Networked Control System Protocols
Modeling & Analysis
using Stochastic Impulsive Systems

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 disclaimer:  This is an overview, technical details in papers referenced in bottom right corner… http://www.ece.ucsb.edu/~hespanha

Examples
• feedback over shared communication network
• estimation using remote sensor

Analysis tools
• Stochastic Hybrid Systems driven by renewal processes
• Lyapunov-based analysis of Stochastic Hybrid Systems
Deterministic Hybrid Systems

\[ \dot{x} = f_1(x) \]
\[ g_1(x) \geq 0? \]
\[ x \mapsto \phi_1(x) \]

\[ \dot{x} = f_2(x) \]
\[ g_2(x) \geq 0? \]
\[ x \mapsto \phi_2(x) \]

\[ \dot{x} = f_3(x) \]
\[ g_3(x) \geq 0? \]
\[ x \mapsto \phi_3(x) \]

\[ \dot{x} = f_4(x) \]
\[ g_4(x) \geq 0? \]
\[ x \mapsto \phi_4(x) \]

Continuous dynamics

Guard conditions

Reset-maps
Stochastic Hybrid Systems

\[ \dot{x} = f_1(x) \]

\[ \dot{x} = f_3(x) \]

\[ \dot{x} = f_2(x) \]

- \( \lambda_1(x)dt \)
- \( \lambda_2(x)dt \)
- \( \lambda_3(x)dt \)
- \( \lambda_4(x)dt \)

\( \phi_1(x) \)
\( \phi_2(x) \)
\( \phi_3(x) \)
\( \phi_4(x) \)

Continuous dynamics

Transition intensities
(probability of transition in interval \((t, t+dt]\))

Reset-maps
Stochastic Hybrid Systems

Special case: When all $\lambda_\ell$ are constant

$\Rightarrow x(t)$ is a Markov process &
times between jumps are exponentially distributed

closely related to the so called
Markovian Jump Systems
[Costa, Fragoso, Boukas, Loparo, Lee, Dullerud]
Example I: Networked Control System

\[
\begin{align*}
\text{process:} & \quad \dot{x}_P = A_P x_P + C_P u \\
& \quad y = C_P x_P + D_P u
\end{align*}
\]

\[
\begin{align*}
\text{controller:} & \quad \dot{x}_C = A_C x_C + C_C \hat{y} \\
& \quad \hat{y} = C_C x_C + D_C \hat{y}
\end{align*}
\]

round-robin network access:

\[
\begin{align*}
\dot{\hat{y}} &= 0 \\
\hat{y}(t_k) &= \begin{bmatrix} \hat{y}_1(t_k) \\ \hat{y}_2(t_k) \end{bmatrix} = \begin{cases} \\
&\begin{bmatrix} y_1(t_k^-) \\ y_2(t_k^-) \end{bmatrix} & k \text{ odd} \\
&\begin{bmatrix} \hat{y}_1(t_k^-) \\ \hat{y}_2(t_k^-) \end{bmatrix} & k \text{ even}
\end{cases}
\end{align*}
\]
Example I: Networked Control System

**process:**  
\[ \dot{x}_P = A_P x_P + C_P u \]  
\[ y = C_P x_P + D_P u \]

**controller:**  
\[ \dot{x}_C = A_C x_C + C_C \hat{y} \]  
\[ \hat{y} = C_C x_C + D_C \hat{y} \]

What if the network is not available at a sample time \( t_k \)?

