Sampling and Quantization

Lecture #5

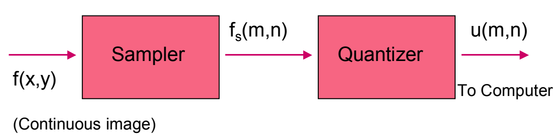
January 20, 2004

Sampling and Quantization

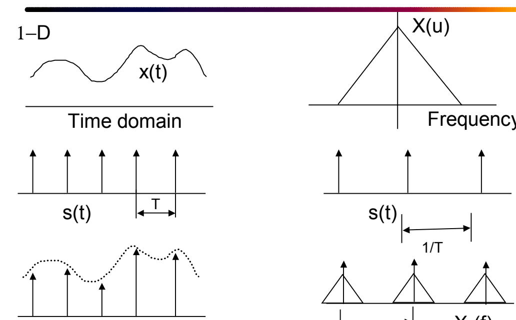
- Spatial Resolution (Sampling)
  - Determines the smallest perceivable image detail.
  - What is the *best* sampling rate?
- Gray-level resolution (Quantization)
  - Smallest discernible change in the gray level value.
  - Is there an optimal quantizer?

Image sampling and quantization

In 2-D



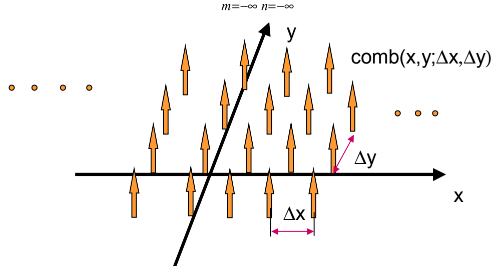
1-D



$x_s(t) = x(t)\ s(t) = \Sigma\, x(kt)\ \delta\ (t\text{-}kT)$

2-D: Comb function

$$\mathrm{Comb}(x, y; \Delta x, \Delta y) \cong \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$



Sampled Image

$$f_s(x, y) = f(x, y)\ \mathrm{comb}\ (x, y\ ;\ \Delta x, \Delta y)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m\Delta x, n\Delta y)\ \delta(x - m\Delta x, y - n\Delta y)$$

$$\mathrm{comb}(x, y; \Delta x, \Delta y) \overset{\Im}{\longleftrightarrow} \mathrm{COMB}\ (u, v) = \frac{1}{\Delta x \Delta y} \mathrm{comb}(u, v\ ; \frac{1}{\Delta x}, \frac{1}{\Delta y})$$

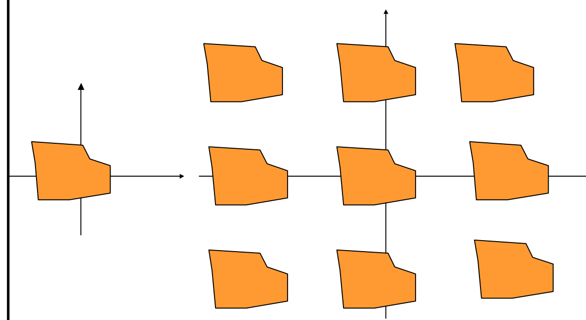
Sampled Spectrum

$$F_s(u,v) = F(u,v) * \text{COMB}(u,v)$$

$$= \frac{1}{\Delta x \Delta y} \sum_{k,l=-\infty}^{\infty}\sum F(u,v) * \delta\left(u - \frac{k}{\Delta x}, v - \frac{l}{\Delta y}\right)$$

$$= \frac{1}{\Delta x \Delta y} \sum_{k,l=-\infty}^{\infty}\sum F\left(u - \frac{k}{\Delta x}, v - \frac{l}{\Delta y}\right)$$

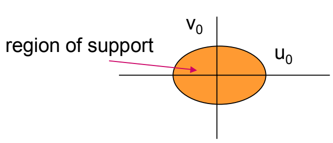
Sampled Spectrum: Example



Bandlimited Images

A function f(x,y) is said to be band limited if the Fourier transform
$F(u,v) = 0$ for $|u| > u_0, |v| > v_0$

u_0, v_0 → Band width of the image in the x- and y- directions



Foldover Frequencies

Sampling frequencies:
Let u_s and v_s be the sampling frequencies

Then $u_s > 2u_0$; $v_s > 2v_0$

or $\Delta x < 1/2u_0$; $\Delta y < 1/2v_0$

Frequencies above half the sampling frequencies are called fold over frequencies.

