HW #3

DUE: Monday, October 29, 2007

Q1. (**Optional–extra credit**)

Recall our discussion on sampling random fields. We would like to reconstruct a random field $f(\mathbf{r})$ from its samples $f(\mathbf{r}_{m,n})$ as $f(\mathbf{r}) = \sum_{m,n} f(\mathbf{r}_{m,n})g(\mathbf{r} - \mathbf{r}_{m,n})$ show that a necessary and sufficient for an error free reconstruction is given by

$$R_{ff}(\mathbf{r}) = \sum_{m,n} R_{ff}(\mathbf{r}_{m,n})g(\mathbf{r} - \mathbf{r}_{m,n}).$$

Q2. **Standard Sampling:**

A picture $\mathcal{R}$ is divided into MN rectangular regions. The pixel sample value in each of their sub-regions is then the average gray level within that region. Write this as sampling using orthonormal basis functions. What are the $\phi_{m,n}(x,y)$? What are the corresponding $a_{m,n}$?.

Q3. The image $f(x,y) = 4\cos 4\pi x \cos 6\pi y$ is sampled with $\Delta x = \Delta y = 0.5$ and $\Delta x = \Delta y = 0.2$. The reconstruction filter is an ideal low-pass filter with bandwidths $(\frac{1}{2}\Delta x, \frac{1}{2}\Delta y)$. What is the reconstructed image in each case?

Q4. **(A. K. Jain)**

In the class, we derived an expression for the reconstructed signal from its samples using the sinc interpolation function (for $g(x,y)$)

$$\tilde{f}(x,y) = \sum_m \sum_n f(m\Delta x, n\Delta y)\mathrm{sinc}(xu_s - m)\mathrm{sinc}(yv_s - n)$$

Show that the above equation is an orthogonal expansion for a bandlimited function such that the least squares error

$$\sigma^2_{l.s.} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left| f(x,y) - \sum_m \sum_n a(m,n)\phi_m(x)\psi_n(y) \right|^2 dxdy$$

where $\phi_m(x) \triangleq \mathrm{sinc}(xu_s - m)$, $\psi_n(y) \triangleq \mathrm{sinc}(yv_s - n)$, is minimized to zero when $a(m,n) = f(m\Delta x, n\Delta y)$.

Q5. **Optimal Sampling: (A. K. Jain)**

A real random field $f(x,y)$, defined on a square $[-L, L] \times [-L, L]$, with auto correlation function $R(x,y;x',y') \triangleq \mathrm{E}[f(x,y)f(x',y')]$, $-L \le x, y, x', y' \le L$, is sampled by a set of orthogonal functions $\phi_{m,n}(x,y)$ to obtain the samples $a_{m,n}$ such that the reconstructed function $\tilde{f}_{M,N}(x,y) \triangleq \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} a_{m,n}\phi_{m,n}(x,y)$ minimizes the mean square error

$$\sigma^2_{M,N} \triangleq \int_{-L}^{L}\int_{-L}^{L} \mathrm{E}[|f(x,y) - \tilde{f}_{M,N}(x,y)|^2]dxdy.$$

Let $\phi_{m,n}(x,y)$ be a set of complete orthonormal functions obtained by solving the eigenvalue integral equation

$$\int_{-L}^{L}\int_{-L}^{L} R(x,y;x',y')\phi_{m,n}(x',y')dx'dy' = \lambda_{m,n}\phi_{m,n}(x,y), \quad -L \le x,y \le L$$

a. Show that $\{a_{m,n}\}$ are orthogonal random variables, i.e.,

$$\mathrm{E}[a_{m,n}a_{m',n'}] = \lambda_{m,n}\delta(m-m', n-n')$$

b. Show that $\sigma^2_{M,N}$ is minimized when $\{\phi_{m,n}\}$ are chosen to correspond to the largest MN eigenvalues and the minimized error is $\sigma^2_{M,N} = \sum_{m=M}^{\infty}\sum_{n=N}^{\infty}\lambda_{m,n}$. The preceding series representation for the random field $f(x,y)$ is called KL series expansion.

Q6. **Interlaced Sampling: (A. K. Jain)**

Consider the spectrum shown in Figure 1, which can be tightly enclosed by a diamond-shaped region. On a rectangular sampling grid G_1, the Nyquist sampling intervals would be $\Delta x = \Delta y \triangleq \Delta_1 = 1$. If the sampling grid G_2 is chosen, which is a 45° of G_1 but with intersample distance of Δ_2, the spectrum of the sampled image will repeat on a grid similar to G_2 (with spacing $1/\Delta_2$)(Figure 1e). Therefore, if $\Delta_2 = \sqrt{2}$, there will be no aliasing, but the sampling density has been reduced by half. Thus if an image does not contain the high frequencies in both the dimensions simultaneously, then its sampling rate can be reduced by a factor of 2. The interlaced sampling grid G_2 in the Figure 1c can be written as a superposition of rectangular grids, i.e.,

$$g(x,y) = \sum_m\sum_n \delta(x-2m, y-2n) + \sum_m\sum_n \delta(x-2m-1, y-2n-1)$$

