Part VI
Function Evaluation

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Appendix: Past, Present, and Future
About This Presentation

This presentation is intended to support the use of the textbook *Computer Arithmetic: Algorithms and Hardware Designs* (Oxford U. Press, 2nd ed., 2010, ISBN 978-0-19-532848-6). It is updated regularly by the author as part of his teaching of the graduate course ECE 252B, Computer Arithmetic, at the University of California, Santa Barbara. Instructors can use these slides freely in classroom teaching and for other educational purposes. Unauthorized uses are strictly prohibited. © Behrooz Parhami

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<td></td>
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VI Function Evaluation

Learn hardware algorithms for evaluating useful functions
- Divisionlike square-rooting algorithms
- Evaluating $\sin x$, $\tanh x$, $\ln x$, ... by series expansion
- Function evaluation via convergence computation
- Use of tables: the ultimate in simplicity and flexibility

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May 2015

Computer Arithmetic, Function Evaluation

Slide 4
Chapter Goals

Learning algorithms and implementations for both digit-at-a-time and convergence square-rooting

Chapter Highlights

Square-rooting part of IEEE 754 standard
Digit-recurrence (divisionlike) algorithms
Convergence or iterative schemes
Square-rooting not special case of division
Square-Rooting Methods: Topics

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<td>21.5 Square-Rooting by Convergence</td>
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<td>21.6 Fast Hardware Square-Rooters</td>
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</table>
21.1 The Pencil-and-Paper Algorithm

Notation for our discussion of division algorithms:

- $z$ Radicand $z_{2k-1}z_{2k-2} \ldots z_3z_2z_1z_0$
- $q$ Square root $q_{k-1}q_{k-2} \ldots q_1q_0$
- $s$ Remainder, $z - q^2$ $s_ks_{k-1}s_{k-2} \ldots s_1s_0$

Remainder range, $0 \leq s \leq 2q$ ($k + 1$ digits)

Justification: $s \geq 2q + 1$ would lead to $z = q^2 + s \geq (q + 1)^2$

Fig. 21.3 Binary square-rooting in dot notation.
Example of Decimal Square-Rooting

Check: $308^2 + 377 = 94,864 + 377 = 95,241$

\[ \begin{array}{cccc}
q_2 & q_1 & q_0 & \leftarrow q \\
\sqrt{5241} & 9 & 5 & 2 & 4 & 1 & \leftarrow z \\
9 & \frac{6q_1 \times q_1}{9} & \leq 52 \\
5 & 2 & 4 & 1 & \Rightarrow 60q_0 \times q_0 \leq 5241 \\
4 & 8 & 6 & 4 & \Rightarrow s = (377)_{\text{ten}} \\
0 & 3 & 7 & 7 & \Rightarrow q = (308)_{\text{ten}}
\end{array} \]

Root digit

Partial root

$q^{(0)} = 0$
$q^{(1)} = 3$
$q^{(2)} = 30$
$q^{(3)} = 308$

“sixty plus $q_1$”

Fig. 21.1 Extracting the square root of a decimal integer using the pencil-and-paper algorithm.
Square-Rooting as Division with Unknown Divisor

$q_3 \ q_2 \ q_1 \ q_0$

$q_3 \ q_2 \ q_1 \ q_0 \ \\ q_3 \ q_2 \ q_1 \ q_0 \ \\ z_7 \ z_6 \ z_5 \ z_4 \ z_3 \ z_2 \ z_1 \ z_0

$q_3$ depends only on $z_7 \ z_6$

Justification: For $\varepsilon \neq 0$, the square of $(q_3 + \varepsilon)r^3$ is $q_3^2r^6 + \varepsilon(2q_3 + \varepsilon)r^6$, leading to a change in $z_7 \ z_6$

Similarly, $q_2$ depends only on $z_7 \ z_6 \ z_5 \ z_4$, and so on
Root Digit Selection Rule

The root thus far is denoted by $q^{(i)} = (q_{k-1}q_{k-2} \cdots q_{k-i})_{10}$

Attaching the next digit $q_{k-i-1}$, partial root becomes $q^{(i+1)} = 10q^{(i)} + q_{k-i-1}$

The square of $q^{(i+1)}$ is $100(q^{(i)})^2 + 20q^{(i)}q_{k-i-1} + (q_{k-i-1})^2$

$100(q^{(i)})^2 = (10q^{(i)})^2$ subtracted from partial remainder in previous steps

Must subtract $(10(2q^{(i)}) + q_{k-i-1}) \times q_{k-i-1}$ to get the new partial remainder

More generally, in radix $r$, must subtract $(r(2q^{(i)}) + q_{k-i-1}) \times q_{k-i-1}$

In radix 2, must subtract $(4q^{(i)} + q_{k-i-1}) \times q_{k-i-1}$, which is

$4q^{(i)} + 1$ for $q_{k-i-1} = 1$, and 0 otherwise

Thus, we use $(q_{k-1}q_{k-2} \cdots q_{k-l}01)_{2\text{wo}}$ in a trial subtraction
Example of Binary Square-Rooting

Check: $10^2 + 18 = 118 = (0111 0110)_{\text{two}}$

<table>
<thead>
<tr>
<th>Root digit</th>
<th>Partial root</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_3 = 1$</td>
<td>$q^{(0)} = 0$</td>
</tr>
<tr>
<td>$q_2 = 0$</td>
<td>$q^{(1)} = 1$</td>
</tr>
<tr>
<td>$q_1 = 1$</td>
<td>$q^{(2)} = 10$</td>
</tr>
<tr>
<td>$q_0 = 0$</td>
<td>$q^{(3)} = 101$</td>
</tr>
</tbody>
</table>

$s = (18)_{\text{ten}}$  
$q = (1010)_{\text{two}} = (10)_{\text{ten}}$

Fig. 21.2 Extracting the square root of a binary integer using the pencil-and-paper algorithm.
21.2 Restoring Shift/Subtract Algorithm

Consistent with the IEEE 754 floating-point standard, we formulate our algorithms for a radicand in the range $1 \leq z < 4$ (after possible 1-bit shift for an odd exponent).

| $1 \leq z < 4$ | Radicand | $z_1z_0.z_{-1}z_{-2} \ldots z_{-l}$ |
| $1 \leq q < 2$ | Square root | $1.q_{-1}q_{-2} \ldots q_{-l}$ |
| $0 \leq s < 4$ | Remainder | $s_1s_0.s_{-1}s_{-2} \ldots s_{-l}$ |

Binary square-rooting is defined by the recurrence

$$s(j) = 2s(j-1) - q_j(2q^{(j-1)} + 2^{-j}q_j)$$

with $s^{(0)} = z - 1$, $q^{(0)} = 1$, $s^{(j)} = s$

where $q^{(j)}$ is the root up to its $(-j)$th digit; thus $q = q^{(l)}$.

To choose the next root digit $q_j \in \{0, 1\}$, subtract from $2s^{(j-1)}$ the value

$$2q^{(j-1)} + 2^{-j} = (1q_{-1}^{(j-1)} \cdot q_{-2}^{(j-1)} \ldots q_{-j+1}^{(j-1)} 0 1)_{\text{two}}$$

A negative trial difference means $q_j = 0$.
Finding the Sq. Root of \( z = 1.110110 \) via the Restoring Algorithm

Fig. 21.4 Example of sequential binary square-rooting using the restoring algorithm.

<table>
<thead>
<tr>
<th>( z ) (radicand = 118/64)</th>
<th>0 1 1 1 0 1 1 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^{(0)} = z - 1 )</td>
<td>0 0 0 1 1 0 1 1 0</td>
</tr>
<tr>
<td>( 2s^{(0)} )</td>
<td>0 0 1 1 0 1 1 0 0</td>
</tr>
<tr>
<td>( -[2 \times (1.)+2^{-1}] )</td>
<td>1 0 0 1</td>
</tr>
<tr>
<td>( s^{(1)} ) = ( 2s^{(0)} ) Restore</td>
<td>1 1 1 0 0 1 1 0 0</td>
</tr>
<tr>
<td>( 2s^{(1)} )</td>
<td>0 1 1 0 1 1 0 0 0</td>
</tr>
<tr>
<td>( -[2 \times (1.0)+2^{-2}] )</td>
<td>1 0 0 1</td>
</tr>
<tr>
<td>( s^{(2)} )</td>
<td>0 1 0 0 1 0 0 0 0</td>
</tr>
<tr>
<td>( 2s^{(2)} )</td>
<td>0 1 0 0 1 0 0 0 0</td>
</tr>
<tr>
<td>( -[2 \times (1.01)+2^{-3}] )</td>
<td>1 0 1 0 1</td>
</tr>
<tr>
<td>( s^{(3)} ) = ( 2s^{(2)} ) Restore</td>
<td>1 1 1 1 0 1 0 0 0</td>
</tr>
<tr>
<td>( 2s^{(3)} )</td>
<td>0 1 0 0 1 0 0 0 0</td>
</tr>
<tr>
<td>( -[2 \times (1.010)+2^{-4}] )</td>
<td>1 0 1 0 1</td>
</tr>
<tr>
<td>( s^{(4)} )</td>
<td>0 1 1 1 1 1 0 0 0</td>
</tr>
<tr>
<td>( 2s^{(4)} )</td>
<td>0 1 1 1 1 1 1 0 0</td>
</tr>
<tr>
<td>( -[2 \times (1.0101)+2^{-5}] )</td>
<td>1 0 1 0 1</td>
</tr>
<tr>
<td>( s^{(5)} )</td>
<td>0 1 0 0 1 1 1 0 0</td>
</tr>
<tr>
<td>( 2s^{(5)} )</td>
<td>0 1 0 0 1 1 1 1 0</td>
</tr>
<tr>
<td>( -[2 \times (1.01011)+2^{-6}] )</td>
<td>1 0 1 0 1 0 1</td>
</tr>
<tr>
<td>( s^{(6)} ) = ( 2s^{(5)} ) Restore</td>
<td>1 1 1 1 0 1 1 1 1</td>
</tr>
<tr>
<td>( s ) (remainder = 156/64)</td>
<td>0 0 0 0 0 1 0 0 0 0</td>
</tr>
<tr>
<td>( q ) (root = 86/64)</td>
<td>1 0 1 0 1 1 0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Root digit</th>
<th>Partial root</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_0 = 1 )</td>
<td>1.</td>
</tr>
<tr>
<td>( q_{-1} = 0 )</td>
<td>1.0</td>
</tr>
<tr>
<td>( q_{-2} = 1 )</td>
<td>1.01</td>
</tr>
<tr>
<td>( q_{-3} = 0 )</td>
<td>1.010</td>
</tr>
<tr>
<td>( q_{-4} = 1 )</td>
<td>1.0101</td>
</tr>
<tr>
<td>( q_{-5} = 1 )</td>
<td>1.01011</td>
</tr>
<tr>
<td>( q_{-6} = 0 )</td>
<td>1.010110</td>
</tr>
</tbody>
</table>

\( q_{-7} = 1 \), so round up
Hardware for Restoring Square-Rooting

Fig. 13.5 Shift/subtract sequential restoring divider (for comparison).

