ECE 178 Hwk #3 Solutions

1)

(a) $y[n] = T[x[n]] = x[n]+2x[n+1]+3$

Test Linearity: check if $T[ax_1[n]+ bx_2[n]] = ay_1[n]+ by_2[n]$

$T[ax_1[n]+ bx_2[n]] = ax_1[n]+ bx_2[n]+ 2ax_1[n+1]+ 2bx_2[n+1]+3$ **Eq. 1**
$ay_1[n]+ by_2[n] = ax_1[n]+ 2ax_1[n+1]+3a+ bx_2[n]+ 2bx_2[n]+3b$ **Eq. 2**

Since **Eq. 1** does not equal **Eq. 2** the system is not linear.

Test Shift Invariance: check if $T[x[n-n_0]] = y[n-n_0]$

$T[x[n-n_0]] = x[n-n_0] +2\ x[n-n_0+1]+3$ **Eq. 3**
$y[n-n_0] = x[n-n_0] +2\ x[n-n_0+1]+3$ **Eq. 4**

Since **Eq. 3** equals **Eq. 4** then the system is shift invariant.

(b) $y[m,n] = \exp(-x[m,n])$

Test Linearity: check if $T[ax_1[m,n]+ bx_2[m,n]] = ay_1[m,n]+ by_2[m,n]$

$T[ax_1[m,n]+ bx_2[m,n]] = \exp(-ax_1[m,n]- bx_2[m,n])$ **Eq. 5**
$ay_1[m,n]+ by_2[m,n] = a*\exp(-x_1[m,n])+b*\exp(-x_2[m,n])$ **Eq. 6**

Since **Eq. 5** does not equal **Eq. 6** the system is not linear.

Test Shift Invariance: check if $T[x[m-m_0,n-n_0]] = y[m-m_0,n-n_0]$

$T[x[m-m_0,n-n_0]] = \exp(-x[m-m_0,n-n_0])$ **Eq. 7**
$y[m-m_0,n-n_0] = \exp(-x[m-m_0,n-n_0])$ **Eq. 8**

Since **Eq. 7** equals **Eq. 8** the system is shift invariant.

(c) $y[m,n] = m*n*x[m,n]$

Test Linearity: check if $T[ax_1[m,n]+ bx_2[m,n]] = ay_1[m,n]+ by_2[m,n]$

$T[ax_1[m,n]+ bx_2[m,n]] = a*m*n*x_1[m,n]+ b*m*n*x_2[m,n]$ **Eq. 9**
$ay_1[m,n]+ by_2[m,n] = a*m*n*x_1[m,n]+ b*m*n*x_2[m,n]$ **Eq. 10**

Since **Eq. 9** equals **Eq. 10** the system is linear.

Test Shift Invariance: check if $T[x[m-m_0,n-n_0]] = y[m-m_0,n-n_0]$

$T[x[m-m_0,n-n_0]] = m*n*x[m-m_0,n-n_0]$ **Eq. 11**
$y[m-m_0,n-n_0] = (m-m_0)*(n-n_0)* x[m-m_0,n-n_0]$ **Eq. 12**

Since **Eq. 11** does not equal **Eq. 12** the system is shift variant, not invariant.

(d) $y[m,n] = x[m-m_0,n-n_0]$

Test Linearity: check if $T[ax_1[m,n]+ bx_2[m,n]] = ay_1[m,n]+ by_2[m,n]$

$T[ax_1[m,n]+ bx_2[m,n]] = ax_1[m-m_0,n-n_0]+ bx_2[m-m_0,n-n_0]$ **Eq. 13**
$ay_1[m,n]+ by_2[m,n] = ax_1[m-m_0,n-n_0]+ bx_2[m-m_0,n-n_0]$ **Eq. 14**

Since **Eq. 13** equals **Eq. 14** the system is linear

Test Shift Invariance: check if $T[x[m-m_0,n-n_0]] = y[m-m_0,n-n_0]$

$T[x[m-m_1,n-n_1]] = x[m-m_0-m_1,n-n_0-n_1]$ **Eq. 15**
$y[m-m_0,n-n_0] = x[m-m_0-m_1,n-n_0-n_1]$ **Eq. 16**

Since **Eq. 15** equals **Eq. 16** the system is shift invariant

(e) $y[m,n] = \sum_{m'=-1}^{1}\sum_{n'=-1}^{1} x[m',n']$

Test Linearity: check if $T[ax_1[m,n]+ bx_2[m,n]] = ay_1[m,n]+ by_2[m,n]$

$$T[ax_1[m,n]+bx_2[m,n]] = \sum_{m'=-1}^{1}\sum_{n'=-1}^{1} ax_1[m',n']+bx_2[m',n'] = a\sum_{m'=-1}^{1}\sum_{n'=-1}^{1} x_1[m',n'] + b\sum_{m'=-1}^{1}\sum_{n'=-1}^{1} x_2[m',n']$$

Eq. 17 above

$$ay_1[m,n]+by_2[m,n] = a\sum_{m'=-1}^{1}\sum_{n'=-1}^{1} x_1[m',n'] + b\sum_{m'=-1}^{1}\sum_{n'=-1}^{1} x_2[m',n'] \quad \textbf{Eq. 18}$$

Since **Eq. 17** equals **Eq. 18** the system is linear

Test Shift Invariance: check if $T[x[m-m_0,n-n_0]] = y[m-m_0,n-n_0]$

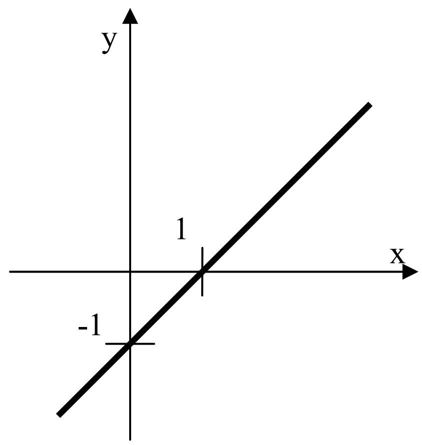
$$T[x[m-m_0,n-n_0]] = \sum_{m'=-1}^{1}\sum_{n'=-1}^{1} x[m'-m_0,n'-n_0] \quad \textbf{Eq. 19}$$

$$y[m-m_0,n-n_0] = \sum_{m'=-1}^{1}\sum_{n'=-1}^{1} x[m',n'] \quad \textbf{Eq. 20}$$

Since **Eq. 19** equals **Eq. 20** the system is shift variant, not invariant.