Verify Figure 1e by showing the Fourier transform of this array is

$$G(u,v) = \frac{1}{2}\sum_{k+l=\text{even}} \delta(u-u_0, v-v_0), \quad u_0 = v_0 = \frac{1}{2}$$

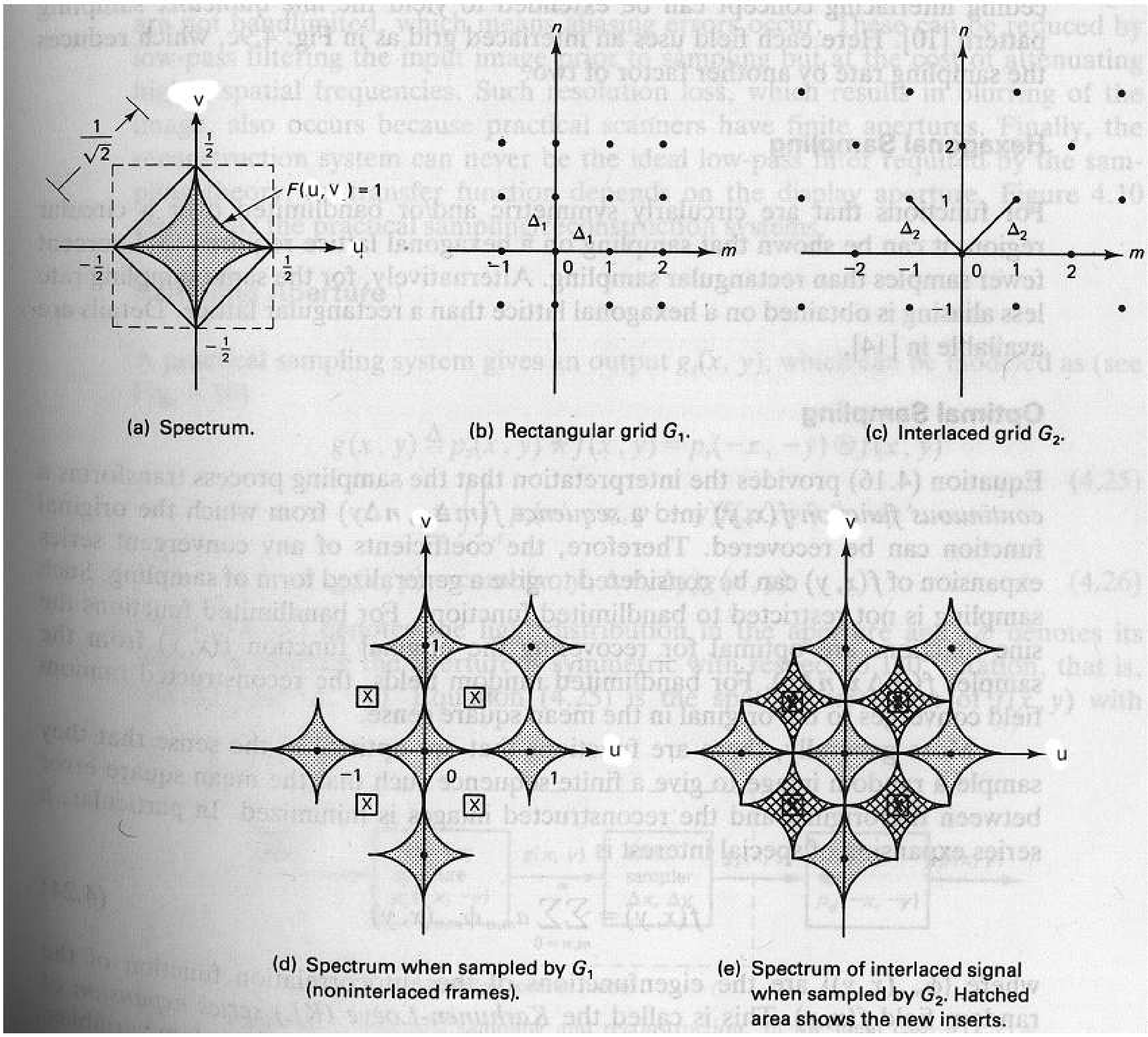
(a) Spectrum.
F(u, v) = 1
(b) Rectangular grid G_1.
(c) Interlaced grid G_2.
(d) Spectrum when sampled by G_1 (noninterlaced frames).
(e) Spectrum of interlaced signal when sampled by G_2. Hatched area shows the new inserts.

Figure 1: