2D Fourier Transform

2-D DFT & Properties

Fourier Transform - review

1-D:

$$F(u) \equiv \Im\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

$$f(x) \equiv \Im^{-1}\{F(u)\} = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

2-D:

$$F(u,v) = \iint f(x,y) e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$f(x,y) = \iint F(u,v) e^{j2\pi(ux+vy)}\, du\, dv$$

2D FT: Properties

Linearity: a f(x,y) + b g(x,y) ⟺ a F(u,v) + b G(u,v)

Convolution: f(x,y) ☆ g(x,y) = F(u,v) G(u,v)

Multiplication: f(x,y) g(x,y) = F(u,v) ☆G(u,v)

Separable functions: Suppose f(x,y) = g(x) h(y), Then
F(u,v)=G(u)H(v)

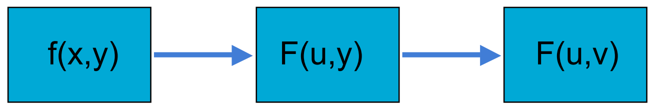
Shifting: $f(x \pm x_0, y \pm y_0)$ ⟷ $\exp[2\pi j\,(\pm x_0 u \pm y_0 v)]\, F(u,v)$

Separability of the FT

$$F(u,v) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\, ux} dx \right] e^{-j2\pi\, vy} dy$$

$$= \int_{-\infty}^{\infty} F(u,y)\, e^{-j2\pi\, vy} dy$$

Separability (contd.)



Fourier Transform along X.

Fourier Transform along Y.

We can implement the 2D Fourier transform as a sequence of 1-D Fourier transform operations.

Eigenfunctions of LSI Systems

A function f(x,y) is an Eigenfunction of a system T if
T[f(x,y)] = α f(x,y) for some constant (Possibly complex) α.

For LSI systems, complex exponentials of the form exp{ j2π (ux+vy) }, for any (u,v), are the Eigenfunctions.

Impulse Response and Eigenfunctions

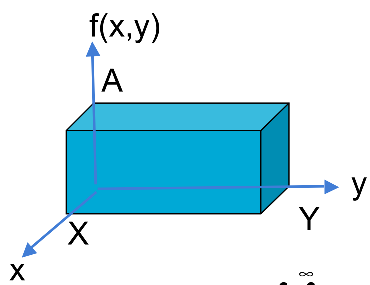
Consider a LSI system with impulse response h(x,y).
Its output to the complex exponential is

$$g(x,y) = \int\int_{-\infty}^{\infty} h(x-s, y-t) e^{j2\pi(us+vt)} ds\, dt$$

$$= \iint h(\bar{x},\bar{y}) e^{j2\pi(ux+vy)} e^{-j2\pi(u\bar{x}+v\bar{y})} d\bar{x}\, d\bar{y}$$

$$= H(u,v) e^{j2\pi(ux+vy)}$$

2-D FT: Example

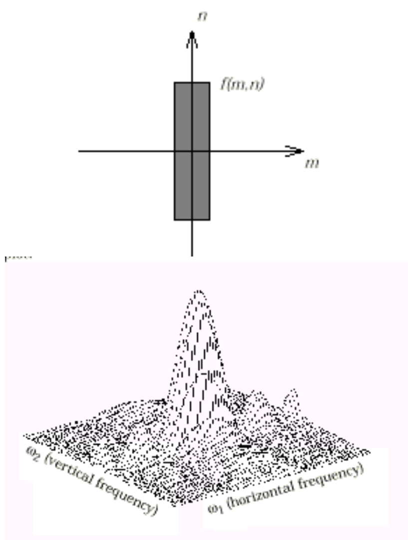


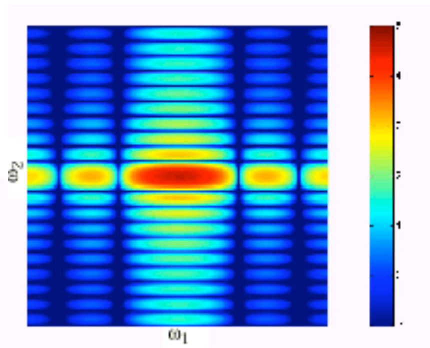
$$F(u,v) = \int\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\, dy$$

$$= A\int_0^X e^{-j2\pi ux} dx \int_0^Y e^{-j2\pi vy} dy$$

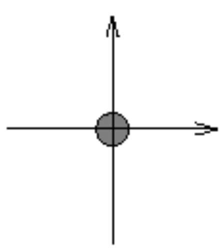
$$= AXY\left[\frac{\sin \pi uX}{\pi uX}\right]\left[\frac{\sin \pi vY}{\pi vY}\right] e^{-j\pi(uX+vY)}$$

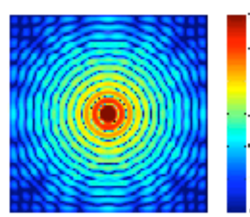
Example (contd.)





Example2





Discrete Fourier Transform

Consider a sequence {u(n), n=0,1,2,....., N-1}. The DFT of u(n) is

$$v(k) = \sum_{n=0}^{N-1} u(n) W_N^{kn} , \qquad k = 0,1,\ldots\ldots, N-1$$

Where $W_N = e^{-j\frac{2\pi}{N}}$, and the inverse is given by

$$u(n) = \frac{1}{N}\sum_{k=0}^{N-1} v(k) W_N^{-kn}, \quad n = 0,1,\ldots, N-1$$

2-D DFT

Often it is convenient to consider a symmetric transform:

$$v(k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} u(n) W_N^{kn} \quad \text{and}$$

$$u(n) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} v(k) W_N^{-kn}$$

In 2-D: consider a NXN image

$$v(k,l) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} u(m,n)\, W_N^{km} W_N^{ln},$$

$$u(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} v(k,l)\, W_N^{-km-ln}$$

2D DFT -- PROPERTIES

- Separability
- Translation
- Scaling
- Periodicity and Conjugate Symmetry
- Rotation
- convolution

Separability

$$v(k,l) = \frac{1}{N} \sum_{m=0}^{N-1} W_N^{km} \sum_{n=0}^{N-1} u(m,n)\, W_N^{ln}$$

$$= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} v(m,l)\, W_N^{km}$$

For each ‘m’, v(m,l) is the 1-D DFT with frequency values l = 0,1,....., N-1

Separability

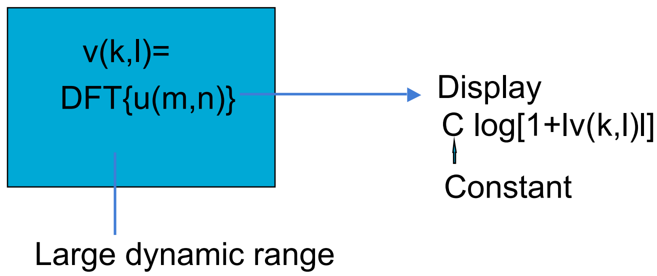
The DFT of a 2-D array can be obtained by first taking the 1-D DFT of each row (or column) and then taking the 1-D DFT of each column (or row).

<u>It does not matter if the order of operation is reversed.</u>

Translation

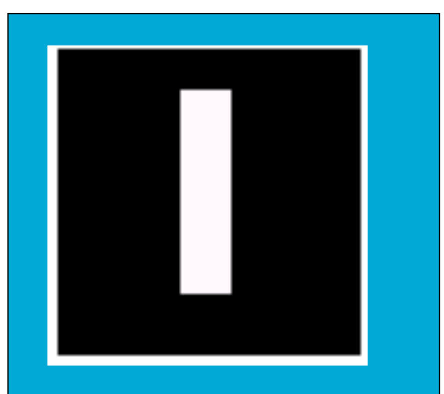
$$u(m-m',n-n') \leftrightarrow v(k,l)e^{-j2\pi\frac{(km'+ln')}{N}}$$

Displaying the DFT: Scaling



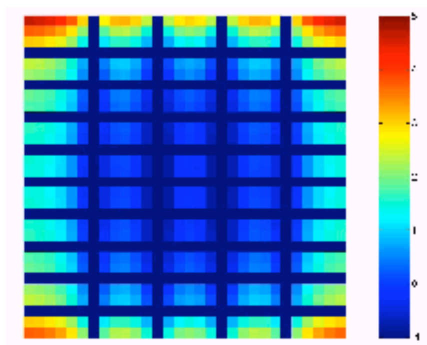
In MATLAB

```
f = zeros(30,30);
f(5:24,13:17)=1;
imshow(f, 'notruesize');
```

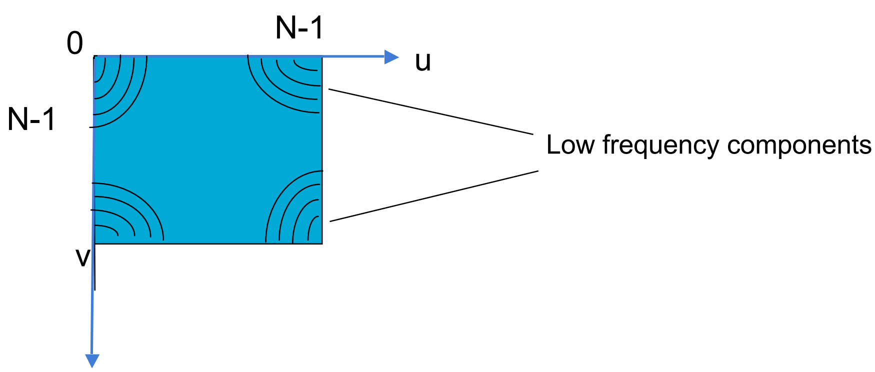


In Matlab(2)

```
F =fft2(f);
F2 = log(abs(F));
imshow(F2, [-1, 5], 'notruesize');
colormap(jet); colorbar;
```



Displaying the DFT



Displaying (again) & Shifting

$$u(m,n)e^{\frac{j2\pi(k'm+l'n)}{N}} \leftrightarrow v(k-k', l-l') \text{ and}$$

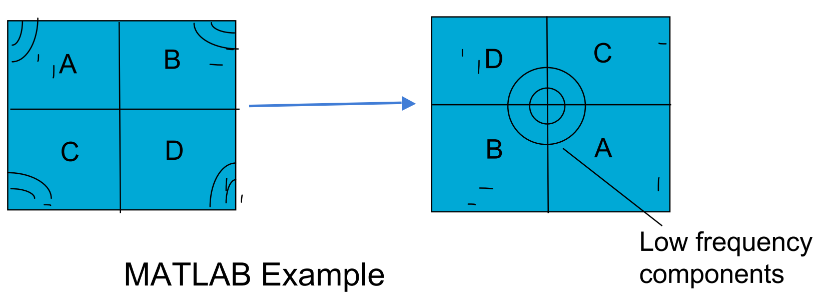
$$u(m,n)(-1)^{m+n} \leftrightarrow v\left(k-\frac{N}{2}, l-\frac{N}{2}\right)$$

The origin of the F{u(m,n)} can be moved to the center of the array (N X N square) by first multiplying u(m,n) by $(-1)^{m+n}$ and then taking the Fourier transform.

Note: Shifting does not affect the magnitude of the Fourier transform.

Displaying DFT

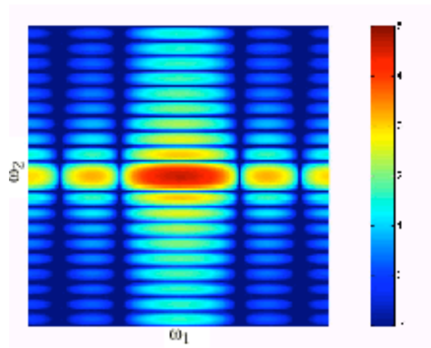
$$|v(k,l)\, e^{-j2\pi[km'+ln']/N}| = |v(k,l)|$$



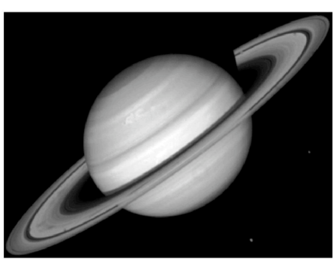
MATLAB Example

In Matlab(3): FFTSHIFT

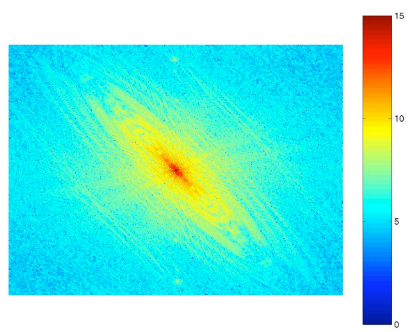
F2= fftshift(F);
imshow (log(abs(F2),..)



Another example



Original image



Its centered DFT magnitude

Periodicity & Conjugate Symmetry

$u(m,n) \overset{F}{\longleftrightarrow} v(k,l)$

$v(k,l) = v(k+N, l) = v(k, l+N) = v(k+N, l+N)$

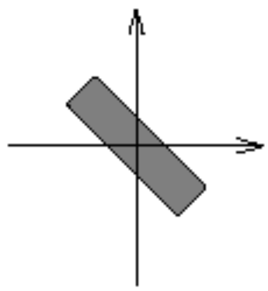
If $u(m,n)$ is real, $v(k,l)$ also exhibits conjugate symmetry
$v(k,l) = v^*(-k, -l)$ or $|v(k,l)| = |v(-k, -l)|$

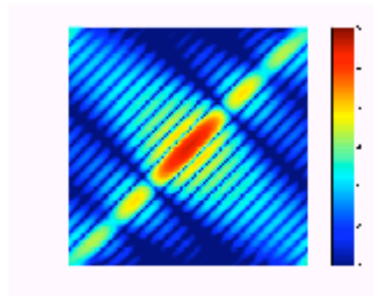
Rotation

(continuous case)

If you rotate the image $u(m,n)$ by an angle θ, its F.T also gets rotated by the same angle.

