Game Theory Lecture #12

Outline:

• Auctions
• Mechanism Design
• Vickrey-Clarke-Groves Mechanism
Optimizing Social Welfare

• **Goal:** Entice players to select outcome which optimizes social welfare

• **Examples:**
  – Traffic Networks
  – Auctions
  – Cost Sharing
  – Matching

• **Technical Challenges:**
  – Social planner has minimal means with respect to enticing players
  – Players have private information not available to social planner

• **Approach:** Augment players’ utility functions so that NE corresponds to optimal joint action profile

• **Problems:** Nash equilibrium reasonable prediction of behavior in one-shot setting?

• **Revised Approach:** Augment players’ utility functions so that each player’s *dominant strategy* results in the optimal joint action profile

• **Features:**
  – A player should never play a dominated strategy
  – Rare that game has a dominant strategy
  – Shaping dominant strategy more challenging than shaping NE.
Example: First price sealed bid auction

• Setup:
  – Players have internal valuations of item: \( v_1 > v_2 > ... > v_n \)
  – Players make bids: \( b_1, b_2, ..., b_n \)
  – Highest bidder wins and pays highest bid

• Player \( i \) payoff: Let \( \bar{b} = \max \{b_{-i}\} \)
  – If \( b_i > \bar{b} \): \( v_i - b_i \)
  – If \( b_i < \bar{b} \): 0

Assume for convenience that ties never happen.

• Claim: There is no dominant strategy in first price sealed bid auctions

• Cases:
  – \( b_i > v_i \): This strategy is always dominated by setting \( b_i = v_i \)
  – \( b_i = v_i \): This strategy is dominated by setting \( b_i = v_i/2 \)
  – \( b_i < v_i \): Is there another bid \( b_i' \) which dominates \( b_i \)?
    * \( b_i' > b_i \): The bid \( b_i \) performs better when \( \bar{b} < b_i \).
    * \( b_i' < b_i \): The bid \( b_i \) performs better when \( b_i' < \bar{b} < b_i \).

• Conclusion: The strategy \( b_i \) is not dominated by any other strategy \( b_i' \).
Example: Second price sealed bid auction

• Setup:
  – Players have internal valuations of item: $v_1 > v_2 > \ldots > v_n$
  – Players make bids: $b_1, b_2, \ldots, b_n$
  – Highest bidder wins and pays second highest bid

• Player $i$ payoff: Let $\bar{b} = \max \{b_{-i}\}$
  – If $b_i > \bar{b}$: $v_i - \bar{b}$
  – If $b_i < \bar{b}$: 0

  Assume for convenience that ties never happen.

• Claim: $b_i = v_i$ is a dominant strategy for player $i$

• Consequence:
  – All bidders bid their true value…
  – The bidder with the highest value is sure to win…
  – The auction allocates the prize efficiently

• Known as the Vickrey Auction

• Conclusion still hold for English Auctions where bids are continually updated.
Efficient Mechanisms

- Definition: An efficient mechanism is a game which induces the players to truthfully reveal their values and which results in at the utilitarian social choice.

- The Vickrey Auction is an efficient mechanism under certain circumstances:
  - No externalities.
  - “Private” values

- Example: Externalities
  - Three bidders \(\{x, y, z\}\).
  - Three possible allocations \(\{X, Y, Z\}\) where \(X\) indicates object given to player \(x\)
  - Player specific valuations of allocations:

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- Bidder \(y\) has a negative externality when \(z\) gets the object. This is a negative externality.
- Consequence: Bidder \(y\) does not have a dominant strategy.

- Consequence: Vickrey auction is not an efficient mechanism under externalities.
Overcoming Externalities

• Example revisited:
  – Three bidders \( \{x, y, z\} \).
  – Three possible allocations \( \{X, Y, Z\} \) where \( X \) indicates object given to player \( x \)
  – Player specific valuations of allocations:
    \[
    \begin{array}{ccc}
    & X & Y & Z \\
    x & v_x & 0 & 0 \\
    y & 0 & v_y & -5 \\
    z & 0 & 0 & v_z \\
    \end{array}
    \]

• The following modified auction in an efficient mechanism.
  – Subtract 5 for \( z \)'s bid. Set \( \hat{b}_z = b_z - 5 \).
  – Award the object to the highest bidder when using \( \hat{b}_z \) for the bid of \( z \)
  – If \( x \) or \( y \) wins, they pay the highest losing bid using \( \hat{b}_z \)
  – If \( z \) wins, she pays the highest losing bid plus 5

• Problems:
  – What if system designer does not know the level of externality?
  – Does approach extend to other problems?

