Sampling and Quantization

Lecture #5

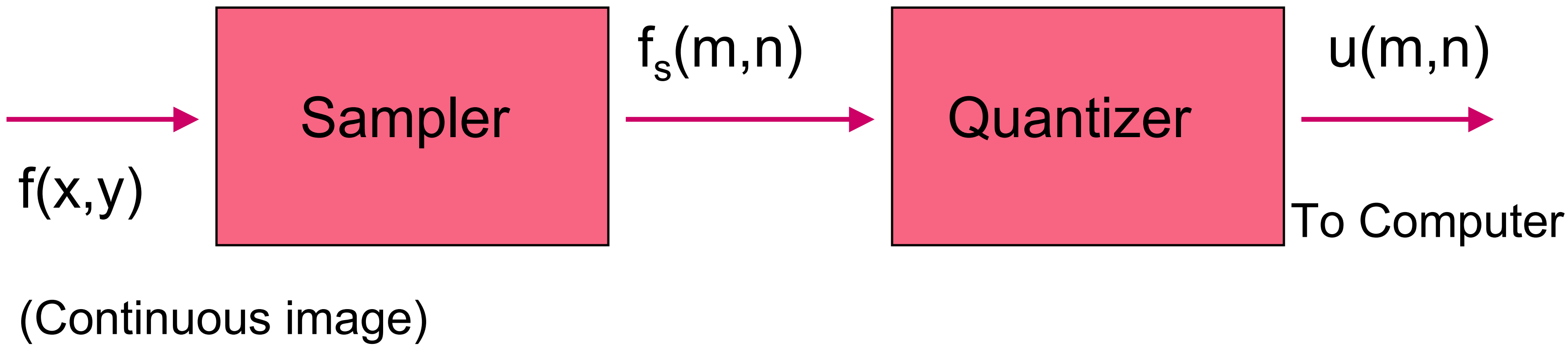
January 20, 2004

Sampling and Quantization

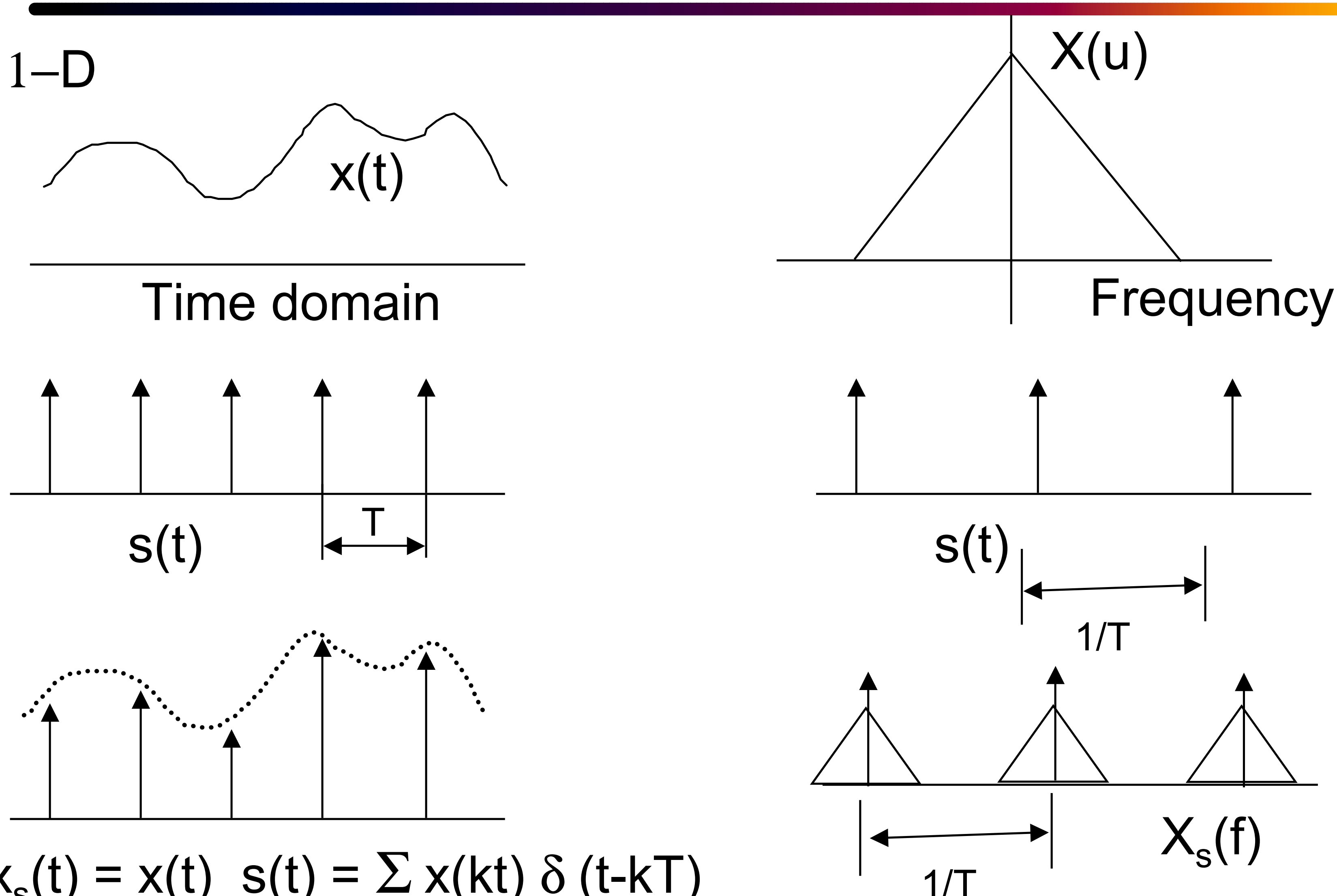
- Spatial Resolution (Sampling)
  - Determines the smallest perceivable image detail.
  - What is the *best* sampling rate?
- Gray-level resolution (Quantization)
  - Smallest discernible change in the gray level value.
  - Is there an optimal quantizer?

