2–Port Parameters

Two-ways of describing device:

A. Equivalent - Circuit-Model

- Physically based
- Includes bias dependence
- Includes frequency dependence
- Includes size dependence - scalability
- Ideal for IC design
- Weakness: Model necessarily simplified; some errors. Thus, weak for highly resonant designs

B. 2–Port Model

- Matrix of tabular data vs. frequency
- Need one matrix for each bias point and device size
- Clumsy – huge data sets required
- Traditional microwave method
- Exact

2 Port descriptions

These are black box (mathematical) descriptions.

Inside might be a transistor, a FET, a transmission line, or just about anything.

The terminal characteristics are $V_1$, $V_2$, $I_1$ & $I_2$ – there are 2 degrees of freedom.
**Admittance Parameters**

\[
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} =
\begin{bmatrix}
Y_{11} & Y_{12} \\
Y_{21} & Y_{22}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
\]

Example: Simple FET Model

By inspection:

\[
Y = \begin{bmatrix}
 j\omega C_{gs} + j\omega C_{gd} & -j\omega C_{gd} \\
 g_m - j\omega C_{gd} & G_{ds} + j\omega C_{gd}
\end{bmatrix}
\]

Easy!

\[
Y_{11} = \left. \frac{I_1}{V_1} \right|_{V_2=0} \quad Y_{12} = \left. \frac{I_1}{V_2} \right|_{V_1=0}
\]
Impedance Parameters

\[
\begin{bmatrix}
V_1 \\
V_2 \\
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22} \\
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
\end{bmatrix}
\]

Example

\[ Z = \begin{bmatrix}
R_1 + R_3 & R_3 \\
R_3 & R_2 + R_3 \\
\end{bmatrix} \]

By inspection

\[ Z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2 = 0} \quad Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2 = 0} \]

But, \( y, z, \) and \( h \) parameters are not suitable for high frequency measurement.

Problem: How can you get a true open or short at the circuit terminals? Any real short is inductive. Any real open is capacitive.

To make matters worse, if you are trying to measure a high freq. active device, a short or open can make it oscillate!

Solution: Use termination in \( Z_0 \) instead!

Broadband.
Not very sensitive to parasitic \( L, C \)
Kills reflections.

Redefine parameters to use fwd. and rev. voltage waves.

Measurement can use directional couplers.
S–Parameters

Note that $Z_0$ must be defined. We don’t really need transmission lines.

Our objective now is to de-mystify S-parameters – they are easy!

Recall

$$V(x) = V^+(x) + V^-(x)$$

phasor quantities.

$$I(x) = \frac{V^+(x)}{Z_0} - \frac{V^-(x)}{Z_0}$$

amplitude, not rms values.

We can normalize the amplitude of waves to $Z_0$:

$$a(x) = \frac{V^+(x)}{\sqrt{Z_0}}$$

forward wave

$$b(x) = \frac{V^-(x)}{\sqrt{Z_0}}$$

reverse wave

Why? So that $\frac{1}{2} a(x)a^*(x) = $ power in forward wave.

if $a = 1.414$ then power in wave is 1 watt. (or $a_{rms} = 1$)
likewise, \( b(x)b^*(x)/2 \) is the power in the reverse wave

So, in terms of total voltage \( V(x) \) and current \( I(x) \),

\[
v(x) = \frac{V(x)}{\sqrt{Z_0}} = a(x) + b(x)
\]

\[
i(x) = \sqrt{Z_0} I(x) = a(x) - b(x)
\]

or,

\[
a(x) = \frac{1}{2} [v(x) + i(x)] = \frac{1}{2\sqrt{Z_0}} [V(x) + Z_0I(x)]
\]

\[
b(x) = \frac{1}{2} [v(x) - i(x)] = \frac{1}{2\sqrt{Z_0}} [V(x) - Z_0I(x)]
\]

**Reflection**

So, how is \( \Gamma \) defined in terms of the S parameters? At port 1,

\[
\Gamma_1 = \frac{b_1}{a_1}
\]

But,

\[
b_1 = S_{11}a_1 + S_{12}a_2
\]

We need to eliminate \( a_2 \). How?