1\(^{st}\) wait until network becomes available
2\(^{nd}\) send (old) data from original sampling of continuous-time output
or
2\(^{nd}\) send (latest) data from current sampling of continuous-time output

⇒ intersampling times \( t_{k+1} - t_k \) typically become random variables
Example I: Networked Control System

Suppose $t_{k+1} - t_k \sim \text{i.i.d., exponentially distributed}$

$$
\begin{bmatrix}
\hat{y}_1(t_k) \\
\hat{y}_2(t_k)
\end{bmatrix} = 
\begin{bmatrix}
y_1(t_k^-) \\
y_2(t_k^-)
\end{bmatrix}
$$

$$
\dot{x} = A x
$$

$$
x \mapsto J_{\text{odd}} x
$$

$$
x \mapsto J_{\text{even}} x
$$

$$
x :=
\begin{bmatrix}
x_P \\
x_C \\
\hat{y}
\end{bmatrix}
$$
Example I: Networked Control System

\[ \hat{y} \rightarrow \text{controller} \rightarrow u \rightarrow \text{process} \]

very unrealistic!

at best, \( t_{k+1} - t_k \sim \text{i.i.d., constant + exponential} \)

but then \( x(t) \) is not a Markov process…

Suppose \( t_{k+1} - t_k \sim \text{i.i.d., exponentially distributed} \)
Example I: Networked Control System

Suppose $t_{k+1} - t_k \sim \text{i.i.d.}$, with cumulative distribution function $F(\cdot)$

The aggregate state $(x, \tau)$ is a Markov process
Impulsive Syst. driven by Renewal Proc.

impulsive system ≡ same continuous dynamics for all modes

\[ N(t) \equiv \# \text{ of jumps before time } t \]

renewal process
(iid inter-increment times)
Impulsive Syst. driven by Renewal Proc.

impulsive system $\equiv$ same continuous dynamics for all modes

\[ N(t) = \text{# of jumps before time } t \]

renewal process (iid inter-increment times)

**Theorem:** (for simplicity, conditions stated for equal reset matrices: $J_1 = J_2 \in \mathbb{R}^{n \times n}$)

system is stochastically stable, i.e.,

\[ \int_0^\infty E[\| x(t) \|^2] dt < \infty \]

\[ \uparrow \]

\[ \exists P > 0 : E_F(T) \left[ e^{A'T} J' P J e^{A'T} \right] - P < 0 \]

LMI on $P_{n \times n}$

and

spectral radius condition on $n^2 \times n^2$ matrix

\[ \sigma \left( E_F(T) \left[ e^{A'T} J' \otimes e^{A'T} J' \right] \right) < 1 \]

Kronecker product

[ACC’09]
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Example II: Estimation through network

\[
\dot{x} = Ax + B\bar{w}
\]

encoder

white noise disturbance

\[x(t_1) \quad x(t_2)\]

encoder logic \(\equiv\) determines when to send measurements to the network

decoder logic \(\equiv\) determines how to incorporate received measurements

for simplicity:
- full-state available
- no measurement noise
- no quantization
- no transmission delays

state-estimator

\[\hat{x} = A\hat{x}\]

decoder
encoder logic \equiv \text{determines } \textit{when} \text{ to send measurements to the network}

1. keep track of remote estimate $\hat{x}$
2. send measurements stochastically
3. probability of sending data increases as $\hat{x}$ deviates from $x$

decoder logic \equiv \text{determines } \textit{how} \text{ to incorporate received measurements}

4. upon reception of $x(t_k)$, reset $\hat{x}(t_k)$ to $x(t_k)$

[similar ideas pursued by Astrom, Tilbury, Hristu, Kumar, Basar]
Example II: Remote estimation

$$\dot{x} = Ax + B\dot{w}$$

encoder

$$x(t_1) \quad x(t_2)$$

packet-switched network

decoder

$$\hat{x} = A\hat{x}$$

state-estimator

Error dynamics: $$e := x - \hat{x}$$

$$\lambda(e) \ dt$$

prob. of sending data in $$[t, t+dt)$$

depends on current error $$e$$

$$\dot{e} = Ae + B\dot{w}$$

$$e \rightarrow 0$$

reset error to zero

for simplicity:
- full-state available
- no measurement noise
- no quantization
- no transmission delays