Image sampling and quantization

In 2-D



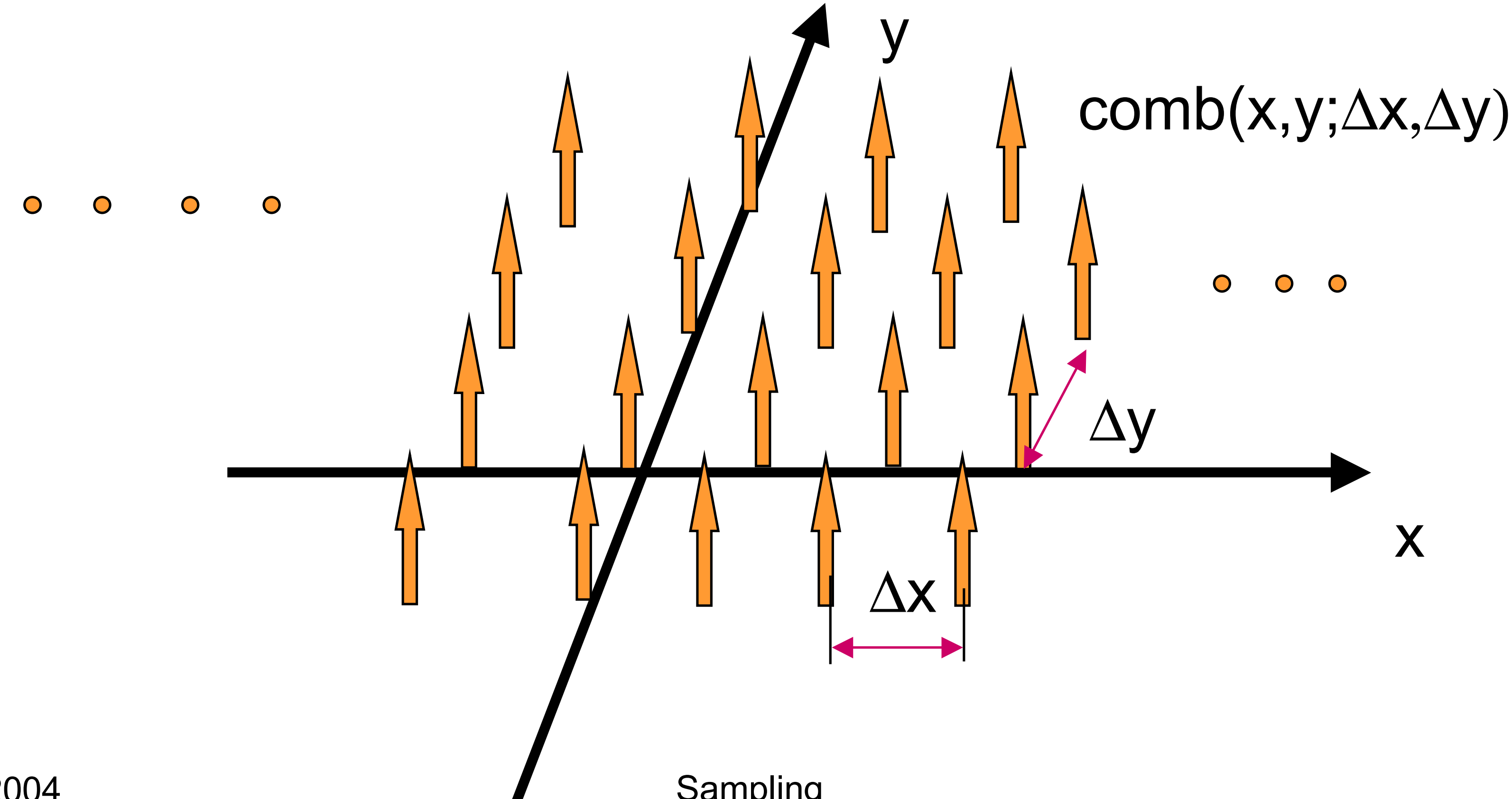
1-D



$x_s(t) = x(t)\ \ s(t) = \Sigma\, x(kt)\, \delta\, (t\text{-}kT)$

2-D: Comb function

$$\mathrm{Comb}(x,y;\Delta x,\Delta y) \cong \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m\Delta x, y-n\Delta y)$$



Sampled Image

$$f_s(x,y) = f(x,y)\ \mathrm{comb}\ (x,y\ ;\ \Delta x, \Delta y)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m\Delta x, n\Delta y)\ \delta(x - m\Delta x, y - n\Delta y)$$

$$\mathrm{comb}(x,y;\Delta x,\Delta y) \overset{\Im}{\longleftrightarrow} \mathrm{COMB}\ (u,v) =$$

$$\frac{1}{\Delta x \Delta y} \mathrm{comb}(u,v\ ;\frac{1}{\Delta x},\frac{1}{\Delta y})$$

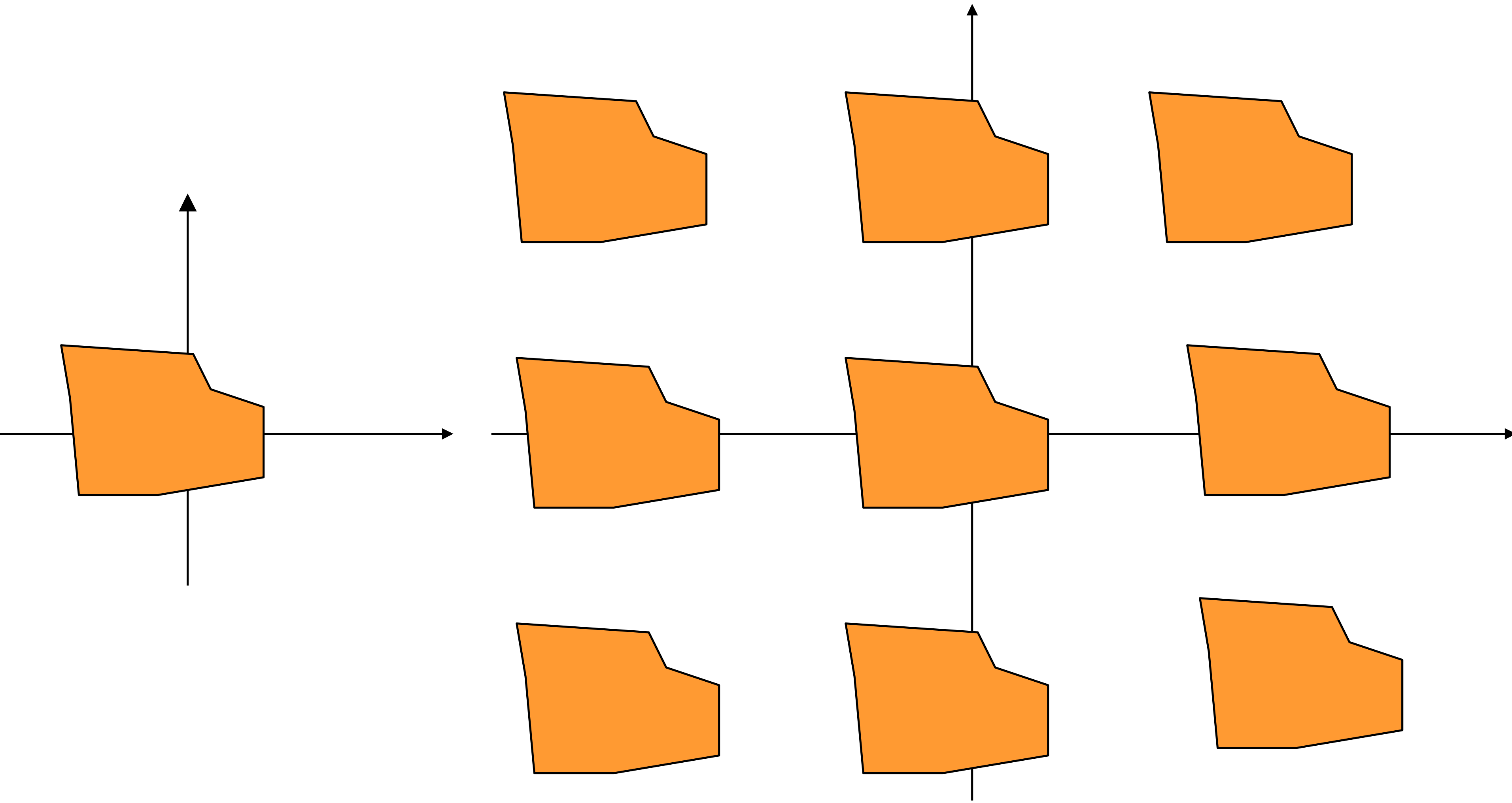
Sampled Spectrum

$$F_s(u,v) = F(u,v) * \mathrm{COMB}(u,v)$$

$$= \frac{1}{\Delta x \Delta y} \sum_{k,l=-\infty}^{\infty} \sum F(u,v) * \delta\left(u - \frac{k}{\Delta x}, v - \frac{l}{\Delta y}\right)$$

$$= \frac{1}{\Delta x \Delta y} \sum_{k,l=-\infty}^{\infty} \sum F\left(u - \frac{k}{\Delta x}, v - \frac{l}{\Delta y}\right)$$