• Question: Is it possible to construct an efficient mechanism that works for a broad class of problems?
General Framework and Definitions

- General framework: Social choice
  - A set of $n$-individuals $N = \{1, \ldots, n\}$.
  - A set $X$ of alternatives from which to choose.
  - $v_i(x)$ is the value to $i$ from alternative $x \in X$ being chosen.
  - Monetary transfer scheme $t = (t_1, \ldots, t_n)$.
- Definition: Utilitarian alternative
  $$x^* \in \arg\max_{x \in X} \sum_{i \in N} v_i(x).$$
- Definition: Marginal contribution player $i$
  $$\sum_{j \neq i} v_j(x^*) - \sum_{j \neq i} v_j(x_{-i}^*).$$
  where
  $$x_{-i}^* \in \arg\max_{x \in X} \sum_{j \neq i} v_j(x).$$
  Note that $x^*$ and $x_{-i}^*$ may very well be different.
- Flavor of forthcoming mechanism:
  - Players report valuation functions $\hat{v}_i$ simultaneously. Note these may be different than $v_i$.
  - Use report valuation functions $\hat{v}_i$ to determine alternative.
  - Use marginal contributions to determine prices.
The Vickrey-Clarke-Groves Mechanism

- The players: \( N = \{1, \ldots, n\} \).
- The actions: Each player will report a valuation function \( \hat{v}_i \)
  - Announcements of valuation functions are simultaneous.
  - Note that \( v_i \) is player \( i \)'s true valuation function
  - Players need not inform truthfully.
- Selection of alternative: The utilitarian alternative is chosen relative to the submitted valuations \( \hat{v} = (\hat{v}_1, \ldots, \hat{v}_n) \), i.e.,
  \[
  x^*(\hat{v}) \in \arg \max_{x \in X} \sum_{i \in N} \hat{v}_i(x).
  \]
  Note that the selected alternative is not dependent on the true valuations \( v_i \).
- Monetary transfers: Price are determined by evaluating marginal contributions according to reported valuations
  \[
  t_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(x^*(\hat{v})) - \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_{-i})).
  \]
- Utility functions:
  \[
  U_i(\hat{v}_i, \hat{v}_{-i}) = v_i(x(\hat{v})) + t_i(\hat{v}).
  \]
- **Theorem:** The VCG mechanism is efficient.
  - All individuals have a dominant strategy to announce their true valuations.
  - When they do so, the utilitarian alternative is enacted by the VCG mechanism.
The Vickrey Auction Revisited

- The players: \( N = \{1, \ldots, n\} \).
- Set of alternatives: \( X = \{1, \ldots, n\} \) where \( x = \{i\} \) means object awarded to agent \( i \).
- The actions: Each player will report a value \( \hat{v}_i \) for each outcome \( x \in X \). Here, \( \hat{v}_i(x) = 0 \) for all \( x \neq i \).
- Selection of alternative: The object goes to the highest bidder, i.e.,
  \[
  x^*(\hat{v}) = \arg \max_{x \in X} \sum_{i \in \mathcal{P}} \hat{v}_i(x),
  = \arg \max_{i \in N} \hat{v}_i(i)
  \]
- Monetary transfers: Price are determined by evaluating marginal contributions according to reported valuations. For player \( i = \arg \max_{i \in \mathcal{P}} \hat{v}_i(i) \), we have
  \[
  t_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(x^*(\hat{v})) - \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_{-i}))
  = 0 - \max_{j \neq i} \hat{v}_j(i)
  \]
  For player \( j \neq \arg \max_{i \in N} \hat{v}_i(i) \), we have
  \[
  t_j(\hat{v}) = \sum_{k \neq j} \hat{v}_k(x^*(\hat{v})) - \sum_{k \neq j} \hat{v}_k(x^*(\hat{v}_{-j}))
  = \max_i \hat{v}_i(i) - \max_i \hat{v}_i(i)
  = 0.
  \]
- Utility functions:
  \[
  U_i(\hat{v}_i, \hat{v}_{-i}) = v_i(x(\hat{v})) + t_i(\hat{v}).
  \]
- Fact: Vickrey auction is special class of VCG mechanism.
Proof

• Want to show that announcing truthfully is dominant strategy

• If the others announce \( \hat{v}_{-i} \) and \( i \) announces \( \hat{v}_i \), \( i \)'s utility is

\[
v_i(x^*(\hat{v}_i, \hat{v}_{-i})) + t_i(\hat{v}_i, \hat{v}_{-i})
\]

and by substituting the VCG transfer formula for \( t_i \)

\[
v_i(x^*(\hat{v}_i, \hat{v}_{-i})) + \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_i, \hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_{-i}))
\]

• Hence, player \( i \)'s best response to \( \hat{v}_{-i} \) is

\[
\arg \max_{\hat{v}_i} v_i(x^*(\hat{v}_i, \hat{v}_{-i})) + \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_i, \hat{v}_{-i}))
\]

• For the moment, suppose \( i \) could choose the alternative \( x \) directly. If this was the case, he would choose the \( x \) that

\[
\arg \max_{x \in X} v_i(x) + \sum_{j \neq i} \hat{v}_j(x)
\]

which is precisely \( x = x^*(v_i, \hat{v}_{-i}) \).

• But \( i \) cannot choose \( x \) directly. Rather he choose \( \hat{v}_i \) and then \( x^*(\hat{v}_i, \hat{v}_{-i}) \) will be chosen.

• By announcing truthfully, i.e., \( \hat{v}_i = v_i \), he ensures that \( x^*(v_i, \hat{v}_i) \) will be chosen.

• Hence, announcing truthfully is a dominant strategy.