Fig. 21.5 Sequential shift/subtract restoring square-rooter.
Rounding the Square Root

In fractional square-rooting, the remainder is not needed.

To round the result, we can produce an extra digit $q_{-l-1}$:

- Truncate for $q_{-l-1} = 0$,
- Round up for $q_{-l-1} = 1$.

Midway case, $q_{-l-1} = 1$ followed by all 0s, impossible (Prob. 21.11)

**Example:** In Fig. 21.4, we had

$$(01.110110)_2 = (1.010111)_2^2 + (10.011100)/64$$

An extra iteration produces $q_{-7} = 1$

So the root is rounded up to $q = (1.010111)_2 = 87/64$

The rounded-up value is closer to the root than the truncated version:

- Original: $118/64 = (86/64)^2 + 156/(64)^2$
- Rounded: $118/64 = (87/64)^2 - 17/(64)^2$
21.3 Binary Nonrestoring Algorithm

As in nonrestoring division, nonrestoring square-rooting implies:

- Root digits in \{-1, 1\}
- On-the-fly conversion to binary
- Possible final correction

The case \(q_{-j} = 1\) (nonnegative partial remainder), is handled as in the restoring algorithm; i.e., it leads to the trial subtraction of

\[q_{-j}[2q^{(j-1)} + 2^{-j}q_{-j}] = 2q^{(j-1)} + 2^{-j}\]

For \(q_{-j} = -1\), we must subtract

\[q_{-j}[2q^{(j-1)} + 2^{-j}q_{-j}] = -[2q^{(j-1)} - 2^{-j}]\]

which is equivalent to adding \(2q^{(j-1)} - 2^{-j}\)

Slight complication, compared with nonrestoring division

This term cannot be formed by concatenation
Finding the Sq. Root of $z = 1.110110$ via the Nonrestoring Algorithm

Fig. 21.6 Example of nonrestoring binary square-rooting.
Some Details for Nonrestoring Square-Rooting

Depending on the sign of the partial remainder, add:

(positive) Add $2q^{(j-1)} + 2^{-j}$  Concatenate 01 to the end of $q^{(j-1)}$

(negative) Sub. $2q^{(j-1)} - 2^{-j}$  Cannot be formed by concatenation

**Solution:** We keep $q^{(j-1)}$ and $q^{(j-1)} - 2^{-j+1}$ in registers $Q$ (partial root) and $Q^*$ (diminished partial root), respectively. Then:

$q_{-j} = 1$ Subtract $2q^{(j-1)} + 2^{-j}$ formed by shifting $Q$ 01

$q_{-j} = -1$ Add $2q^{(j-1)} - 2^{-j}$ formed by shifting $Q^*$11

Updating rules for $Q$ and $Q^*$ registers:

$q_{-j} = 1 \quad \Rightarrow \quad Q := Q \ 1 \quad Q^* := Q \ 0$

$q_{-j} = -1 \quad \Rightarrow \quad Q := Q^*1 \quad Q^* := Q^*0$

Additional rule for SRT-like algorithm that allow $q_{-j} = 0$ as well:

$q_{-j} = 0 \quad \Rightarrow \quad Q := Q \ 0 \quad Q^* := Q^*1$
21.4 High-Radix Square-Rooting

Basic recurrence for fractional radix-

\[
s^{(j)} = rs^{(j-1)} - q_j(2q^{(j-1)} + r^{-j}q_j)
\]

As in radix-2 nonrestoring algorithm, we can use two registers

Q and Q* to hold \( q^{(j-1)} \) and its diminished version \( q^{(j-1)} - r^{-j+1} \), respectively, suitably updating them in each step.
An Implementation of Radix-4 Square-Rooting

$r = 4$, root digit set $[-2, 2]$

$Q^*$ holds $q^{(j-1)} - 4^{-j+1} = q^{(j-1)} - 2^{-2j+2}$. Then, one of the following values must be subtracted from, or added to, the shifted partial remainder $rs^{(j-1)}$

- $q_{-j} = 2$: Subtract $4q^{(j-1)} + 2^{-2j+2}$ double-shift $Q$ 010
- $q_{-j} = 1$: Subtract $2q^{(j-1)} + 2^{-2j}$ shift $Q$ 001
- $q_{-j} = -1$: Add $2q^{(j-1)} - 2^{-2j}$ shift $Q^*$ 111
- $q_{-j} = -2$: Add $4q^{(j-1)} - 2^{-2j+2}$ double-shift $Q^*$ 110

Updating rules for $Q$ and $Q^*$ registers:

- $q_{-j} = 2 \implies Q := Q 10 \quad Q^* := Q 01$
- $q_{-j} = 1 \implies Q := Q 01 \quad Q^* := Q 00$
- $q_{-j} = 0 \implies Q := Q 00 \quad Q^* := Q*11$
- $q_{-j} = -1 \implies Q := Q*11 \quad Q^* := Q*10$
- $q_{-j} = -2 \implies Q := Q*10 \quad Q^* := Q*01$

Note that the root is obtained in binary form (no conversion needed!)
Keeping the Partial Remainder in Carry-Save Form

As in fast division, root digit selection can be based on a few bits of the shifted partial remainder $4s^{(j-1)}$ and of the partial root $q^{(j-1)}$

This would allow us to keep $s$ in carry-save form

One extra bit of each component of $s$ (sum and carry) must be examined

Can use the same lookup table for quotient digit and root digit selection

To see how, compare recurrences for radix-4 division and square-rooting:

Division: $s^{(j)} = 4s^{(j-1)} - q_{-j}d$

Square-rooting: $s^{(j)} = 4s^{(j-1)} - q_{-j}(2q^{(j-1)} + 4^{-j}q_{-j})$

To keep magnitudes of partial remainders for division and square-rooting comparable, we can perform radix-4 square-rooting using the digit set

$\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$

Can convert from the digit set above to the digit set $[-2, 2]$, or directly to binary, with no extra computation
21.5 Square-Rooting by Convergence

Newton-Raphson method

Choose \( f(x) = x^2 - z \) with a root at \( x = \sqrt{z} \)

\[
x^{(i+1)} = x^{(i)} - \frac{f(x^{(i)})}{f'(x^{(i)})}
\]

\[
x^{(i+1)} = 0.5(x^{(i)} + z/x^{(i)})
\]

Each iteration: division, addition, 1-bit shift

Convergence is quadratic

For \( 0.5 \leq z < 1 \), a good starting approximation is \( (1 + z)/2 \)

This approximation needs no arithmetic

The error is 0 at \( z = 1 \) and has a max of 6.07\% at \( z = 0.5 \)

The hardware approximation method of Schwarz and Flynn, using the tree circuit of a fast multiplier, can provide a much better approximation (e.g., to 16 bits, needing only two iterations for 64 bits of precision)
Initial Approximation Using Table Lookup

Table-lookup can yield a better starting estimate $x^{(0)}$ for $\sqrt{z}$

For example, with an initial estimate accurate to within $2^{-8}$, three iterations suffice to increase the accuracy of the root to 64 bits

$$x^{(i+1)} = 0.5(x^{(i)} + z / x^{(i)})$$

Example 21.1: Compute the square root of $z = (2.4)_{10}$

- $x^{(0)}$ read out from table = 1.5 accurate to $10^{-1}$
- $x^{(1)} = 0.5(x^{(0)} + 2.4 / x^{(0)}) = 1.550 000 000$ accurate to $10^{-2}$
- $x^{(2)} = 0.5(x^{(1)} + 2.4 / x^{(1)}) = 1.549 193 548$ accurate to $10^{-4}$
- $x^{(3)} = 0.5(x^{(2)} + 2.4 / x^{(2)}) = 1.549 193 338$ accurate to $10^{-8}$

Check: $(1.549 193 338)^2 = 2.399 999 999$
Convergence Square-Rooting without Division

Rewrite the square-root recurrence as:

\[ x^{(i+1)} = x^{(i)} + 0.5 \frac{1}{x^{(i)}} (z - (x^{(i)})^2) = x^{(i)} + 0.5 \gamma(x^{(i)})(z - (x^{(i)})^2) \]

where \( \gamma(x^{(i)}) \) is an approximation to \( 1/x^{(i)} \) obtained by a simple circuit or read out from a table.

Because of the approximation used in lieu of the exact value of \( 1/x^{(i)} \), convergence rate will be less than quadratic.

**Alternative:** Use the recurrence above, but find the reciprocal iteratively; thus interlacing the two computations.

Using the function \( f(y) = 1/y - x \) to compute \( 1/x \), we get:

\[ x^{(i+1)} = 0.5(x^{(i)} + z/y^{(i)}) \]

\[ y^{(i+1)} = y^{(i)}(2 - x^{(i)}y^{(i)}) \]

Convergence is less than quadratic but better than linear.

3 multiplications, 2 additions, and a 1-bit shift per iteration.
Example for Division-Free Square-Rooting

\[ x^{(i+1)} = 0.5(x^{(i)} + z \cdot y^{(i)}) \]
\[ y^{(i+1)} = y^{(i)}(2 - x^{(i)} y^{(i)}) \]

Example 21.2: Compute \( \sqrt{1.4} \), beginning with \( x^{(0)} = y^{(0)} = 1 \)

\[
\begin{align*}
x^{(1)} & = 0.5(x^{(0)} + 1.4 \cdot y^{(0)}) & = 1.200\ 000\ 000 \\
y^{(1)} & = y^{(0)}(2 - x^{(0)} y^{(0)}) & = 1.000\ 000\ 000 \\
x^{(2)} & = 0.5(x^{(1)} + 1.4 \cdot y^{(1)}) & = 1.300\ 000\ 000 \\
y^{(2)} & = y^{(1)}(2 - x^{(1)} y^{(1)}) & = 0.800\ 000\ 000 \\
x^{(3)} & = 0.5(x^{(2)} + 1.4 \cdot y^{(2)}) & = 1.210\ 000\ 000 \\
y^{(3)} & = y^{(2)}(2 - x^{(2)} y^{(2)}) & = 0.768\ 000\ 000 \\
x^{(4)} & = 0.5(x^{(3)} + 1.4 \cdot y^{(3)}) & = 1.142\ 600\ 000 \\
y^{(4)} & = y^{(3)}(2 - x^{(3)} y^{(3)}) & = 0.822\ 312\ 960 \\
x^{(5)} & = 0.5(x^{(4)} + 1.4 \cdot y^{(4)}) & = 1.146\ 919\ 072 \\
y^{(5)} & = y^{(4)}(2 - x^{(4)} y^{(4)}) & = 0.872\ 001\ 394 \\
x^{(6)} & = 0.5(x^{(5)} + 1.4 \cdot y^{(5)}) & = 1.183\ 860\ 512 \approx \sqrt{1.4} \\
\end{align*}
\]

Check: \((1.183\ 860\ 512)^2 = 1.401\ 525\ 712\)
Another Division-Free Convergence Scheme

Based on computing $1/\sqrt{z}$, which is then multiplied by $z$ to obtain $\sqrt{z}$

The function $f(x) = 1/x^2 - z$ has a root at $x = 1/\sqrt{z}$ \quad (f'(x) = -2/x^3)

$$x^{(i+1)} = 0.5x^{(i)}(3 - z(x^{(i)})^2)$$

Quadratic convergence

**Example 21.3:** Compute the square root of $z = (0.5678)_{\text{ten}}$

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Equation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^{(0)}$</td>
<td>read out from table</td>
<td>1.3</td>
</tr>
<tr>
<td>$x^{(1)}$</td>
<td>$0.5x^{(0)}(3 - 0.5678(x^{(0)})^2)$</td>
<td>1.326 271 700</td>
</tr>
<tr>
<td>$x^{(2)}$</td>
<td>$0.5x^{(1)}(3 - 0.5678(x^{(1)})^2)$</td>
<td>1.327 095 128</td>
</tr>
<tr>
<td>$\sqrt{z}$</td>
<td>$z \times x^{(2)}$</td>
<td>0.753 524 613</td>
</tr>
</tbody>
</table>

Cray 2 supercomputer used this method. Initially, instead of $x^{(0)}$, the two values $1.5x^{(0)}$ and $0.5(x^{(0)})^3$ are read out from a table, requiring only 1 multiplication in the first iteration. The value $x^{(1)}$ thus obtained is accurate to within half the machine precision, so only one other iteration is needed (in all, 5 multiplications, 2 additions, 2 shifts)
21.6 Fast Hardware Square-Rooters

Combinational hardware square-rooter serve two purposes:
1. Approximation to start up or speed up convergence methods
2. Replace digit recurrence or convergence methods altogether

\[ \sqrt{z} \approx 1.5 \]
\[ \sqrt{z} \approx 1 + \frac{z}{4} \]
\[ \sqrt{z} \approx \frac{7}{8} + \frac{z}{4} \]
\[ \sqrt{z} \approx \frac{17}{24} + \frac{z}{3} \]

Best linear approx.

More subranges
Better approx in each

Fig. 21.7 Plot of the function \( \sqrt{z} \) for \( 1 \leq z < 4 \).
Nonrestoring Array Square-Rooters

Array square-rooters can be derived from the dot-notation representation in much the same way as array dividers.

Fig. 21.8   Nonrestoring array square-rooter built of controlled add/subtract cells incorporating full adders (FAs) and XOR gates.
Understanding the Array Square-Rooter Design

Partial root, transferred diagonally from row to row, is appended with:
01 if the last root digit was 1; with 11 if the last root digit was 0
Nonrestoring Array Square-Rooter in Action

Check: $118/256 = (10/16)^2 + (-3/256)$? Note that the answer is approximate (to within 1 $ulp$) due to there being no final correction.
Digit-at-a-Time Version of the Previous Example

In this example, z is $\frac{1}{4}$ of that in Fig. 21.6. Subtraction (addition) uses the term $2q + 2^{-i} (2q - 2^{-i})$.

<table>
<thead>
<tr>
<th>Root digit</th>
<th>Partial root</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_{-1}$ = 1</td>
<td>$q = .1$</td>
</tr>
<tr>
<td>$q_{-2}$ = 0</td>
<td>$q = .10$</td>
</tr>
<tr>
<td>$q_{-3}$ = 1</td>
<td>$q = .101$</td>
</tr>
<tr>
<td>$q_{-4}$ = 0</td>
<td>$q = .1010$</td>
</tr>
</tbody>
</table>

\[
z = 118/256 \quad 0.11110110
\]

\[
s^{(0)} = z \quad 0.011110110
\]

\[
2s^{(0)} \quad 0.001110110
\]

\[
-(2q + 2^{-1}) \quad 1.11
\]

\[
s^{(1)} \quad 0.0110110
\]

\[
2s^{(1)} \quad 0.001110110
\]

\[
-(2q + 2^{-2}) \quad 1.011
\]

\[
s^{(2)} \quad 1.1100110
\]

\[
2s^{(2)} \quad 1.1100110
\]

\[
+(2q - 2^{-3}) \quad 0.111
\]

\[
s^{(3)} \quad 0.0000010
\]

\[
2s^{(3)} \quad 0.0000010
\]

\[
-(2q + 2^{-4}) \quad 1.01011
\]

\[
s^{(4)} \quad 1.01101
\]
Square Rooting Is Not a Special Case of Division

**Multiplier**

\[ a \times x = p \]

**Multiplier**

\[ x \times x = x^2 \]

**Divider**

\[ z \div d = q \]

**Divider**

\[ z \div q = z^{1/2} \]

Multiplier, with both inputs connected to same value, becomes a squarer.

But, direct realization of squarer leads to simpler and faster circuit.

Divider can’t be used as square-rooter via feedback connection.

Direct square-rooter realization does not lead to simpler or faster circuit.
22 The CORDIC Algorithms

Chapter Goals

Learning a useful convergence method for evaluating trigonometric and other functions

Chapter Highlights

Basic CORDIC idea: rotate a vector with end point at \((x, y) = (1, 0)\) by the angle \(z\) to put its end point at \((\cos z, \sin z)\)

Other functions evaluated similarly

Complexity comparable to division
The CORDIC Algorithms: Topics

<table>
<thead>
<tr>
<th>Topics in This Chapter</th>
</tr>
</thead>
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<td>22.2 Basic CORDIC Iterations</td>
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<td>22.3 CORDIC Hardware</td>
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<tr>
<td>22.4 Generalized CORDIC</td>
</tr>
<tr>
<td>22.5 Using the CORDIC Method</td>
</tr>
<tr>
<td>22.6 An Algebraic Formulation</td>
</tr>
</tbody>
</table>
22.1 Rotations and Pseudorotations

Evaluation of trigonometric, hyperbolic, and other common functions, such as log and exp, is needed in many computations.

It comes as a surprise to most people that such elementary functions can be evaluated in time that is comparable to division time or a fairly small multiple of it.

Some groups advocate including these functions in IEEE 754, thus requiring that they be evaluated exactly, except for the final rounding.

Progress has been made toward such properly rounded elementary functions, but the cost of achieving this goal is still prohibitive.

CORDIC is a low-cost method that achieves the reasonable accuracy of about 1 $ulp$, but does not guarantee proper rounding.
Key Ideas on which CORDIC Is Based

If we have a computationally efficient way of rotating a vector, we can evaluate \( \cos, \sin, \) and \( \tan^{-1} \) functions.

Rotation by an arbitrary angle is difficult, so we:

- Perform pseudorotations that require simpler operations.
- Use special angles to synthesize the desired angle \( z \)

\[
Z = \alpha^{(1)} + \alpha^{(2)} + \ldots + \alpha^{(m)}
\]

COordinate Rotation Digital Computer used this method in the 1950s; modern electronic calculators also use it.

Start at \((1, 0)\)
- Rotate by \(z\)
- Get \(\cos z, \sin z\)

Start at \((1, y)\)
- Rotate until \(y = 0\)
- Rotation amount is \(\tan^{-1}y\)
Rotating a Vector \((x^{(i)}, y^{(i)})\) by the Angle \(\alpha^{(i)}\)

\[
x^{(i+1)} = x^{(i)} \cos \alpha^{(i)} - y^{(i)} \sin \alpha^{(i)} = \frac{(x^{(i)} - y^{(i)} \tan \alpha^{(i)})}{(1 + \tan^2 \alpha^{(i)})^{1/2}} \\
y^{(i+1)} = y^{(i)} \cos \alpha^{(i)} + x^{(i)} \sin \alpha^{(i)} = \frac{(y^{(i)} + x^{(i)} \tan \alpha^{(i)})}{(1 + \tan^2 \alpha^{(i)})^{1/2}} \\
z^{(i+1)} = z^{(i)} - \alpha^{(i)}
\]

Recall that \(\cos \theta = 1 / (1 + \tan^2 \theta)^{1/2}\)

**Our strategy:** Eliminate the terms \((1 + \tan^2 \alpha^{(i)})^{1/2}\) and choose the angles \(\alpha^{(i)}\) so that \(\tan \alpha^{(i)}\) is a power of 2; need two shift-adds

Fig. 22.1  A pseudorotation step in CORDIC
Pseudorotating a Vector \((x^{(i)}, y^{(i)})\) by the Angle \(\alpha^{(i)}\)

\[
\begin{align*}
x^{(i+1)} &= x^{(i)} - y^{(i)} \tan \alpha^{(i)} \\
y^{(i+1)} &= y^{(i)} + x^{(i)} \tan \alpha^{(i)} \\
z^{(i+1)} &= z^{(i)} - \alpha^{(i)}
\end{align*}
\]

**Pseudorotation:** Whereas a real rotation does not change the length \(R^{(i)}\) of the vector, a pseudorotation step increases its length to:

\[
R^{(i+1)} = R^{(i)} / \cos \alpha^{(i)} = R^{(i)} (1 + \tan^2 \alpha^{(i)})^{1/2}
\]
A Sequence of Rotations or Pseudorotations

After \( m \) real rotations by \( \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)} \), given \( x^{(0)} = x, y^{(0)} = y, \) and \( z^{(0)} = z \)

\[
\begin{align*}
x^{(m)} &= x \cos(\sum \alpha^{(i)}) - y \sin(\sum \alpha^{(i)}) \\
y^{(m)} &= y \cos(\sum \alpha^{(i)}) + x \sin(\sum \alpha^{(i)}) \\
z^{(m)} &= z - (\sum \alpha^{(i)})
\end{align*}
\]

After \( m \) pseudorotations by \( \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)} \), given \( x^{(0)} = x, y^{(0)} = y, \) and \( z^{(0)} = z \)

\[
\begin{align*}
x^{(m)} &= K(x \cos(\sum \alpha^{(i)}) - y \sin(\sum \alpha^{(i)})) \\
y^{(m)} &= K(y \cos(\sum \alpha^{(i)}) + x \sin(\sum \alpha^{(i)})) \\
z^{(m)} &= z - (\sum \alpha^{(i)})
\end{align*}
\]

where \( K = \prod (1 + \tan^2 \alpha^{(i)})^{1/2} \) is a constant if angles of rotation are always the same, differing only in sign or direction

**Question:** Can we find a set of angles so that any angle can be synthesized from all of them with appropriate signs?
22.2 Basic CORDIC Iterations

CORDIC iteration: In step $i$, we pseudorotate by an angle whose tangent is $d_i 2^{-i}$ (the angle $e^{(i)}$ is fixed, only direction $d_i$ is to be picked)

$x^{(i+1)} = x^{(i)} - d_i y^{(i)} 2^{-i}$
$y^{(i+1)} = y^{(i)} + d_i x^{(i)} 2^{-i}$
$z^{(i+1)} = z^{(i)} - d_i \tan^{-1} 2^{-i}$

$= z^{(i)} - d_i e^{(i)}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$e^{(i)}$ in degrees (approximate)</th>
<th>$e^{(i)}$ in radians (precise)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>45.0</td>
<td>0.785 398 163</td>
</tr>
<tr>
<td>1</td>
<td>26.6</td>
<td>0.463 647 609</td>
</tr>
<tr>
<td>2</td>
<td>14.0</td>
<td>0.244 978 663</td>
</tr>
<tr>
<td>3</td>
<td>7.1</td>
<td>0.124 354 994</td>
</tr>
<tr>
<td>4</td>
<td>3.6</td>
<td>0.062 418 810</td>
</tr>
<tr>
<td>5</td>
<td>1.8</td>
<td>0.031 239 833</td>
</tr>
<tr>
<td>6</td>
<td>0.9</td>
<td>0.015 623 728</td>
</tr>
<tr>
<td>7</td>
<td>0.4</td>
<td>0.007 812 341</td>
</tr>
<tr>
<td>8</td>
<td>0.2</td>
<td>0.003 906 230</td>
</tr>
<tr>
<td>9</td>
<td>0.1</td>
<td>0.001 953 123</td>
</tr>
</tbody>
</table>

Table 22.1 Value of the function $e^{(i)} = \tan^{-1} 2^{-i}$, in degrees and radians, for $0 \leq i \leq 9$

Example: $30^\circ$ angle

$30.0 \approx 45.0 - 26.6 + 14.0$
$- 7.1 + 3.6 + 1.8$
$- 0.9 + 0.4 - 0.2$
$+ 0.1$

$= 30.1$
Choosing the Angles to Force $z$ to Zero

$$x^{(i+1)} = x^{(i)} - d_i y^{(i)} 2^{-i}$$
$$y^{(i+1)} = y^{(i)} + d_i x^{(i)} 2^{-i}$$
$$z^{(i+1)} = z^{(i)} - d_i \tan^{-1} \frac{2^{-i}}{} = z^{(i)} - d_i e^{(i)}$$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$z^{(i)}$</th>
<th>$- d_i e^{(i)}$</th>
<th>$z^{(i+1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>+30.0</td>
<td>−45.0</td>
<td>+30.0</td>
</tr>
<tr>
<td>1</td>
<td>−15.0</td>
<td>+26.6</td>
<td>−11.6</td>
</tr>
<tr>
<td>2</td>
<td>+11.6</td>
<td>−14.0</td>
<td>−2.4</td>
</tr>
<tr>
<td>3</td>
<td>−2.4</td>
<td>+7.1</td>
<td>+4.7</td>
</tr>
<tr>
<td>4</td>
<td>+4.7</td>
<td>−3.6</td>
<td>+1.1</td>
</tr>
<tr>
<td>5</td>
<td>+1.1</td>
<td>−1.8</td>
<td>−0.7</td>
</tr>
<tr>
<td>6</td>
<td>−0.7</td>
<td>+0.9</td>
<td>+0.2</td>
</tr>
<tr>
<td>7</td>
<td>+0.2</td>
<td>−0.4</td>
<td>−0.2</td>
</tr>
<tr>
<td>8</td>
<td>−0.2</td>
<td>+0.2</td>
<td>+0.0</td>
</tr>
<tr>
<td>9</td>
<td>+0.0</td>
<td>−0.1</td>
<td>−0.1</td>
</tr>
</tbody>
</table>

Fig. 22.2 The first three of 10 pseudorotations leading from $(x^{(0)}, y^{(0)})$ to $(x^{(10)}, 0)$ in rotating by $+30^\circ$.

Table 22.2 Choosing the signs of the rotation angles in order to force $z$ to 0.
Why Any Angle Can Be Formed from Our List

**Analogy:** Paying a certain amount while using all currency denominations (in positive or negative direction) exactly once; red values are fictitious.

- $20  $10  $5  $3  $2  $1  $.50  $.25  $.20  $.10  $.05  $.03  $.02  $.01

**Example:** Pay $12.50

$20 – $10 + $5 – $3 + $2 – $1 – $.50 + $.25 – $.20 – $.10 + $.05 + $.03 – $.02 – $.01

Convergence is possible as long as each denomination is no greater than the sum of all denominations that follow it.

Domain of convergence: −$42.16 to +$42.16

We can guarantee convergence with actual denominations if we allow multiple steps at some values:

- $20  $10  $5  $2  $2  $1  $.50  $.25  $.10  $.10  $.05  $.01  $.01  $.01  $.01

**Example:** Pay $12.50

$20 – $10 + $5 – $2 – $2 + $1 + $.50 + $.25 – $.10 – $.10 – $.05 + $.01 – $.01 + $.01 – $.01

We will see later that in hyperbolic CORDIC, convergence is guaranteed only if certain “angles” are used twice.
Angle Recoding

The selection of angles during pseudorotations can be viewed as recoding the angle in a specific number system.

For example, an angle of $30^\circ$ is recoded as the following digit string, with each digit being 1 or -1:

$$
\begin{array}{cccccccccccc}
45.0 & 26.6 & 14.0 & 7.1 & 3.6 & 1.8 & 0.9 & 0.4 & 0.2 & 0.1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1
\end{array}
$$

The money-exchange analogy also lends itself to this recoding view.

For example, a payment of $12.50 is recoded as:

$$
\begin{array}{cccccccccccc}
$20 & $10 & $5 & $3 & $2 & $1 & $.50 & $.25 & $.20 & $.10 & $.05 & $.03 & $.02 & $.01 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1
\end{array}
$$
Using CORDIC in Rotation Mode

\[
\begin{align*}
    x^{(i+1)} &= x^{(i)} - d_i \cdot y^{(i)} \cdot 2^{-i} \\
    y^{(i+1)} &= y^{(i)} + d_i \cdot x^{(i)} \cdot 2^{-i} \\
    z^{(i+1)} &= z^{(i)} - d_i \cdot \tan^{-1} 2^{-i} \\
    &= z^{(i)} - d_i \cdot e^{(i)}
\end{align*}
\]

Make \( z \) converge to 0 by choosing \( d_i = \text{sign}(z^{(i)}) \)

For \( k \) bits of precision in results, \( k \) CORDIC iterations are needed, because \( \tan^{-1} 2^{-i} \approx 2^{-i} \) for large \( i \)

\[
\begin{align*}
    x^{(m)} &= K(x \cos z - y \sin z) \\
    y^{(m)} &= K(y \cos z + x \sin z) \\
    z^{(m)} &= 0
\end{align*}
\]

where \( K = 1.646\ 760\ 258\ 121 \ldots \)

Start with \( x = 1/K = 0.607\ 252\ 935 \ldots \) and \( y = 0 \) to find \( \cos z \) and \( \sin z \)

Convergence of \( z \) to 0 is possible because each of the angles in our list is more than half the previous one or, equivalently, each is less than the sum of all the angles that follow it

Domain of convergence is \(-99.7^\circ \leq z \leq 99.7^\circ\), where 99.7\(^\circ\) is the sum of all the angles in our list; the domain contains \([-\pi/2, \pi/2]\) radians
Using CORDIC in Vectoring Mode

\[ x^{(i+1)} = x^{(i)} - d_i y^{(i)} 2^{-i} \]
\[ y^{(i+1)} = y^{(i)} + d_i x^{(i)} 2^{-i} \]
\[ z^{(i+1)} = z^{(i)} - d_i \tan^{-1} 2^{-i} \]
\[ = z^{(i)} - d_i e^{(i)} \]

Make \( y \) converge to 0 by choosing 
\[ d_i = - \operatorname{sign}(x^{(i)} y^{(i)}) \]

For \( k \) bits of precision in results, 
\( k \) CORDIC iterations are needed, 
because \( \tan^{-1} 2^{-i} \approx 2^{-i} \) for large \( i \)

\[ x^{(m)} = K(x^2 + y^2)^{1/2} \]
\[ y^{(m)} = 0 \]
\[ z^{(m)} = z + \tan^{-1}(y/x) \]

where \( K = 1.646760258121 \ldots \)

Start with 
\( x = 1 \) and \( z = 0 \) 
to find \( \tan^{-1} y \)

Even though the computation above always converges, one can use the relationship \( \tan^{-1}(1/y) = \pi/2 - \tan^{-1} y \) to limit the range of fixed-point numbers encountered

Other trig functions: \( \tan z \) obtained from \( \sin z \) and \( \cos z \) via division; 
inverse sine and cosine (\( \sin^{-1} z \) and \( \cos^{-1} z \)) discussed later
22.3 CORDIC Hardware

Fig. 22.3 Hardware elements needed for the CORDIC method.

\[ x^{(i+1)} = x^{(i)} - d_i y^{(i)} 2^{-i} \]

\[ y^{(i+1)} = y^{(i)} + d_i x^{(i)} 2^{-i} \]

\[ z^{(i+1)} = z^{(i)} - d_i \tan^{-1} 2^{-i} \]

= \[ z^{(i)} - d_i e^{(i)} \]

If very high speed is not needed (as in a calculator), a single adder and one shifter would suffice.

\[ k \] table entries for \[ k \] bits of precision.
22.4 Generalized CORDIC

\[
\begin{align*}
x^{(i+1)} &= x^{(i)} - \mu_d y^{(i)} 2^{-i} \\
y^{(i+1)} &= y^{(i)} + d_i x^{(i)} 2^{-i} \\
z^{(i+1)} &= z^{(i)} - d_i e^{(i)}
\end{align*}
\]

- \(\mu = 1\) Circular rotations (basic CORDIC) \(e^{(i)} = \tan^{-1} 2^{-i}\)
- \(\mu = 0\) Linear rotations \(e^{(i)} = 2^{-i}\)
- \(\mu = -1\) Hyperbolic rotations \(e^{(i)} = \tanh^{-1} 2^{-i}\)

Fig. 22.4 Circular, linear, and hyperbolic CORDIC.
22.5 Using the CORDIC Method

\[ x(i+1) = x(i) - \mu \, d_i \, y(i) \, 2^{-i} \]
\[ y(i+1) = y(i) + d_i \, x(i) \, 2^{-i} \]
\[ z(i+1) = z(i) - d_i \, e(i) \]

\[ \mu \in \{-1, 0, 1\} \]
\[ d_i \in \{-1, 1\} \]
\[ K = 1.646760258121... \]
\[ 1/K = .607252935009... \]
\[ K' = .8281593609602... \]
\[ 1/K' = 1.207497067763... \]

### Table: Summary of Generalized CORDIC Algorithms

<table>
<thead>
<tr>
<th>Mode</th>
<th>Rotation: ( d_i = \text{sign}(z(i)), \ z^{(i)} \to 0 )</th>
<th>Vectoring: ( d_i = -\text{sign}(x(i) , y^{(i)}), \ y^{(i)} \to 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu = 1 ) Circular</td>
<td>( x \to K(x \cos z - y \sin z) )</td>
<td>( x \to K \sqrt{x^2 + y^2} )</td>
</tr>
<tr>
<td></td>
<td>( y \to K(y \cos z + x \sin z) )</td>
<td>( y \to 0 )</td>
</tr>
<tr>
<td></td>
<td>( z \to 0 )</td>
<td>( z + \tan^{-1}(y/x) )</td>
</tr>
</tbody>
</table>
| \( e^{(i)} = \tan^{-1}2^{-i} \) | For cos & sin, set \( x = 1/K, y = 0 \) \[ \tan z = \sin z / \cos z \] | For \( \tan^{-1} \), set \( x = 1, z = 0 \) \[ \cos^{-1}w = \tan^{-1}[\sqrt{1 - w^2}/w] \]
| \( \mu = 0 \) Linear | \( x \to x \) | \( x \to \sqrt{x^2 - y^2} \) |
| | \( y \to y + xz \) | \( y \to 0 \) |
| | \( z \to 0 \) | \( z + y/x \) |
| \( e^{(i)} = 2^{-i} \) | For multiplication, set \( y = 0 \) | For division, set \( z = 0 \) |
| \( \mu = -1 \) Hyperbolic | \( x \to K'(x \cosh z - y \sinh z) \) | \( x \to K' \sqrt{x^2 - y^2} \) |
| | \( y \to K'(y \cosh z + x \sinh z) \) | \( y \to 0 \) |
| | \( z \to 0 \) | \( z + \tanh^{-1}(y/x) \) |
| \( e^{(i)} = \tanh^{-1}2^{-i} \) | For cosh & sinh, set \( x = 1/K', y = 0 \) \[ \tanh z = \sinh z / \cosh z \] | For \( \tanh^{-1} \), set \( x = 1, z = 0 \) \[ \ln w = 2 \tanh^{-1}[(w - 1)/(w + 1)] \]
| | \[ \exp(z) = \sinh z + \cosh z \] | \[ \gamma w = \sqrt{(w + 1/4)^2 - (w - 1/4)^2} \] |
| | \[ w^t = \exp(t \ln w) \] | \[ \cosh^{-1}w = \ln(w + \sqrt{1 - w^2}) \] |
| \( \mu \in \{-1, 0, 1\} \) | \[ \sinh^{-1}w = \ln(w + \sqrt{1 + w^2}) \] | |

Note → In executing the iterations for \( \mu = -1 \), steps 4, 13, 40, 121, ... \( j \), \( 3j + 1 \), ... must be repeated. These repetitions are incorporated in the constant \( K' \) below.
CORDIC Speedup Methods

Skipping some rotations
Must keep track of expansion via the recurrence:
\[(K^{(i+1)})^2 = (K^{(i)})^2 (1 \pm 2^{-2^i})\]
This additional work makes variable-factor CORDIC less cost-effective than constant-factor CORDIC

Early termination
Do the first \(k/2\) iterations as usual, then combine the remaining \(k/2\) into a single multiplicative step:

For very small \(z\), we have \(\tan^{-1} z \approx z \approx \tan z\)
Expansion factor not an issue because contribution of the ignored terms is provably less than \(ulp\)

High-radix CORDIC
The hardware for the radix-4 version of CORDIC is quite similar to Fig. 22.3
22.6 An Algebraic Formulation

Because

\[ \cos z + j \sin z = e^{jz} \quad \text{where} \quad j = \sqrt{-1} \]

\( \cos z \) and \( \sin z \) can be computed via evaluating the complex exponential function \( e^{jz} \)

This leads to an alternate derivation of CORDIC iterations

Details in the text
23 Variations in Function Evaluation

Chapter Goals

Learning alternate computation methods (convergence and otherwise) for some functions computable through CORDIC

Chapter Highlights

Reasons for needing alternate methods:
Achieve higher performance or precision
Allow speed/cost tradeoffs
Optimizations, fit to diverse technologies
# Variations in Function Evaluation: Topics

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<th>Topics in This Chapter</th>
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<td>23.2 Computing Logarithms</td>
</tr>
<tr>
<td>23.3 Exponentiation</td>
</tr>
<tr>
<td>23.4 Division and Square-Rooting, Again</td>
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23.1 Normalization and Range Reduction

\[ u^{(i+1)} = f(u^{(i)}, v^{(i)}) \quad \text{Constant} \quad u^{(i+1)} = f(u^{(i)}, v^{(i)}, w^{(i)}) \]
\[ v^{(i+1)} = g(u^{(i)}, v^{(i)}) \quad \text{Desired} \quad v^{(i+1)} = g(u^{(i)}, v^{(i)}, w^{(i)}) \]
\[ w^{(i+1)} = h(u^{(i)}, v^{(i)}, w^{(i)}) \]

Guide the iteration such that one of the values converges to a constant (usually 0 or 1); this is known as *normalization*

The other value then converges to the desired function

*Additive normalization*: Normalize \( u \) via addition of terms to it

*Multiplicative normalization*: Normalize \( u \) via multiplication of terms

Additive normalization is more desirable, unless the multiplicative terms are of the form \( 1 \pm 2^a \) (shift-add) or multiplication leads to much faster convergence compared with addition
Convergence Methods You Already Know

**CORDIC**  
Example of additive normalization

\[
x^{(i+1)} = x^{(i)} - \mu d_i y^{(i)} 2^{-i}
\]
\[
y^{(i+1)} = y^{(i)} + d_i x^{(i)} 2^{-i}
\]
\[
z^{(i+1)} = z^{(i)} - d_i e^{(i)}
\]

Force \(y\) or \(z\) to 0 by adding terms to it

Force \(d\) to 1 by multiplying terms with it

\[
d^{(i+1)} = d^{(i)} (2 - d^{(i)})
\]
Set \(d^{(0)} = d\); iterate until \(d^{(m)} \approx 1\)

\[
z^{(i+1)} = z^{(i)} (2 - d^{(i)})
\]
Set \(z^{(0)} = z\); obtain \(z/d = q \approx z^{(m)}\)

**Division by repeated multiplications**  
Example of multiplicative normalization
Range Reduction

\[-2\pi \quad -3\pi/2 \quad -\pi \quad -\pi/2 \quad 0 \quad \pi/2 \quad \pi \quad 3\pi/2 \quad 2\pi\]

CORDIC’s conv. domain
-99.7° to 99.7°

\[\cos(z - \pi) = -\cos z\]
Adding \(\pi\) to the argument flips the function sign

\[\cos(2\pi + z) = \cos z\]
 Subtracting multiples of 2\(\pi\) from the argument does not change the function value

Must be careful: A slight error in the value of \(\pi\) is amplified when a large multiple of 2\(\pi\) is added to, or subtracted from, the argument

Example: Compute \(\cos(1.125 \times 2^{47})\)

Additive range reduction: see the CORDIC example above

Multiplicative range reduction: applicable to the log function, e.g.
23.2 Computing Logarithms

\begin{align*}
x^{(i+1)} &= x^{(i)} c^{(i)} = x^{(i)} (1 + d_i 2^{-i}) & \text{Force } x^{(m)} \text{ to 1} \\
y^{(i+1)} &= y^{(i)} - \ln c^{(i)} = y^{(i)} - \ln (1 + d_i 2^{-i}) & y^{(m)} \text{ converges to } y + \ln x
\end{align*}

Read out from table

Why does this multiplicative normalization method work?

\begin{align*}
x^{(m)} &= x \prod c^{(i)} \approx 1 \quad \Rightarrow \quad \prod c^{(i)} \approx 1/x \\
y^{(m)} &= y - \sum \ln c^{(i)} = y - \ln (\prod c^{(i)}) = y - \ln(1/x) \approx y + \ln x
\end{align*}

Convergence domain: \(1/\prod (1 + 2^{-i}) \leq x \leq 1/\prod (1 - 2^{-i})\) or \(0.21 \leq x \leq 3.45\)

Number of iterations: \(k\), for \(k\) bits of precision; for large \(i\), \(\ln(1 \pm 2^{-i}) \approx \pm 2^{-i}\)

Use directly for \(x \in [1, 2)\). For \(x = 2^q s\), we have:
\[
\ln x = q \ln 2 + \ln s = 0.693147180 q + \ln s
\]

Radix-4 version can be devised
Computing Binary Logarithms via Squaring

For $x \in [1, 2)$, $\log_2 x$ is a fractional number $y = (\cdot y_{-1} y_{-2} y_{-3} \ldots y_{-l})_{\text{two}}$

$x = 2^y = 2^{(\cdot y_{-1} y_{-2} y_{-3} \ldots y_{-l})_{\text{two}}}$

$x^2 = 2^{2y} = 2^{(y_{-1} y_{-2} y_{-3} \ldots y_{-l})_{\text{two}}} \implies y_{-1} = 1 \text{ iff } x^2 \geq 2$

Once $y_{-1}$ has been determined, if $y_{-1} = 0$, we are back at the original situation; otherwise, divide both sides of the equation above by 2 to get:

$x^{2/2} = 2^{(1 \cdot y_{-2} y_{-3} \ldots y_{-l})_{\text{two}} /2} = 2^{(y_{-2} y_{-3} \ldots y_{-l})_{\text{two}}}$

Generalization to base $b$:

$x = b^{(\cdot y_{-1} y_{-2} y_{-3} \ldots y_{-l})_{\text{two}}}$

$y_{-1} = 1 \text{ iff } x^2 \geq b$

**Fig. 23.1** Hardware elements needed for computing $\log_2 x$. 

---

May 2015

Computer Arithmetic, Function Evaluation
23.3 Exponentiation

Computing $e^x$

Read out from table

$x^{(i+1)} = x^{(i)} - \ln c^{(i)} = x^{(i)} - \ln(1 + d_i 2^{-i})$

Force $x^{(m)}$ to 0

$y^{(i+1)} = y^{(i)} c^{(i)} = y^{(i)} (1 + d_i 2^{-i})$

$y^{(m)}$ converges to $y e^x$

$d_i \in \{-1, 0, 1\}$

Why does this additive normalization method work?

$x^{(m)} = x - \sum \ln c^{(i)} \approx 0 \quad \Rightarrow \quad \sum \ln c^{(i)} \approx x$

$y^{(m)} = y \prod c^{(i)} = y \exp(\ln \prod c^{(i)}) = y \exp(\sum \ln c^{(i)}) \approx y e^x$

Convergence domain: $\sum \ln (1 - 2^{-i}) \leq x \leq \sum \ln (1 + 2^{-i})$ or $-1.24 \leq x \leq 1.56$

Number of iterations: $k$, for $k$ bits of precision; for large $i$, $\ln(1 \pm 2^{-i}) \approx \pm 2^{-i}$

Can eliminate half the iterations because

$\ln(1 + \varepsilon) = \varepsilon - \varepsilon^2/2 + \varepsilon^3/3 - \ldots \approx \varepsilon$ for $\varepsilon^2 < \text{ulp}$

and we may write $y^{(k)} = y^{(k/2)} (1 + x^{(k/2)})$

Radix-4 version can be devised
General Exponentiation, or Computing $x^y$

$x^y = (e^{\ln x})^y = e^{y \ln x}$  

So, compute natural log, multiply, exponentiate

**Method is prone to inaccuracies**

When $y$ is an integer, we can exponentiate by repeated multiplication (need to consider only positive $y$; for negative $y$, compute reciprocal)

In particular, when $y$ is a constant, the methods used are reminiscent of multiplication by constants (Section 9.5)

**Example:** $x^{25} = (((((x)^2x)^2)^2x)^2x)^2$  
[4 squarings and 2 multiplications]

Noting that $25 = (1 1 0 0 1)_2$, leads to a general procedure

**Computing $x^y$, when $y$ is an unsigned integer**

- Initialize the partial result to 1
- Scan the binary representation of $y$, starting at its MSB, and repeat
  - If the current bit is 1, multiply the partial result by $x$
  - If the current bit is 0, do not change the partial result
- Square the partial result before the next step (if any)
Faster Exponentiation via Recoding

**Example:**  \( x^{31} = (((((x^2)x)^2)x)^2)x \)  [4 squarings and 4 multiplications]

Note that \( 31 = (1 1 1 1 1)_{\text{two}} = (1 0 0 0 0 -1)_{\text{two}} \)

\[ x^{31} = (((((x^2)^2)^2)^2) / x \]  [5 squarings and 1 division]

**Computing** \( x^y \), when \( y \) is an integer encoded in BSD format

Initialize the partial result to 1
Scan the binary representation of \( y \), starting at its MSB, and repeat
If the current digit is 1, multiply the partial result by \( x \)
If the current digit is 0, do not change the partial result
If the current digit is \(-1\), divide the partial result by \( x \)
Square the partial result before the next step (if any)

**Radix-4 example:**  \( 31 = (1 1 1 1 1)_{\text{two}} = (1 0 0 0 0 0 -1)_{\text{two}} = (2 0 -1)_{\text{four}} \)

\[ x^{31} = (((((x^4)^4)^4) / x \]  [Can you formulate the general procedure?]
23.4 Division and Square-Rooting, Again

Computing \( q = z/d \)  

\[
s^{(i+1)} = s^{(i)} - \gamma^{(i)} d \\
q^{(i+1)} = q^{(i)} + \gamma^{(i)}
\]

In digit-recurrence division, \( \gamma^{(i)} \) is the next quotient digit and the addition for \( q \) turns into concatenation; more generally, \( \gamma^{(i)} \) can be any estimate for the difference between the partial quotient \( q^{(i)} \) and the final quotient \( q \).