2) Sketch $\delta(x-y-1)$

The function only has value when x-y-1 = 0 so the graph is



3) The image f(x,y) = 4 cos(4πx)cos(4πy) is to be sampled such that one can reconstruct the signal from its samples without any errors. Suggest a sampling scheme. How do you propose to reconstruct from the samples? Is your Sampling scheme optimal?

Use trigonometric identity cos(a)cos(b) = [cos(a+b) + cos(a-b)]/2

f(x,y) = 4*1/2*[cos(2π(2x+3y))+cos(2π(2x-3y))]
F(u.v) = 2*2*[δ(u+2,u+3)+ δ(u-2,u-3)+ δ(u+2,u-3)+ δ(u-2,u+3)]

Now pick a sampling spacing such that the sampling frequency is twice the highest frequency component in that given direction.

u_s=1/Δx, and u_s>2*2 therefore Δx < 1/4
v_s=1/Δy, and v_s>2*3 therefore Δy < 1/6

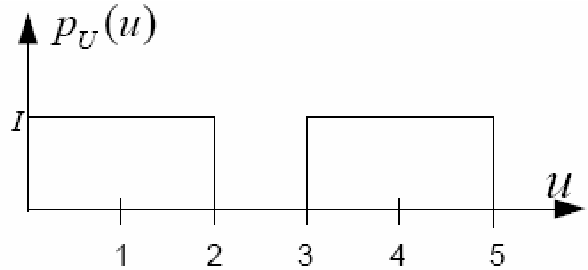
To satisfy the nyquist sampling rate we can pick our sampling grid as Δx = 1/4 and Δy = 1/6. Plugging back into the above equations we get u_s = 4 and v_s = 6. Plugging these into the sampling equation derived in class we get:

$$F(u,v) = 96\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta(u+2-4k, v+3-6k) + \delta(u-2-4k, v-3-6k)\ldots$$
$$+\delta(u+2-4k, v-3-6k) + \delta(u-2-4k, v+3-6k)$$

With this sampling scheme pick a lowpass filter that will perfectly reconstruct the signal. This is done by using an ideal lowpass filter based on the sampling grid picked. The frequency response of the reconstruction filter H_r is shown below.

$$H_r(u,v)=\begin{Bmatrix} \frac{1}{24} & \begin{matrix} -2\le u\le 2 \\ -3\le v\le 3 \end{matrix} \\ 0 & otherwise \end{Bmatrix}$$

4)



-MMSE quantizer:

$$r_k=\frac{\int_{t_k}^{t_{k+1}} u p_U(u)du}{\int_{t_k}^{t_{k+1}} p_U(u)du},\ t_k=\frac{r_{k-1}+r_k}{2}$$

$$-\int_u p_U(u)du=1 \Rightarrow I=\frac{1}{4}$$

-pdf symmetry: $\boxed{t_1=0}$, $\boxed{t_3=2.5}$, $\boxed{t_5=5}$, $t_5-t_4=t_2-t_1 \Rightarrow t_4=5-t_2$ (1)

$$-\ r_1=\frac{\int_0^{t_2} u\frac{1}{4}du}{0\int_{t_k}^{t2}\frac{1}{4}du}=\frac{t_2}{2}\Rightarrow r_1=\frac{t_2}{2},\ t_2=\frac{r_1+r_2}{2}\ (2) \qquad -r_2=\frac{\int_{t_2}^{t_3} u p_U(u)du}{\int_{t_2}^{t_3} p_U(u)du}=\frac{\int_{t_2}^{2} u\frac{1}{4}du}{\int_{t_2}^{t_2}\frac{1}{4}du}=\frac{2+t_2}{2}\ (3)$$

$$-\ (2)\ \&\ (3) \Rightarrow t_2=\frac{r_1+r_2}{2}\Rightarrow t_2=\frac{1}{2}\left(\frac{t_2}{2}+\frac{2+t_2}{2}\right)\Rightarrow \boxed{t_2=1} \quad -\ (1)\Rightarrow t_4=t_5-t_2\Rightarrow \boxed{t_4=4}$$

$$-\ (2)\ \ r_1=\frac{t_2}{2}\Rightarrow \boxed{r_1=\frac{1}{2}} \quad -\ (3)\ \ r_2=\frac{2+t_2}{2}\Rightarrow \boxed{r_2=\frac{3}{2}}$$

$$-\ \text{pdf symmetry: } r_3=5-r_2\Rightarrow \boxed{r_3=\frac{7}{2}},\ r_4=5-r_1\Rightarrow \boxed{r_3=\frac{9}{2}}$$

- This 4-level MMSE quantizer combines two uniform 2-level quantizers ([0,2]&[3,5]), and $\Delta=1$

$$\Rightarrow \varepsilon=\frac{\Delta^2}{12}\Rightarrow \boxed{\varepsilon=\frac{1}{12}}$$

Note that the t_1,t_3 and t_5 are chosen because the pdf $p_u(u)$ is symmetric about the point 2.5.