Sampled Spectrum: Example

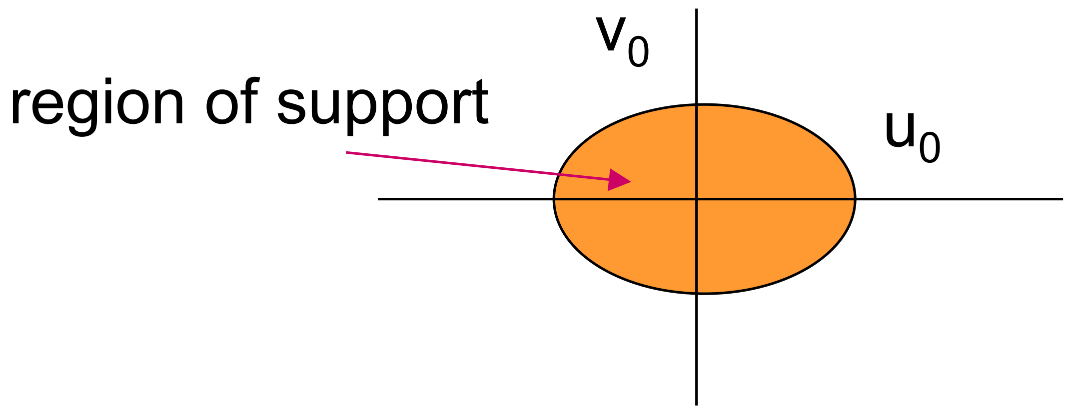


Bandlimited Images

A function f(x,y) is said to be band limited if the Fourier transform

$F(u,v) = 0$ for $|u| > u_0, \; |v| > v_0$

u_0, v_0 ➡ Band width of the image in the x- and y- directions



Foldover Frequencies

Sampling frequencies:
Let u_s and v_s be the sampling frequencies

Then $u_s > 2u_0$; $v_s > 2v_0$

or $\Delta x < 1/2u_0$; $\Delta y < 1/2v_0$

Frequencies above half the sampling frequencies are called fold over frequencies.

Sampling Theorem

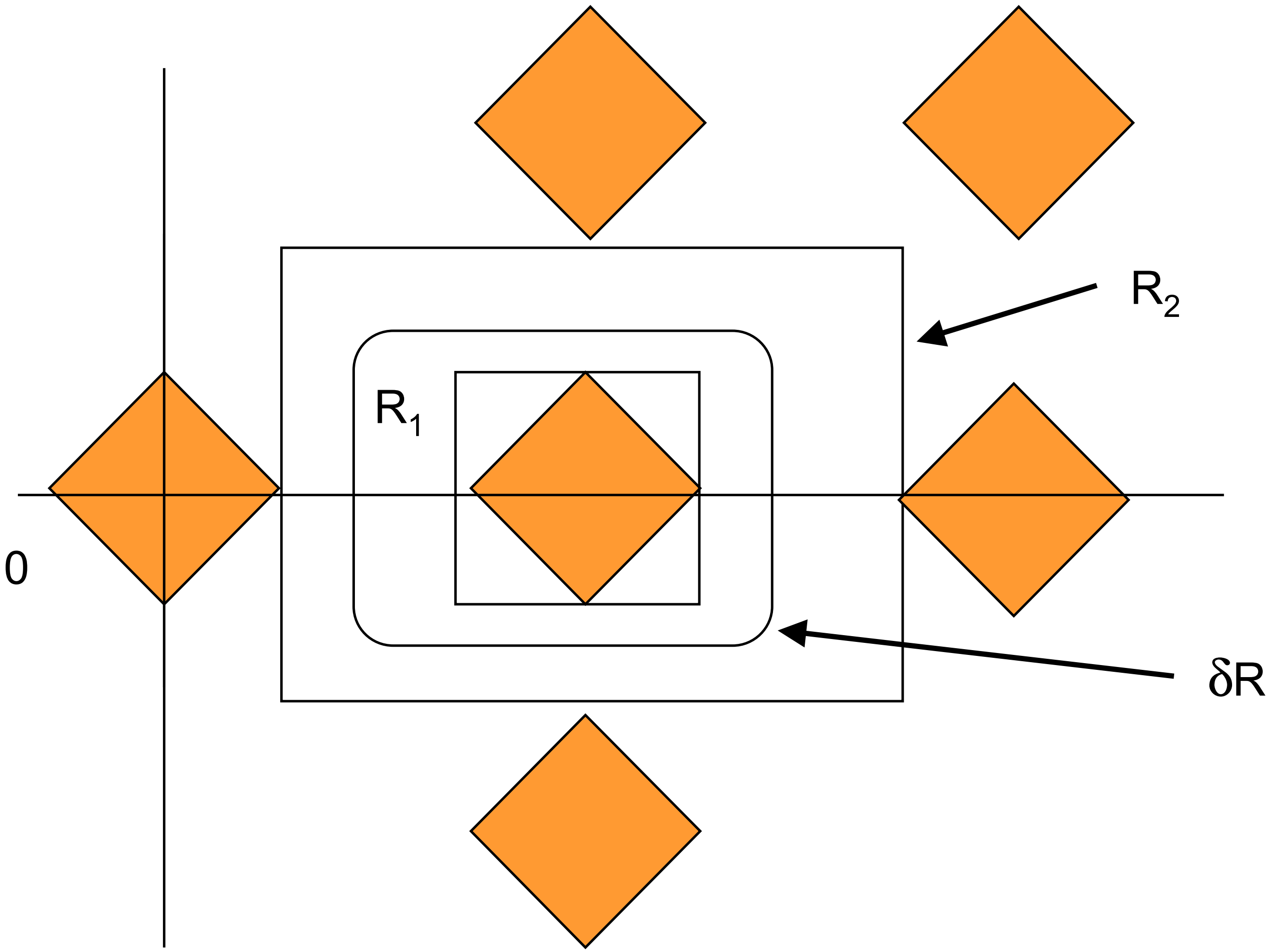
A band limited image f(x,y) with F(u,v) as its Fourier transform; and F(u,v) = 0 $|u| > u_0$ $|v| > v_0$; and sampled uniformly on a rectangular grid with spacing Δx and Δy, can be recovered without error from the sample values f(m Δx, n Δy) provided the sampling rate is greater than the nyquist rate.

