. send measurements stochastically
3. probability of sending data increases as \(\hat{x}\) deviates from \(x\)

[related ideas pursued by Astrom, Basar, Hristu, Kumar, Tilbury]

**Error Dynamics**

Error dynamics: \(e := x - \hat{x}\)

\[
\dot{e} = Ae + B\dot{w}
\]

\(\lambda(e)\ dt\) \quad\text{prob. of sending data in } (t, t+dt) \text{ depends on current error } e\)

\(e \rightarrow 0\) \quad\text{reset error to zero}
ODE – Lie Derivative

\[ \dot{x} = f(x) \quad x \in \mathbb{R}^n \]

Given scalar-valued function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \)

\[
\frac{dV(x(t))}{dt} = \frac{\partial V(x(t))}{\partial x} f(x(t))
\]

Basis of “Lyapunov” formal arguments to establish boundedness and stability…

E.g., picking \( V(x) := \|x\|^2 \)

\[
\frac{dV(x(t))}{dt} = \frac{\partial V}{\partial x} f(x) \leq 0 \quad \Rightarrow \quad V(x(t)) = \|x(t)\|^2 \leq \|x(0)\|^2
\]

\[ \|x\|^2 \text{ remains bounded along trajectories!} \]

Generator of a Stochastic Hybrid System

Given scalar-valued function \( V : \mathcal{Q} \times \mathbb{R}^n \rightarrow \mathbb{R} \)

\[
\frac{d}{dt} E\left[ V(q(t), x(t)) \right] = E\left[ (LV)(q(t), x(t)) \right]
\]

where

\[ (LV)(q, x) := \frac{\partial V}{\partial x} (q, x) f_q(x) \quad \text{(Lie derivative)} \]

\[
+ \sum_{\ell=1}^{m} \lambda_\ell(q, x) \left( V\left( \phi_\ell(q, x) \right) - V(q, x) \right) \quad \text{(Reset term)}
\]

\[
+ \frac{1}{2} \text{trace} \left( g_q(x)^T \frac{\partial^2 V}{\partial x^2} g_q(x) \right) \quad \text{(Diffusion term)}
\]

\( x, q \) are discontinuous, but the expected value is differentiable

Dynkin’s formula

(in differential form)
Lyapunov Analysis – SHSs

\[
\begin{align*}
\dot{x} &= f(x) + g(x)w \\
& \quad \lambda(x)dt \\
\end{align*}
\]

\[
\frac{d}{dt} E[V(x(t))] = E[(LV)(x(t))]
\]

class-K functions:
(Zero at zero & mon. increasing)

\[
\begin{align*}
\alpha_1(||x||) &\leq V(x) \leq \alpha_2(||x||) \\
LV(x) &\leq -\alpha_3(||x||)
\end{align*}
\]

\[
\begin{align*}
\{V(x) \geq 0 \quad LV(x) &\leq -W(x) \} \Rightarrow \int_0^\infty E[W(x(t))]dt < \infty \\
\{V(x) \geq W(x) \geq 0 \quad LV(x) &\leq -\mu V + c \} \Rightarrow E[W(x(t))] \leq e^{-\mu t}V(x_0) + \frac{c}{\mu}
\end{align*}
\]

Almost sure (a.s.) asymptotic stability

\[
\begin{align*}
P(\exists t : \|x(t)\| \geq M) &\leq \frac{\alpha_2(||x_0||)}{\alpha_1(M)} \\
P(x(t) \to 0) &= 1
\end{align*}
\]

Stochastic stability (mean square when \(W(x) = \|x\|^2\))

Exponential stability (mean square when \(W(x) = \|x\|^2\))

Example #2: Remote estimation

\[
\begin{align*}
\dot{e} &= Ae + B\dot{w} \\
\lambda(e)dt
\end{align*}
\]

Error dynamics in remote estimation

Dynkin’s formula

\[
\frac{d}{dt} E[V(e(t))] = E[(LV)(e(t))]
\]

\[
(LV)(e) := \frac{\partial V}{\partial e} Ae + \lambda(e)\left(V(0) - V(e)\right) + \frac{1}{2} \text{trace} \left(B^T \frac{\partial^2 V}{\partial e^2} B \right)
\]

For constant rate: \(\lambda(e) = \gamma\) (exp. distributed inter-jump times) using \(V(e) = e'Pe\)

1. \(E[e] \to 0\) if and only if \(\gamma > \Re[\lambda_i(A)], \forall i\)
2. \(E[\|e\|^m]\) bounded if and only if \(\gamma > m \Re[\lambda_i(A)], \forall i\)

For radially unbounded rate: \(\lambda(e)\) (reactive transmissions) using \(V(e) = \|e\|^2\)

3. \(E[e] \to 0\) (always)
4. \(E[\|e\|^m]\) bounded \(\forall m\)

Getting more moments bounded requires higher comm. rates

Moreover, one can achieve the same \(E[\|e\|^2]\) with less communication than with a constant rate or periodic transmissions...

[Chu et al., 2006]
Can we pick an intensity $\lambda(\cdot)$ to obtain the desired distribution for the $t_k$?

YES
Converse Lyapunov Stability

Theorem:
System is mean-square exponentially stable, i.e.,
\[ \lim_{t \to \infty} \mathbb{E}[\|x(t)\|^2] \text{ exp. fast } = 0 \]

\[ \exists P(\tau) \text{ such that defining } V(x, \tau) = x'P(\tau)x \]

\[ \begin{cases} c_1 I \leq P(\tau) \leq c_2 I \\ (LV)(x, \tau) \leq -\epsilon V(x, \tau) \end{cases} \Rightarrow V \text{ is positive definite} \]

\[ \frac{d}{dt} \mathbb{E}[V(x, \tau)] \leq -\epsilon \mathbb{E}[V(x, \tau)] \]

(motivates choices for Lyapunov function for nonlinear systems)

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Gene expression ≡ process by which a gene (encoded in the DNA) produces proteins:

\[ \begin{align*}
* &\rightarrow \text{mRNA} & \text{transcription (constant rate)} \\
\text{mRNA} &\rightarrow X + \text{mRNA} & \text{translation} \\
\text{mRNA} &\rightarrow * & \text{mRNA decay} \\
X &\rightarrow * & \text{protein decay}
\end{align*} \]

\[ x(t) \equiv \text{number of proteins at time } t \]

\[
\begin{align*}
\text{prob. of one transcription event in } (t, t+dt] &\quad K dt \\
\text{prob. of one decay event in } (t, t+dt] &\quad d x dt
\end{align*}
\]

\# of proteins produced per transcription event
\[
\begin{align*}
x \mapsto x + N \\
x \mapsto x - 1
\end{align*}
\]

equivalent to Gillespie’s stochastic simulation algorithm (SSA)
Example III: (Unregulated) Gene Expression

- transcription (constant rate)
- translation
- mRNA decay
- protein decay

How to go beyond stability/bounds and study the dynamics of means, variances, co-variances, etc.?

- prob. of one transcription event in \((t, t+dt)\)
- prob. of one decay event in \((t, t+dt)\)

\[
K dt \\
\frac{dx}{dt} dt
\]

Moment Dynamics

\[
\frac{d}{dt} E[V(x)] = E[(LV)(x)] \\
(LV)(x) = K \left( V(x + N) - V(x) \right) \\
+ d \frac{x}{dt} \left( V(x - 1) - V(x) \right)
\]
Moment Dynamics

\[
\frac{d}{dt} E[V(x)] = E[(LV)(x)] \\
(LV)(x) = K \left( V(x + N) - V(x) \right) + dx \left( V(x - 1) - V(x) \right)
\]

\[
\frac{dE[x]}{dt} = KE[N] - dE[x] \\
\frac{dE[x^2]}{dt} = KE[N^2] + (2K E[N] + d)E[x] - 2dE[x^2]
\]

One can show that

\[
E[N] = N := \frac{\text{mRNA translation rate}}{\text{mRNA decay rate}}
\]

Var[N] = N^2 - N

(Unregulated) Gene Expression

Thus, at steady-state,

\[
E[x] = \frac{KN}{d}
\]

\[
CV[x] = \frac{\text{StdDev}[x]}{E[x]} = \sqrt{\frac{d}{K}} = \sqrt{\frac{N}{E[x]}}
\]

- measure of stochastic fluctuations in protein level \(x\)
  (normalized by mean population)
- intrinsic noise (solely due to random protein expression/degradation)

\[
N := \frac{\text{mRNA translation rate}}{\text{mRNA decay rate}}
\]
Auto-Regulated Gene Expression

Protein production rate is a function of the current protein molecule count through transcription regulation:

\[ \frac{dx}{dt} = g(x) \]

- Altering the RNA polymerase specificity for a given promoter or set of promoters
- Binding to non-coding sequences on the DNA to impede RNA polymerase's progress

Auto-Regulatory Negative Feedback

negative feedback = protein production rate is a decreasing function of the protein molecule count

- Common form of auto regulation (e.g., half of the repressors in E. Coli)
- Experimentally shown to exhibit noise reduction ability
Moment Dynamics

\[ \frac{d}{dt} E[V(x)] = E[(LV)(x)] \]

\( (LV)(x) = g(x) \left( V(x + N) - V(x) \right) + dx \left( V(x - 1) - V(x) \right) \)

\[ \frac{dE[x]}{dt} = E[N]E[g(x)] - dE[x] \]

\[ \frac{dE[x^2]}{dt} = E[N^2]E[g(x)] + 2E[N]E[g(x)x] + dE[x] - 2dE[x^2] \]

- When \( g(x) \) is an affine function we still get a finite system of linear equations
- When \( g(x) \) is a polynomial, we get a closed but infinite system of linear equation (general property of polynomial SHSs)
- For other \( g(x) \), one generally does not get a closed system of equations

Auto-Regulated Gene Expression

Approximate Analysis Methods
- **Distribution-based**: assume a specific type of distribution (Normal, LogNormal, Poisson, etc.) and force dynamics to be compatible with this type of distribution
- **Large numbers/large volume**: take the limit as volume \( \to \infty \) and assume concentrations do not \( \to 0 \)
- **Derivative matching**: force solutions of approximate dynamics to match exact equation locally in time
- **Linearization**: Linearize transcriptional response around steady-state value of the mean
Auto-Regulatory Negative Feedback

For a transcriptional response approximately linear around steady-state mean

\[ g(x) \approx g(x^*) + g'(x^*)(x - x^*) \]

protein's response-time (with feedback)

\[ CV[x] = \frac{\text{StdDev}[x]}{E[x]} = \sqrt{\frac{T_r}{T_p}} \cdot \frac{N}{E[x]} \]

protein's half-live (response time without feedback)

Negative feedback reduces \( T_r \) with respect to \( T_p \) \( \Rightarrow \) decreases noise

Exogenous Noise

In practice, transcription rate also depends on exogenous species (e.g., RNA polymerase and other enzymes)

\[ g(x, z) = \text{transcriptional response (stochastic rate at which transcription events occur)} \]

exogenous species (with stochastic fluctuations)
Exogenous Noise

\[ CV[x]^2 \approx \frac{T_r}{T_p} \frac{N}{E[x]} + \left( \frac{T_r}{T_p} \right)^2 CV[z]^2 \]

intrinsic noise (as before)

extrinsic noise

CV of extrinsic species

\[ T_r \] = protein’s response-time (with feedback)

\[ T_p \] = protein’s half-live (response time without feedback)

Negative feedback reduces \( T_r \) with respect to \( T_p \)
- attenuates both intrinsic and extrinsic noise
- more efficient at reducing extrinsic noise
- surprisingly good matches with experimental results…
- offers a new technique to discover sources of extrinsic noise (solve for \( CV[z] \) !)

[Singh et al, 2009; related results by Paulsson 2004]

Why Should I Care About SHS ?

1. SHS models that find use in several areas
   - network traffic modeling, networked control systems, distributed estimation, biochemistry, population dynamics in ecosystems

2. The analysis of SHSs is challenging but there are tools available
   - stability conditions for linear time-triggered SHS, Lyapunov methods, moment dynamics, linearization, truncations

3. Lots of work to be done:
   - theory
     - stability/robustness/performance of SHSs
   - networked control systems
     - protocol design to optimize performance & minimize communication resources
   - biology
     - quantitative study of common motifs (modules) in systems biology
     - study of spatial processes
   - other applications…