Sampling Theorem

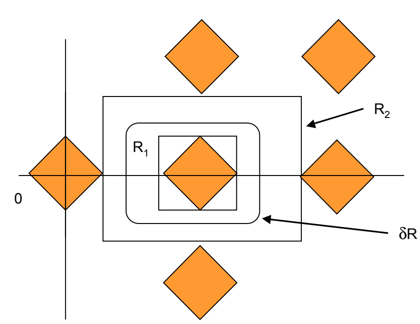
A band limited image f(x,y) with F(u,v) as its Fourier transform; and $F(u,v) = 0$ $|u| > u_0$ $|v| > v_0$; and sampled uniformly on a rectangular grid with spacing Δx and Δy, can be recovered without error from the sample values $f(m\,\Delta x, n\,\Delta y)$ provided the sampling rate is greater than the nyquist rate.

i.e $1/\Delta x = u_s > 2u_0$, $1/\Delta y = v_s > 2v_0$

The reconstructed image is given by the interpolation formula:

$$f(x,y) = \sum_{m,n=-\infty}^{\infty}\sum f(m\,\Delta x, n\,\Delta y)\ \frac{\sin(x u_s - m)\pi}{(x u_s - m)\pi}\ \frac{\sin(y v_s - n)\pi}{(y v_s - n)\pi}$$

Reconstruction



Reconstruction via LPF

F(u,v) can be recovered by a LPF with

$$H(u,v) = \begin{cases} \Delta x\, \Delta y & (u,v) \in R \\ 0 & \text{Other wise} \end{cases}$$

R is any region whose boundary ∂R is contained within the annular ring between the rectangles R_1 and R_2 in the figure. Reconstructed signal is

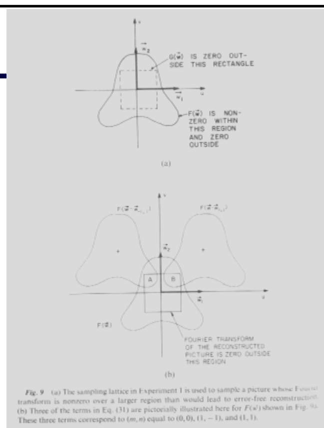
$$\tilde{F}(u,v) = H(u,v)\, F_s(u,v) = F(u,v)$$

$$f(x,y) = \Im^{-1}[F(u,v)]$$

Aliasing

Note: If u_s and v_s are below the Nyquist rate, the periodic replications will overlap, resulting in a distorted spectrum.

This overlapping of successive periods of the spectrum causes the foldover frequencies in the original image to appear as frequencies below $u_s/2$, $v_s/2$ in the sampled image. This is called aliasing.



Example

$$f(x,y) = 2\cos(2\pi(3x+4y))$$

$$F(u,v) = \delta(u-3, v-4) + \delta(u+3, v+4)$$

$$\Rightarrow u_0 = 3, \quad v_0 = 4$$

$$\text{Let} \quad \Delta x = \Delta y = 0.2, \Rightarrow u_s = v_s = \frac{1}{0.2} = 5 \quad < 2u_0, < 2v_0$$

there will be aliasing.

Example:(contd.)

$$F_s(u,v) = 25 \sum_{k,l=-\infty}^{\infty}\sum F(u-ku_s, v-lv_s)$$

$$= 25 \sum_{k,l=-\infty}^{\infty}\sum [\delta(u-3-5k, v-4-5l) + \delta(u+3-5k, v+4-5l)]$$

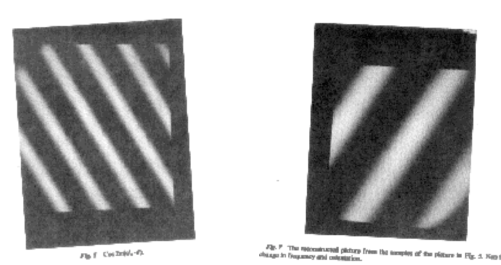
$$\text{Let} \quad H(u,v) = \begin{cases} 1/25 & -2.5 \le u \le 2.5, \quad -2.5 \le u \le 2.5 \\ 0 & \text{Otherwise} \end{cases}$$

$$\therefore F(u,v) = H(u,v)\, F_s(u,v)$$

$$= \delta(u+2, v+1) + \delta(u-2, v-1)$$

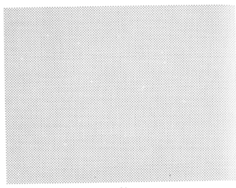
$$\therefore \tilde{f}(x,y) = 2\cos(2\pi(2x+y))$$

Examples

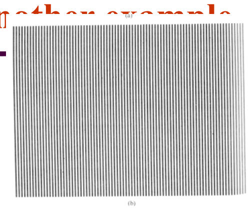


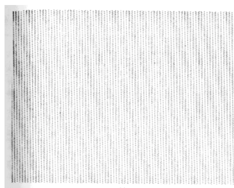
Original and the reconstructed image from samples.

Another example



Sampling filter





sampled image

Aliasing Problems (real images!)