i.e $1/\Delta x = u_s > 2u_0$, $1/\Delta y = v_s > 2v_0$

The reconstructed image is given by the interpolation formula:

$$f(x,y) = \sum_{m,n=-\infty}^{\infty}\sum f(m\,\Delta x, n\,\Delta y)\ \frac{\sin(x\,u_s - m)\pi}{(x\,u_s - m)\pi}\ \frac{\sin(y\,v_s - n)\pi}{(y\,v_s - n)\pi}$$

Reconstruction



Reconstruction via LPF

F(u,v) can be recovered by a LPF with

$$H(u,v)=\begin{cases}\Delta x\ \Delta y & (u,v)\in R\\ 0 & \text{Other wise}\end{cases}$$

R is any region whose boundary ∂R is contained within the annular ring between the rectangles R_1 and R_2 in the figure. Reconstructed signal is

$$\widetilde{F}(u,v)=H(u,v)\,F_s(u,v)=F(u,v)$$

$$f(x,y)=\Im^{-1}[F(u,v)]$$

Aliasing

Note: If u_s and v_s are below the Nyquist rate, the periodic replications will overlap, resulting in a distorted spectrum.

This overlapping of successive periods of the spectrum causes the foldover frequencies in the original image to appear as frequencies below $u_s/2$, $v_s/2$ in the sampled image. This is called aliasing.

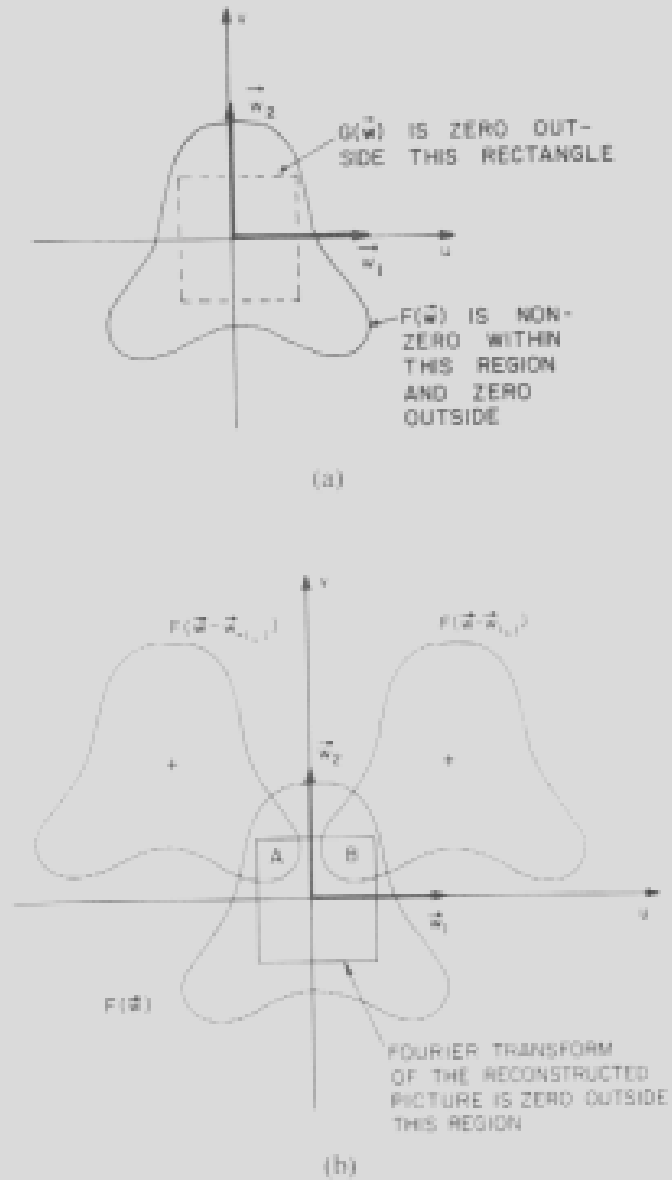


Fig. 9 (a) The sampling lattice in Experiment 1 is used to sample a picture whose Fourier transform is nonzero over a larger region than would lead to error-free reconstruction. (b) Three of the terms in Eq. (31) are pictorially illustrated here for $F(w)$ shown in Fig. 9a. These three terms correspond to (m, n) equal to $(0, 0)$, $(1, -1)$, and $(1, 1)$.

Example

$$f(x,y) = 2\cos(2\pi(3x + 4y))$$

$$F(u,v) = \delta(u-3, v-4) + \delta(u+3, v+4)$$

$$\Rightarrow u_0 = 3, \quad v_0 = 4$$

$$\text{Let} \quad \Delta x = \Delta y = 0.2, \Rightarrow u_s = v_s = \frac{1}{0.2} = 5 \quad < 2u_0, < 2v_0$$

there will be aliasing.

Example:(contd.)

$$\mathrm{F_s}(u,v) = 25 \sum_{k,l=-\infty}^{\infty}\sum F(u-ku_s, v-lv_s)$$

$$= 25 \sum_{k,l=-\infty}^{\infty}\sum [\delta(u-3-5k, v-4-5l) + \delta(u+3-5k, v+4-5l)]$$

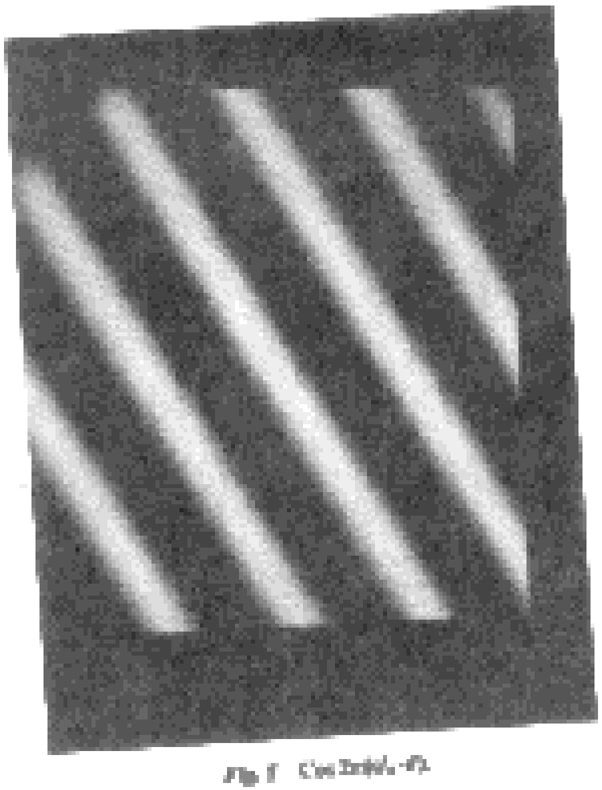
Let $\quad H(u,v) = \begin{cases} 1/25 & -2.5 \le u \le 2.5, \quad -2.5 \le u \le 2.5 \\ 0 & \text{Otherwise} \end{cases}$

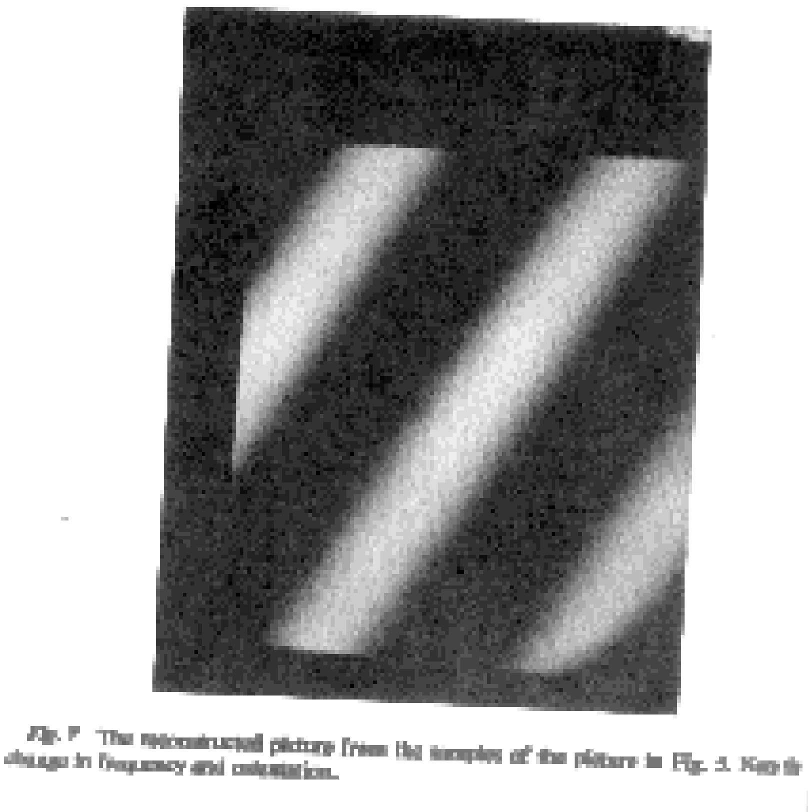
$$\therefore F(u,v) = H(u,v)\, F_s(u,v)$$