[CDC'04, CRC Press'06]
Given scalar-valued function $\psi : \mathbb{R}^n \times [0,\infty) \rightarrow \mathbb{R}$

$$\frac{d}{dt} \psi(x(t), t) = \frac{\partial \psi}{\partial x} f(x) + \frac{\partial \psi}{\partial t}$$

derivative along solution to ODE

$L_f \psi$

Lie derivative of $\psi$

Basis of “Lyapunov” formal arguments to establish boundedness and stability…

E.g., picking $\psi(x, t) := \|x\|^2$

$$\frac{d}{dt} \psi(x(t), t) = \frac{\partial \psi}{\partial x} f(x) + \frac{\partial \psi}{\partial t} \leq 0 \Rightarrow \psi(x(t), t) = \|x(t)\|^2 \leq \|x(0)\|^2$$

$\|x(t)\|$ remains bounded along trajectories!
Generator of a SHS

Given scalar-valued function \( \psi : Q \times \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R} \)

\[
\frac{d}{dt} E[\psi(q, x, t)] = E \left[ (L\psi)(q, x, t) \right]
\]

where

\[
(L\psi)(e, t) = \frac{\partial \psi}{\partial e} Ae + \frac{\partial \psi}{\partial t} + \left[ \psi(0, t) - \psi(e, t) \right] \lambda(e) + \frac{1}{2} \text{trace} \left( B' \frac{\partial^2 \psi}{\partial e^2} B \right)
\]

- Lie derivative
- instantaneous variation
- reset term
- intensity
- diffusion term

\( x \) & \( q \) are discontinuous, but the expected value is differentiable!!!

Prob. of sending data in \([t, t+dt)\) depends on current error \( e \)

Reset error to zero

Disclaimer: see Nonlinear Analysis’05 for technical assumptions
Generator of a SHS

Given scalar-valued function $\lambda(e) \, dt$,

where

$\lambda(e) \, dt$ depends on current error $e$

$\dot{e} = Ae + B\dot{q}$

$x & q$ are discontinuous, but the expected value is differentiable!!!

$x$ & $q$ are discontinuous, but the expected value is differentiable!!!

reset error to zero

generalizes to large classes of Stochastic Hybrid Systems

$e \rightarrow 0$

$\lambda(e)$ intensity

$\psi(0, t) - \psi(e, t)$ intensity

$\frac{1}{2} \text{trace } \left( B' \frac{\partial^2 \psi}{\partial e^2} B \right)$ diffusion term

Disclaimer: see Nonlinear Analysis'05 for technical assumptions
Lyapunov-based stability analysis

error dynamics in remote estimation

For constant rate: $\lambda(e) = \gamma$ (exp. distributed inter-jump times)

1. $E[e] \rightarrow 0$ if and only if $\gamma > \Re[\lambda(A)]$
2. $E[\|e\|^m]$ bounded if and only if $\gamma > m \Re[\lambda(A)]$

getting more moments bounded requires higher comm. rates
Lyapunov-based stability analysis

For constant rate: $\lambda(e) = \gamma$ (exp. distributed inter-jump times)

1. $E[e] \to 0$ if and only if $\gamma > \Re[\lambda(A)]$
2. $E[\|e\|^m]$ bounded if and only if $\gamma > m \Re[\lambda(A)]$

For polynomial rates: $\lambda(e) = (e' Q e)^k$  $Q > 0$, $k > 0$ (reactive transmissions)

1. $E[e] \to 0$ (always)
2. $E[\|e\|^m]$ bounded $\forall m$

Moreover, one can achieve the same $E[\|e\|^2]$ with less communication than with a constant rate or periodic transmissions...

[CDC'04, Birkhauser'06]
Conclusions

1. A simple SHS model that finds use in several areas (networked control systems, network traffic modeling, biochemistry)

2. The analysis of SHSs is challenging but there are tools available (generator, Lyapunov methods, moment dynamics, truncations)

3. Lots of work to be done:
   - theory
     ✓ stability/robustness/performance of SHS
   - networked control systems
     ✓ protocol design to optimize performance & minimize communication resources