If \( Z_L = Z_0 \), \( \Gamma_L = 0 = \frac{a_2}{b_2} \) so, therefore \( a_2 = 0 \) if port 2 is terminated in \( Z_0 \).

\[
\Gamma_1 = \frac{b_1}{a_1}\bigg|_{a_2=0} = S_{11}
\]

Same with at port 2 with \( S_{22} \):

\[
S_{22} = \frac{b_2}{a_2}\bigg|_{a_1=0} = \Gamma_2
\]
Transmission

\[ b_2 = S_{21}a_1 + S_{22}a_2 \]

So, the forward transmission \( S_{21} \) can be found by setting \( a_2 = 0 \) (terminate output)

\[ S_{21} = \left. \frac{b_2}{a_1} \right|_{a_2=0} \]

Reverse transmission, similarly, is found by setting \( a_1 = 0 \) (terminate input in \( Z_0 \))

\[ b_1 = S_{11}a_1 + S_{21}a_2 \]

\[ S_{12} = \left. \frac{b_1}{a_2} \right|_{a_1=0} \]
Some comments on power measurement:

Power can vary over a large range, therefore it is often specified on a logarithmic scale. There must be a point of reference on the scale; the power measurements are usually with reference to 1 mW.

The unit is called $\text{dBm}$ meaning dB relative to 1 mW of power. Thus,

- $0 \text{ dBm} = 1 \text{ mW}$
- $10 \text{ dBm} = 10 \text{ mW}$
- $-10 \text{ dBm} = 0.1 \text{ mW}$
- etc.

To convert mW to dBm:

$$\text{dBm} = 10 \log_{10} (P)$$

To convert dBm to mW:

$$P = 10^{\frac{\text{dBm}}{10}}$$

What is the difference between dB and dBm?

dB is a power ratio – used to describe a gain or loss for example.

$$G = 10 \log_{10} \left( \frac{P_{\text{out}}}{P_{\text{in}}} \right) \quad \text{dB}$$
$$\text{Return Loss} = -20 \log_{10} |\Gamma| \quad \text{dB}$$

But, dB says nothing about the absolute power level. Don’t confuse their usage!
Now, define available power:

\[ P_{AVS} = \text{max power output from a source with impedance } Z_s \text{ that can be absorbed into a load.} \]

let \( Z_S = Z_0, \) \( Z_L = Z_S^* = Z_0 \) (in this case)

because maximum power transfer occurs when we have a conjugate match

\[ P_{load} = P_{AVS} = \frac{1}{8} \frac{V_{gen}^2}{Z_0} \]

Or, in terms of \( a \) and \( b \):

\[ a_1 = \frac{V^+}{\sqrt{Z_0}} \text{ and } b_1 = 0; \quad V^+ = V_{gen} \left( \frac{Z_0}{Z_0 + Z_0} \right) = \frac{V_{gen}}{2} \text{ and } V^- = 0 \]

So,

\[ P_{load} = P_{AVS} = \frac{1}{2} a_1 a_1^* = \frac{V_{gen}^2}{8Z_0} \]
We see that the available power is independent of load impedance. Even if the load is not matched, available power remains constant. Actual power in the load is reduced however.

Generator output power is calibrated and displayed as available power.

Actual Load Power

\[ P_{\text{Load}} = \frac{1}{2} |a_1|^2 - \frac{1}{2} |b_1|^2 = \frac{1}{2} \text{Re} \left( I_1 V_1^* \right) \]

or

\[ P_{\text{Load}} = P_{\text{AVS}} \left( 1 - |S_{11}|^2 \right) \]

Reflected Power

\[ b_1 = a_1 S_{11} \]

\[ P_R = \frac{1}{2} |b_1|^2 = \frac{1}{2} |a_1|^2 |S_{11}|^2 = P_{\text{AVS}} |S_{11}|^2 \]

\[ |S_{11}|^2 = \frac{\text{Power reflected from input}}{\text{Power incident on input}} = \frac{|b_1|^2}{|a_1|^2} \]

\[ |S_{22}|^2 = \frac{\text{Power reflected from network output}}{\text{Power incident on output}} = \frac{|b_2|^2}{|a_2|^2} \]

Similarly,

\[ \frac{1}{2} |a_2|^2 = \text{Power incident on output} \]

\[ \text{= Reflected power from load} \]

\[ \frac{1}{2} |b_2|^2 = \text{Power reflected from input port} \]

\[ \frac{1}{2} |b_2|^2 = \text{Power incident on load from the network} \]
Also, by definition, transducer gain $= \frac{P_{\text{load}}}{P_{\text{avs}}} = G_T$ even if

1. load isn’t matched to network and
2. input of network not matched to generator

Here, $P_{\text{Load}} = |b_2|^2 (1 - |\Gamma_L|^2)$

$S_{21}$ is defined in terms of transducer gain for the special case of where $Z_L = Z_0$:

$$|S_{21}|^2 = \left| \frac{|b_2|^2}{|a_1|^2} \right|_{a_2=0}$$

$$\frac{1}{2} |b_2|^2 = \text{power incident on load (and is absorbed since } \Gamma_L=0)$$

$$\frac{1}{2} |a_1|^2 = \text{source available power}$$

$$|S_{21}|^2 = \text{transducer gain with source and load } Z_0$$

Similarly,

$$|S_{12}|^2 = \text{reverse transducer power gain}$$
Reference Planes

Microwave transistor in package

On board:

Define \( x = 0 \) at both ports

Defining the reference planes differently changes the S-parameters.
The reflection parameters are shifted in phase by twice the electrical length because the incident wave travels twice over this length upon reflection. The transmission parameters have the sum of the electrical lengths, since the transmitted wave must pass through both lengths.
Comment on electrical length:

The microwave literature will say a line is 43° long at 5 GHz. What does this mean?

Electrical length = \( E = \frac{\ell}{\lambda_{ref}} \cdot 360° \)

Recall \( f \cdot \lambda = v \) so \( f_{ref} \cdot \lambda_{ref} = v \)

\[ \rightarrow E = \frac{\ell}{v/f_{ref}} \cdot 360° = \frac{\ell}{v} \cdot f_{ref} \cdot 360° \]

\[ E = T \cdot f_{ref} \cdot 360° \]

a line which is 1 ns long has an electrical length \( E = 360° \) at \( f_{ref} = 1 \text{ GHz} \)

and

an electrical length \( E = 36° \) at \( f_{ref} = 100 \text{ MHz} \)

Why not just say \( T = 1 \text{ ns} \)?

...you should be conversant with both terminologies.

Converting to physical length

\[ f \cdot \lambda_{ref} = v_p \]

\[ \lambda_{ref} = \frac{v_p}{f} \]

thus: physical length = \( \frac{E(\text{deg}) \lambda_{ref}}{360} \) = Electrical length (in wavelengths) \( \lambda_{ref} \)

or:
How to Calculate S-Parameters Quickly

First Comment

\[ S_{11} = \left. \frac{b_1}{a_1} \right|_{a_2=0} \]

\[ b_1 = S_{11}a_1 + S_{12}a_2 \]

(We must kill \( a_2 \) in order to measure or calculate \( S_{11} \))

\[
\begin{array}{c}
\text{S} \\
\rightarrow b_2 \\
\leftarrow a_2 \\
\end{array}
\]

\( \Gamma_L \)

\( Z_L \)

if \( Z_L = Z_0 \), then \( \Gamma_L \) is zero

and so \( a_2 = \Gamma_L b_2 = 0 \).

So

\[
S_{11} = \left. \frac{b_1}{a_1} \right|_{Z_L=Z_0}
\]

So if we say that \( Z_{in} \big|_{Z_L=Z_0} \) is the input impedance with \( Z_0 = Z_L \)

then

\[
S_{11} = \frac{Z_{in} \big|_{Z_L=Z_0} - Z_0}{Z_{in} \big|_{Z_L=Z_0} + Z_0} = \Gamma_{in}
\]

or:

\[
Z_{in} \big|_{Z_L=Z_0} = \frac{1 + S_{11}}{1 - S_{11}}
\]

The same comment clearly applies for \( S_{22} \). The Smith Chart is often used to plot \( S_{11}, S_{22} \).
Example:

Given $Z_0 = 50\Omega$, what is $S_{11}$?

1. **Input Impedance Calculation**

   
   
   $Z_{in}|_{Z_L=Z_0} = 54\Omega$
   
   
   
   $S_{11} = \frac{54 - 50}{54 + 50} = \frac{4}{104}$
   
   Similar arguments give $S_{22} = \frac{4}{104}$.

2. **Finding $S_{21}$**

   
   $S_{21} = \frac{b_2}{a_1}|_{a_2=0}$

   
   
   
   $Z_S = Z_0$
   
   
   $V_{gen}$
   
   $Z_L = Z_0$
What is $a_1$ in this case?