$$= \delta(u+2, v+1) + \delta(u-2, v-1)$$

$$\therefore \tilde{f}(x,y) = 2\cos(2\pi(2x+y))$$

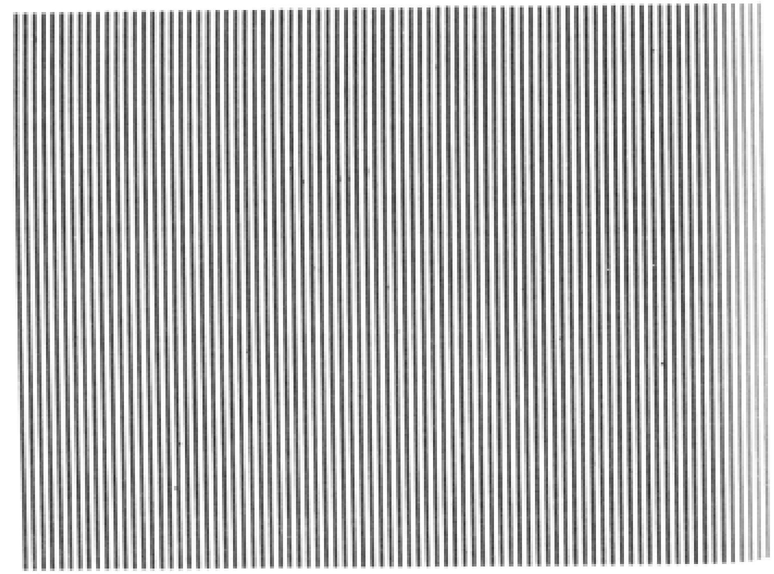
Examples

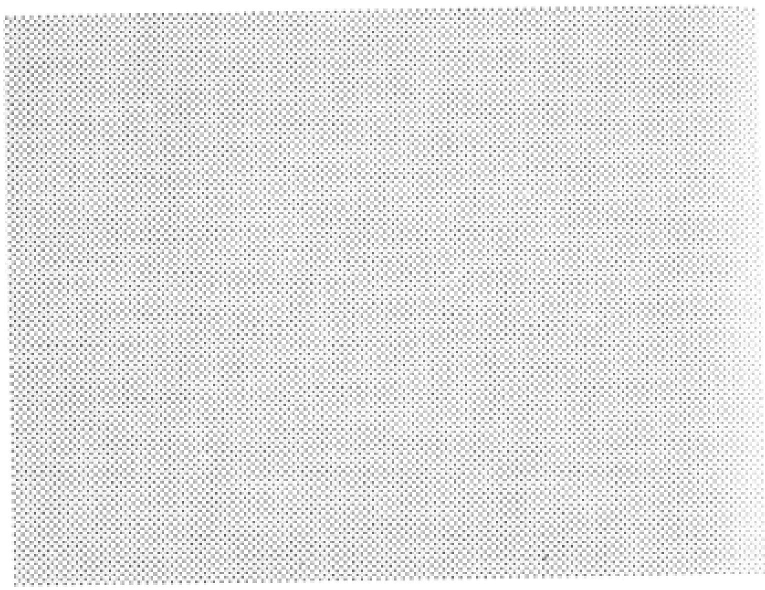




Original and the reconstructed image from samples.