Rotation





Average Value

$$\bar{u} = \frac{1}{N}\sum_m \sum_n u(m,n) = \text{Average}$$

$$v(k,l) = \frac{1}{N}\sum_m \sum_n u(m,n) e^{-j2\pi \frac{km+ln}{N}}$$

$$v(0,0) = \frac{1}{N}\sum_m \sum_n u(m,n) = N\bar{u}$$

$$\text{or } \bar{u} = \frac{v(0,0)}{N} \text{ (Scaled Average)}$$

Convolution (Revisited)

Consider 1-D continuous case

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x')g(x - x')dx'$$

Let $f(x) \leftrightarrow F(u),\ g(x) \leftrightarrow G(u)$

Then $f(x) * g(x) \leftrightarrow F(u)G(u)$

Convolution in Space ⟷ Multiplication in Frequency

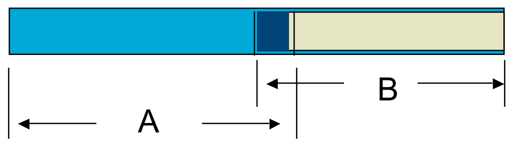
Discrete Convolution

Let us now assume that we discretize f(x) and g(x) into vectors f(n) and g(n) of lengths A and B

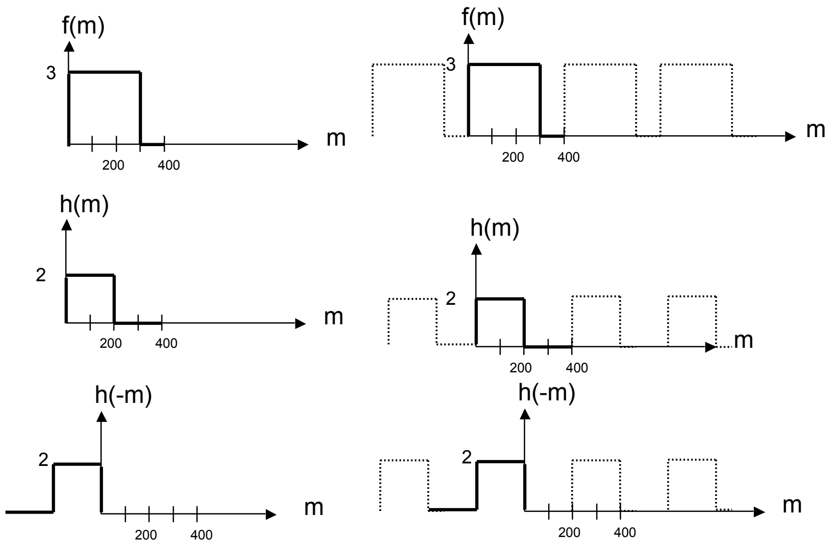
f(n) ⟶ {f(0), f(1),..... f(A-1)}
g(n) ⟶ {g(0), g(1), g(2),....g(B-1)}

(a) DFT and its inverse are periodic functions
(b) Convolving two vectors of length A and B gives a vector of dimension A+B-1. (Linear convolution)

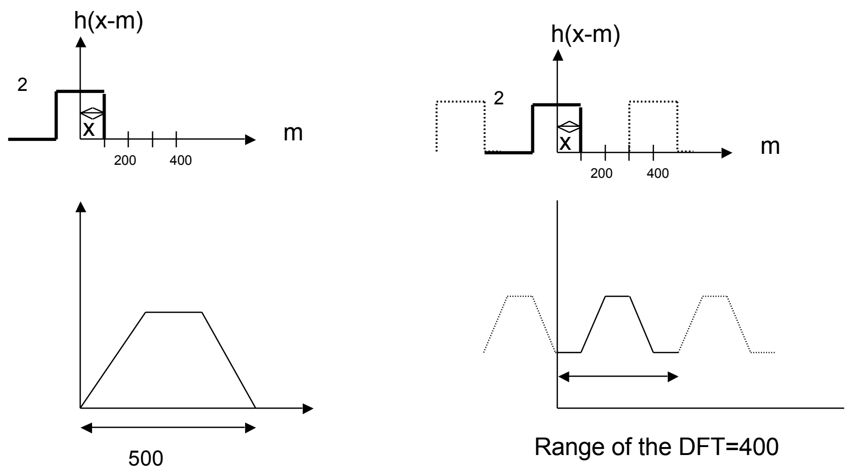
Length of the Convolution



Discrete Convolution: an example (Fig 4.36)



Discrete conv. (cont.)