Because \( s^{(i)} \) becomes successively smaller as it converges to 0, scaled versions of the recurrences above are usually preferred. In the following, \( s^{(i)} \) stands for \( s^{(i)} r^i \) and \( q^{(i)} \) for \( q^{(i)} r^i \):

\[
s^{(i+1)} = rs^{(i)} - \gamma^{(i)} d \\
q^{(i+1)} = rq^{(i)} + \gamma^{(i)}
\]

Set \( s^{(0)} = z \) and keep \( s^{(i)} \) bounded

Set \( q^{(0)} = 0 \) and find \( q^* = q^{(m)} r^{-m} \)

In the scaled version, \( \gamma^{(i)} \) is an estimate for \( r(r^{i-m} q - q^{(i)}) = r^i q^* - q^{(i)} \), where \( q^* = r^{-m} q \) represents the true quotient.
Square-Rooting via Multiplicative Normalization

Idea: If \( z \) is multiplied by a sequence of values \( (c^{(i)})^2 \), chosen so that the product \( z \prod (c^{(i)})^2 \) converges to 1, then \( z \prod c^{(i)} \) converges to \( \sqrt{z} \)

\[
x^{(i+1)} = x^{(i)} (1 + d_i 2^{-i})^2 = x^{(i)} (1 + 2d_i 2^{-i} + d_i^2 2^{-2i})
\]

\( x^{(0)} = z, \quad x^{(m)} \approx 1 \)

\[
y^{(i+1)} = y^{(i)} (1 + d_i 2^{-i})
\]

\( y^{(0)} = z, \quad y^{(m)} \approx \sqrt{z} \)

What remains is to devise a scheme for choosing \( d_i \) values in \( \{-1, 0, 1\} \)

\( d_i = 1 \) for \( x^{(i)} < 1 - \varepsilon = 1 - \alpha 2^{-i} \)

\( d_i = -1 \) for \( x^{(i)} > 1 + \varepsilon = 1 + \alpha 2^{-i} \)

To avoid the need for comparison with a different constant in each step, a scaled version of the first recurrence is used in which \( u^{(i)} = 2^i (x^{(i)} - 1) \):

\[
u^{(i+1)} = 2(u^{(i)} + 2d_i) + 2^{-i+1} (2d_i u^{(i)} + d_i^2) + 2^{-2i+1} d_i^2 u^{(i)}
\]

\( u^{(0)} = z - 1, \quad u^{(m)} \approx 0 \)

\[
y^{(i+1)} = y^{(i)} (1 + d_i 2^{-i})
\]

\( y^{(0)} = z, \quad y^{(m)} \approx \sqrt{z} \)

Radix-4 version can be devised: Digit set \([-2, 2]\) or \([-1, -\frac{1}{2}, 0, \frac{1}{2}, 1]\)
Square-Rooting via Additive Normalization

Idea: If a sequence of values $c^{(i)}$ can be obtained such that $z - (\sum c^{(i)})^2$ converges to 0, then $\sum c^{(i)}$ converges to $\sqrt{z}$

$$x^{(i+1)} = z - (y^{(i+1)})^2 = z - (y^{(i)} + c^{(i)})^2 = x^{(i)} + 2d_i y^{(i)} 2^{-i} - d_i^2 2^{-2i} \quad x^{(0)} = z, \ x^{(m)} \approx 0$$

$$y^{(i+1)} = y^{(i)} + c^{(i)} = y^{(i)} - d_i 2^{-i} \quad y^{(0)} = 0, \ y^{(m)} \approx \sqrt{z}$$

What remains is to devise a scheme for choosing $d_i$ values in \{-1, 0, 1\}

$d_i = 1$ for $x^{(i)} < -\varepsilon = -\alpha 2^{-i}$

$d_i = -1$ for $x^{(i)} > +\varepsilon = +\alpha 2^{-i}$

To avoid the need for comparison with a different constant in each step, a scaled version of the first recurrence may be used in which $u^{(i)} = 2^i x^{(i)}$:

$$u^{(i+1)} = 2(u^{(i)} + 2d_i y^{(i)} - d_i^2 2^{-i}) \quad u^{(0)} = z, \ u^{(i)} \text{ bounded}$$

$$y^{(i+1)} = y^{(i)} - d_i 2^{-i} \quad y^{(0)} = 0, \ y^{(m)} \approx \sqrt{z}$$

Radix-4 version can be devised: Digit set \{-2, 2\} or \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}
23.5 Use of Approximating Functions

Convert the problem of evaluating the function $f$ to that of function $g$ approximating $f$, perhaps with a few pre- and postprocessing operations.

Approximating polynomials need only additions and multiplications.

Polynomial approximations can be derived from various schemes.

The Taylor-series expansion of $f(x)$ about $x = a$ is

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!} (x - a)^j$$

The error due to omitting terms of degree $> m$ is:

$$f^{(m+1)}(a + \mu(x - a)) \frac{(x - a)^{m+1}}{(m + 1)!}$$

Setting $a = 0$ yields the Maclaurin-series expansion

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j$$

and its corresponding error bound:

$$f^{(m+1)}(\mu x) \frac{x^{m+1}}{(m + 1)!}$$

Efficiency in computation can be gained via Horner’s method and incremental evaluation.
### Some Polynomial Approximations (Table 23.1)

<table>
<thead>
<tr>
<th>Func</th>
<th>Polynomial approximation</th>
<th>Conditions</th>
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<tbody>
<tr>
<td>$1/x$</td>
<td>$1 + y + y^2 + y^3 + \cdots + y^i + \cdots$</td>
<td>$0 &lt; x &lt; 2, \ y = 1 - x$</td>
</tr>
<tr>
<td>$e^{x}$</td>
<td>$1 + x/1! + x^2/2! + x^3/3! + \cdots + x^i/i! + \cdots$</td>
<td></td>
</tr>
<tr>
<td>$\ln x$</td>
<td>$-y - y^2/2 - y^3/3 - y^4/4 - \cdots - y^i/i - \cdots$</td>
<td>$0 &lt; x \leq 2, \ y = 1 - x$</td>
</tr>
<tr>
<td>$\ln x$</td>
<td>$2[z + z^3/3 + z^5/5 + \cdots + z^{2i+1}/(2i+1) + \cdots]$</td>
<td>$x &gt; 0, \ z = \frac{x-1}{x+1}$</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$x - x^3/3! + x^5/5! - x^7/7! + \cdots + (-1)^i x^{2i+1}/(2i+1)! + \cdots$</td>
<td></td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$1 - x^2/2! + x^4/4! - x^6/6! + \cdots + (-1)^i x^{2i}/(2i)! + \cdots$</td>
<td></td>
</tr>
<tr>
<td>$\tan^{-1} x$</td>
<td>$x - x^3/3 + x^5/5 - x^7/7 + \cdots + (-1)^i x^{2i+1}/(2i+1) + \cdots$</td>
<td>$-1 &lt; x &lt; 1$</td>
</tr>
<tr>
<td>$\sinh x$</td>
<td>$x + x^3/3! + x^5/5! + x^7/7! + \cdots + x^{2i+1}/(2i+1)! + \cdots$</td>
<td></td>
</tr>
<tr>
<td>$\cosh x$</td>
<td>$1 + x^2/2! + x^4/4! + x^6/6! + \cdots + x^{2i}/(2i)! + \cdots$</td>
<td></td>
</tr>
<tr>
<td>$\tanh^{-1} x$</td>
<td>$x + x^3/3 + x^5/5 + x^7/7 + \cdots + x^{2i+1}/(2i+1) + \cdots$</td>
<td>$-1 &lt; x &lt; 1$</td>
</tr>
</tbody>
</table>
Function Evaluation via Divide-and-Conquer

Let $x$ in $[0, 4)$ be the $(l + 2)$-bit significand of a floating-point number or its shifted version. Divide $x$ into two chunks $x_H$ and $x_L$:

$$x = x_H + 2^{-t} x_L$$

$0 \leq x_H < 4$  \hspace{1cm} $t + 2$ bits

$0 \leq x_L < 1$  \hspace{1cm} $l - t$ bits

The Taylor-series expansion of $f(x)$ about $x = x_H$ is

$$f(x) = \sum_{j=0}^{\infty} f^{(j)}(x_H) (2^{-t} x_L)^j / j!$$

A linear approximation is obtained by taking only the first two terms

$$f(x) \approx f(x_H) + 2^{-t} x_L f'(x_H)$$

If $t$ is not too large, $f$ and/or $f'$ (and other derivatives of $f$, if needed) can be evaluated via table lookup.
Approximation by the Ratio of Two Polynomials

Example, yielding good results for many elementary functions

\[ f(x) \approx \frac{a(5)x^5 + a(4)x^4 + a(3)x^3 + a(2)x^2 + a(1)x + a(0)}{b(5)x^5 + b(4)x^4 + b(3)x^3 + b(2)x^2 + b(1)x + b(0)} \]

Using Horner’s method, such a “rational approximation” needs 10 multiplications, 10 additions, and 1 division
23.6 Merged Arithmetic

Our methods thus far rely on word-level building-block operations such as addition, multiplication, shifting, . . .

Sometimes, we can compute a function of interest directly without breaking it down into conventional operations

Example: merged arithmetic for inner product computation

\[ z = z^{(0)} + x^{(1)}y^{(1)} + x^{(2)}y^{(2)} + x^{(3)}y^{(3)} \]

Fig. 23.2 Merged-arithmetic computation of an inner product followed by accumulation.
Example of Merged Arithmetic Implementation

Example: Inner product computation

\[ z = z^{(0)} + x^{(1)}y^{(1)} + x^{(2)}y^{(2)} + x^{(3)}y^{(3)} \]

Fig. 23.3 Tabular representation of the dot matrix for inner-product computation and its reduction.

\[
\begin{array}{cccccccccc}
1 & 4 & 7 & 10 & 13 & 10 & 7 & 4 & \text{16 FAs} \\
2 & 4 & 6 & 8 & 8 & 6 & 4 & 2 & \text{10 FAs + 1 HA} \\
3 & 4 & 4 & 6 & 6 & 3 & 3 & 1 & \text{9 FAs} \\
1 & 2 & 3 & 4 & 4 & 3 & 2 & 1 & \text{1 HA} \\
1 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & \text{1 HA} \\
2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & \text{5-bit CPA}
\end{array}
\]
Another Merged Arithmetic Example

Approximation of reciprocal ($1/x$) and reciprocal square root ($1/\sqrt{x}$) functions with 29-30 bits of precision, so that a long floating-point result can be obtained with just one iteration at the end [Pine02]

$$f(x) = c + bv + av^2$$

2 adds
2 mult's
1 square

Comparable to a multiplier
Chapter Goals

Learning table lookup techniques for flexible and dense VLSI realization of arithmetic functions

Chapter Highlights

We have used tables to simplify or speedup \( q \) digit selection, convergence methods, . . . . Now come tables as primary computational mechanisms (as stars, not supporting cast)
## Arithmetic by Table Lookup: Topics

### Topics in This Chapter

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<th>Topic</th>
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<td>24.2</td>
<td>Binary-to-Unary Reduction</td>
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<td>Interpolating Memory</td>
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<td>Piecewise Lookup Tables</td>
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<td>24.6</td>
<td>Multipartite Table Methods</td>
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</table>
24.1 Direct and Indirect Table Lookup

Fig. 24.1 Direct table lookup versus table-lookup with pre- and post-processing.
Tables in Supporting and Primary Roles

Tables are used in two ways:

- In supporting role, as in initial estimate for division
- As main computing mechanism

Boundary between two uses is fuzzy

- Pure logic
- Hybrid solutions
- Pure tabular

Previously, we started with the goal of designing logic circuits for particular arithmetic computations and ended up using tables to facilitate or speed up certain steps.

Here, we aim for a tabular implementation and end up using peripheral logic circuits to reduce the table size.

Some solutions can be derived starting at either endpoint.
24.2 Binary-to-Unary Reduction

**Strategy:** Reduce the table size by using an auxiliary unary function to evaluate a desired binary function

**Example 1:** Addition/subtraction in a logarithmic number system; i.e., finding $L_z = \log(x \pm y)$, given $L_x$ and $L_y$

**Solution:** Let $\Delta = L_y - L_x$

\[
L_z = \log(x \pm y) \\
= \log(x (1 \pm y/x)) \\
= \log x + \log(1 \pm y/x) \\
= L_x + \log(1 \pm \log^{-1} \Delta)
\]

Pre-process $\phi^+$ table, $\phi^-$ table

Postprocess $L_x + \phi^+(\Delta)$, $L_x + \phi^-(\Delta)$

$L_z$
Another Example of Binary-to-Unary Reduction

**Example 2:** Multiplication via squaring, $xy = (x + y)^2/4 - (x - y)^2/4$

**Simplification and implementation details**

If $x$ and $y$ are $k$ bits wide, $x + y$ and $x - y$ are $k + 1$ bits wide, leading to two tables of size $2^{k+1} \times 2^k$ (total table size = $2^{k+3} \times k$ bits)

$$(x \pm y)/2 = \lfloor (x \pm y)/2 \rfloor + \varepsilon/2 \quad \varepsilon \in \{0, 1\} \text{ is the LSB}$$

$$(x + y)^2/4 - (x - y)^2/4$$

$$= \left[ \lfloor (x + y)/2 \rfloor + \varepsilon/2 \right]^2 - \left[ \lfloor (x - y)/2 \rfloor + \varepsilon/2 \right]^2$$

$$= \left[ (x + y)/2 \right]^2 - \left[ (x - y)/2 \right]^2 + \varepsilon y$$

Pre-process: compute $x + y$ and $x - y$; drop their LSBs

Table lookup: consult two squaring table(s) of size $2^k \times (2k - 1)$

Post-process: carry-save adder, followed by carry-propagate adder

Can be realized with one adder and one table

Fig. 24.2
24.3 Tables in Bit-Serial Arithmetic

Fig. 24.3 Bit-serial ALU with two tables implemented as multiplexers.

From Memory (64 Kb)

8-bit opcode (f truth table)

Carry bit for addition

Specified by 16-bit addresses

f opcode

Mux

Specified by 2-bit address

3 bits specify a flag and a value to conditionalize the operation

To Memory

Used in Connection Machine 2, an MPP introduced in 1987

Replaces a in memory

Specified by 2-bit address

8-bit opcode (g truth table)

Sum bit for addition
Other Table-Based Bit-Serial Arithmetic Examples

See Section 4.3: Conversion from binary/decimal to RNS

Evaluation of linear expressions (assume unsigned values)

\[ z = ax + by = a \sum x_i 2^i + b \sum y_i 2^i = \sum (ax_i + by_i) 2^i \]

![Diagram of bit-serial evaluation](image)

Fig. 24.4 Bit-serial evaluation of \( z = ax + by \).
24.4 Interpolating Memory

**Linear interpolation:** Computing $f(x)$, $x \in [x_{lo}, x_{hi}]$, from $f(x_{lo})$ and $f(x_{hi})$

$$f(x) = f(x_{lo}) + \frac{x - x_{lo}}{x_{hi} - x_{lo}} [f(x_{hi}) - f(x_{lo})]$$

4 adds, 1 divide, 1 multiply

If the $x_{lo}$ and $x_{hi}$ endpoints are consecutive multiples of a power of 2, the division and two of the additions become trivial

**Example:** Evaluating $\log_2 x$ for $x \in [1, 2)$

$$f(x_{lo}) = \log_2 1 = 0, f(x_{hi}) = \log_2 2 = 1; \text{ thus:}$$

$$\log_2 x \approx x - 1 = \text{Fractional part of } x$$

An improved linear interpolation formula

$$\log_2 x \approx \frac{\ln 2 - \ln(\ln 2) - 1}{2 \ln 2} + (x - 1) = 0.043 \ 036 + \Delta x$$
Hardware Linear Interpolation Scheme

Fig. 24.5 Linear interpolation for computing $f(x)$ and its hardware realization.
Linear Interpolation with Four Subintervals

Fig. 24.6
Linear interpolation for computing \( f(x) \) using 4 subintervals.

Table 24.1
Approximating \( \log_2 x \) for \( x \) in \([1, 2)\) using linear interpolation within 4 subintervals.

<table>
<thead>
<tr>
<th></th>
<th>( x_{lo} )</th>
<th>( x_{hi} )</th>
<th>( a^{(i)} )</th>
<th>( b^{(i)}/4 )</th>
<th>Max error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00</td>
<td>1.25</td>
<td>0.004 487</td>
<td>0.321 928</td>
<td>± 0.004 487</td>
</tr>
<tr>
<td>1</td>
<td>1.25</td>
<td>1.50</td>
<td>0.324 924</td>
<td>0.263 034</td>
<td>± 0.002 996</td>
</tr>
<tr>
<td>2</td>
<td>1.50</td>
<td>1.75</td>
<td>0.587 105</td>
<td>0.222 392</td>
<td>± 0.002 142</td>
</tr>
<tr>
<td>3</td>
<td>1.75</td>
<td>2.00</td>
<td>0.808 962</td>
<td>0.192 645</td>
<td>± 0.001 607</td>
</tr>
</tbody>
</table>
Tradeoffs in Cost, Speed, and Accuracy

Fig. 24.7
Maximum absolute error in computing $\log_2 x$ as a function of number $h$ of address bits for the tables with linear, quadratic (second-degree), and cubic (third-degree) interpolations [Noet89].
Interpolation with Nonuniform Intervals

One way to use interpolation with nonuniform intervals to successively divide ranges and subranges of interest into 2 parts, with finer divisions used where the function exhibits greater curvature (nonlinearity).

In this way, a number of leading bits can be used to decide which subrange is applicable.

The $[0,1)$ range divided into 4 nonuniform intervals
24.5 Piecewise Lookup Tables

To compute a function of a short (single) IEEE floating-point number:

Divide the 26-bit significand \( x \) (2 whole + 24 fractional bits) into 4 sections

\[
x = t + \lambda u + \lambda^2 v + \lambda^3 w
= t + 2^{-6}u + 2^{-12}v + 2^{-18}w
\]

where \( u, v, w \) are 6-bit fractions in \([0, 1)\) and \( t \), with up to 8 bits, is in \([0, 4)\)

Taylor polynomial for \( f(x) \):

\[
f(x) = \sum_{i=0}^{\infty} f^{(i)}(t + \lambda u) \frac{(\lambda^2 v + \lambda^3 w)^i}{i!}
\]

Ignore terms smaller than \( \lambda^5 = 2^{-30} \)

\[
f(x) \approx f(t + \lambda u)
+ (\lambda/2) [f(t + \lambda u + \lambda v) - f(t + \lambda u - \lambda v)]
+ (\lambda^2/2) [f(t + \lambda u + \lambda w) - f(t + \lambda u - \lambda w)]
+ \lambda^4 [(v^2/2)f^{(2)}(t) - (v^3/6)f^{(3)}(t)]
\]

Use 4 additions to form these terms
Read 5 values of \( f \) from tables
Read this last term from a table
Perform 6-operand addition
Modular Reduction, or Computing $z \mod p$

Divide the argument $z$ into a $(b - g)$-bit upper part ($x$) and a $g$-bit lower part ($y$), where $x$ ends with $g$ zeros.

$$(x + y) \mod p = (x \mod p + y \mod p) \mod p$$

Fig. 24.8a  Two-table modular reduction scheme based on divide-and-conquer.
Another Two-Table Modular Reduction Scheme

Divide the argument $z$ into a $(b - h)$-bit upper part ($x$) and an $h$-bit lower part ($y$), where $x$ ends with $h$ zeros.

Explanation to be added

Fig. 24.8b Modular reduction based on successive refinement.
24.6 Multipartite Table Methods

Divide the domain of interest into $2^a$ intervals, each of which is further divided into $2^b$ smaller subintervals.

The trick: Use linear interpolation with an initial value determined for each subinterval and a common slope for each larger interval.

Total table size is $2^{a+b} + 2^{k-b}$, in lieu of $2^k$; width of table entries has been ignored in this comparison.

Fig. 24.9 The bipartite table method.
Generalizing to Tripartite and Higher-Order Tables

Two-part tables have been generalized to multipart (3-part, 4-part, . . . ) tables

Source of figure: www.ens-lyon.fr/LIP/Arenaire/Ware/Multipartite/