We know that:

$$a_1 = \frac{V_1^+}{\sqrt{Z_0}} \quad \text{and} \quad V_1^+ = \frac{V_{\text{gen}}}{2}$$

So,

$$a_1 = \frac{V_{\text{gen}}}{2\sqrt{Z_0}}$$

Consider the load:

$$b_2 = \frac{V_{\text{out}}}{\sqrt{Z_0}} \quad \text{Why?}$$

But, $\Gamma_L = 0$ because $Z_L = Z_0$, so $a_2 = 0$.

$$V_{\text{out}} = V^+ + V^- = \sqrt{Z_0} a_2 + \sqrt{Z_0} b_2 = \sqrt{Z_0} b_2$$

Now, calculate $V_{\text{out}}/V_{\text{gen}}$:

$$V_{\text{out}} = \sqrt{Z_0} b_2 = \sqrt{Z_0} (S_{21}a_1 + S_{22}a_2)$$

But, $a_2 = 0$ because the load impedance = $Z_0$, so

$$V_{\text{out}} = \sqrt{Z_0} S_{21} a_1$$

Substitute for $a_1$:

$$a_1 = \frac{V_{\text{gen}}}{2\sqrt{Z_0}}$$

so,

$$\frac{V_{\text{out}}}{V_{\text{gen}}} = \frac{\sqrt{Z_0} S_{21}}{2\sqrt{Z_0}} = \frac{S_{21}}{2}$$
thus, \( S_{21} = \frac{2V_{\text{out}}}{V_{\text{gen}}} \) when \( Z_L = Z_S = Z_0 \)

Why the factor of 2?

We see that the generator voltage is split between the source and load in the matched case. Here, we see that \( \frac{V_{\text{out}}}{V_{\text{gen}}} = \frac{1}{2} \), but the transducer gain must be equal to 1. \( \frac{P_{\text{LOAD}}}{P_{\text{AVS}}} \) is the transducer gain in this situation. If we insert an amplifier into the network, the signal has been increased by an amount \( S_{21} \).
So, $|S_{21}|^2$ is the FORWARD INSERTION GAIN or FORWARD TRANSDUCER GAIN in a system of impedance $Z_0$.

**EXAMPLE:** Find $S_{21}$

\[ \frac{V_{out}}{V_{gen}} = \frac{50}{104} = 0.48 \]

\[ S_{21} = 0.96 \]

OR, we could let $V_{gen} = 2$. Then, $S_{21} = V_{out}$.

**What about a reference plane extension?**

\[ S_{11}' = S_{11} e^{j\theta_1} \quad S_{11} = \Gamma_{IN}(0) \]

\[ S_{22}' = S_{22} e^{j\theta_2} \quad S_{22} = \Gamma_{OUT}(0) \]

and

\[ \theta_1 = -\beta \ell_1 = -\frac{2\pi}{\lambda} \ell_1 \quad \theta_2 = -\beta \ell_2 = -\frac{2\pi}{\lambda} \ell_2 \]

\[ S'_{21} = S_{21} e^{j(\theta_1 + \theta_2)} = S_{21} e^{-2\pi j(\ell_1 + \ell_2) / \lambda} \]
EXAMPLE: Find the 4 S parameters of the following circuit:

\[ Z_{IN|Z_L=Z_0} = \frac{1}{sC + 1/Z_0} \]

\[ S_{11} = \frac{Z_{IN} - Z_0}{Z_{IN} + Z_0} = \frac{Z_{IN}/Z_0 - 1}{Z_{IN}/Z_0 + 1} \]

turning the crank,

\[ S_{11} = \frac{-j\omega CZ_0/2}{1 + j\omega CZ_0/2} \]

\( S_{22} \) will be the same due to symmetry. Note that we calculated \( Z_{IN} \) with port 2 terminated in \( Z_0 \). This is part of the definition of \( S_{11} \) so is essential.
Now find $S_{21}$: first use Thevenin – Norton transformation:

\[ V_{\text{out}} = \frac{V_{\text{gen}}}{Z_0} \frac{1}{\frac{2}{Z_0} + sC} = I/Y \]

\[ S_{21} = \frac{2V_{\text{out}}}{V_{\text{gen}}} = \frac{1}{1 + j\omega C Z_0 / 2} = S_{12} \]