Zero Imbedding

In order to obtain a convolution theorem for the discrete case, and still be consistent with the periodicity property we need to assume that sequences f( ) and g( ) are periodic with some period M. From (b) it is clear that M>A+B-1 to avoid overlap.

Since this period is greater than A or B, the original sequence length must be increased and this is done by appending zeros at the end. Redefine the extended sequences as

$$f_e(n) = \begin{cases} f(n) & 0 \le n \le A-1 \\ 0 & A \le n \le M-1 \end{cases}$$

$$g_e(n) = \begin{cases} g(n) & 0 \le n \le B-1 \\ 0 & B \le n \le M-1 \end{cases}$$

$$f_e(n) *_c g_e(n) = \sum_{m=0}^{M-1} f_e(m) g_e(n-m)_c$$

where $\left(g(n)\right)_c = g\left[n \text{ Modulo } M\right]$

Note: With n expressed as

$n = n_1 + n_2 N$ where $0 \le n_1 \le N-1$

n modulo N equals n_1

$$x \mod y = x - y\left[{}^{x}\!/_{y}\right] \quad \text{if } y \neq 0$$

$$x \mod 0 = x.$$

$\left[\frac{x}{y}\right]$ is the integer part of ${}^{x}\!/_{y}$

Theorem

The DFT of the circular convolution of two sequences of length N is equal to the product of their DFTs.

If $y(n) = \sum_{m=0}^{N-1} f(n-m)_c g(n)$ then

$$\text{DFT}\left[y(n)\right]_N = \text{DFT}\left[f(n)\right]_N \text{DFT}\left[g(n)\right]_N$$

A linear convolution of two sequences can be obtained via FFT by embedding it into a circular convolution.

2-D Convolution

These results can be similarly extended to 2-D signals.

Let f(m,n) : A x B array
g(m,n) : C x D array
Let M> = A + C -1
N> = B + D -1

For linear convolution using DFT create the extended periodic sequences of period MxN in the 2-D.

Extended (periodic) Sequences

computing convolution is more efficient in the frequency domain.

$$f_e(m,n) = \begin{cases} f(m,n) & 0 \le m \le A-1 \\ & 0 \le n \le B-1 \\ 0 & A \le m \le M-1 \\ & B \le n \le N-1 \end{cases}$$

$$g_e(m,n) = \begin{cases} g(m,n) & 0 \le m \le C-1 \\ & 0 \le n \le D-1 \\ 0 & C \le m \le M-1 \\ & D \le n \le N-1 \end{cases}$$

and the 2 - D linear convolution becomes

$$y(m,n) = \sum_{m'=0}^{M-1} \sum_{n'=0}^{N-1} f_e(m-m',\ n-n')_c\ g_e(m',n')$$

Linear Convolution and DFT: Summary

$y(n) = f(n) * g(n)$

1. Let $M >= A+B-1$ be an integer for which the FFT algorithm is available.

2. Define the zero extended sequences $f_e(n)$, $g_e(n)$.

3. Let $F_e(k) = DFT\{ f_e(n) \}_M$, $G_e(k) = DFT\{ g_e(n) \}_M$. Let $Y_e(k)=F_e(k)G_e(k)$

4. Take the I-DFT of $Y_e(k)$ to obtain $Y_e(n)$.
   Then $Y(n) = Y_e(n)$ for $0 <= n <= A+B-1$

A note on convolution with images

Note: In many cases involving images, we deal with square arrays of size N X N. We normally would like to have the resulting convolved output also as an N X N array.

Conv2 (.) in Matlab

CONV2 Two dimensional convolution.

C = CONV2(A, B) performs the 2-D convolution of matrices A and B. If [ma,na] = size(A) and [mb,nb] = size(B), then size(C) = [ma+mb-1,na+nb-1].

C = CONV2(... ,'shape') returns a subsection of the 2-D convolution with size specified by 'shape':

- 'full' - (default) returns the full 2-D convolution,
- 'same' - returns the central part of the convolution that is the same size as A.
- 'valid' - returns only those parts of the convolution that are computed without the zero-padded edges, size(C) = [ma-mb+1,na-nb+1] when size(A) > size(B).