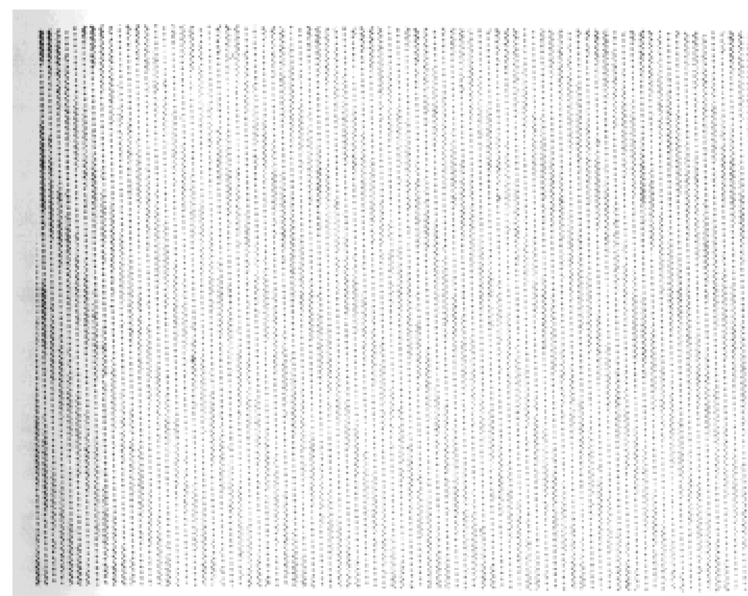
Another example



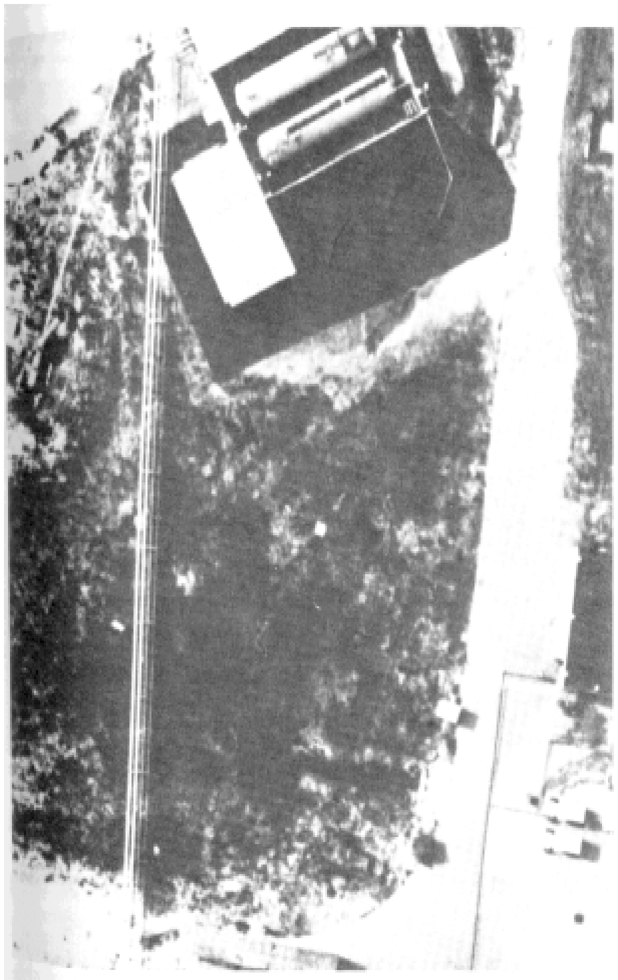


Sampling filter

sampled image



Aliasing Problems (real images!)



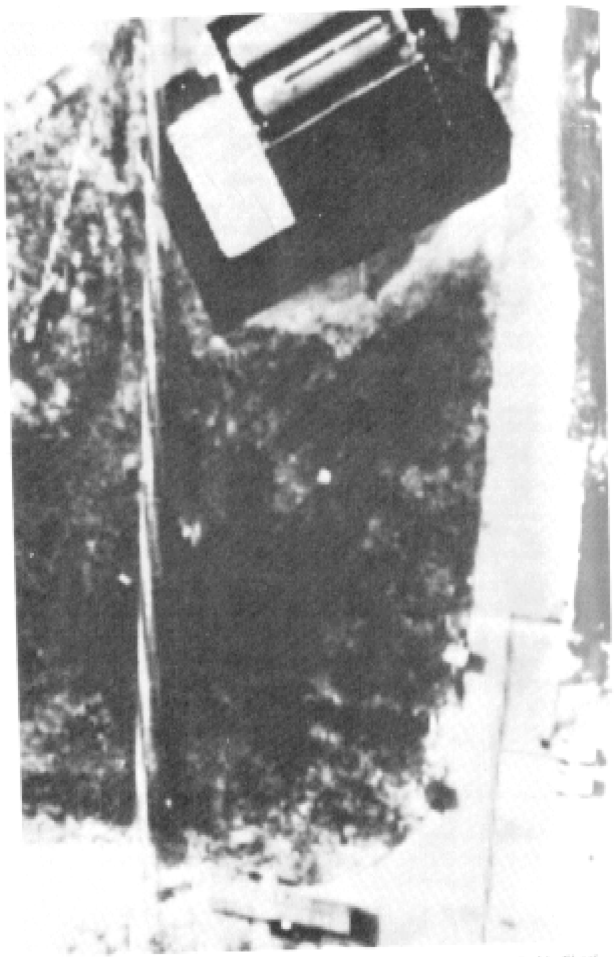


Fig. 10b Picture of Fig. 10a after sampling and reconstruction. (Courtesy Perkin-Elmer