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A Class of Odd-Radix Chordal Ring Networks

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Abstract

An n-node network, with its nodes numbered from $-\lfloor n/2\rfloor$ to $\lceil n/2\rceil - 1$, is a chordal ring with chord lengths $1 = s_0 < s_1 < \cdots < s_{k-1} < n/2$ when an arbitrary node j ($-\lfloor n/2\rfloor \le j < \lceil n/2\rceil$) is connected to each of the $2k$ nodes $j \pm s_i$ mod n ($0 \le i < k$) via an undirected link, where "mod" represents (nearly) symmetric residues in $[-\lfloor n/2\rfloor, \lceil n/2\rceil - 1]$. We study a class of chordal rings in which the chord length s_i is a power of an odd "radix" r, that is, $s_i = r^i$, for $r = 2a + 1 \ge 3$. We show that this class of chordal rings, with their nodes indexed by radix-r integers using the symmetric digit set $[-a, a]$, are easy to analyze and offer a number of advantages in terms of static network parameters and dynamic performance for many application contexts. In particular, these networks allow a very simple optimal (shortest-path) routing algorithm that generates balanced traffic. We then briefly discuss fault tolerance properties of our networks and point out a number of variations and extensions to the basic structure.

Keywords*:* Bisection width, Connectivity, Diameter, Embedding, Fault diameter, Fault tolerance, Hierarchical network, Optimal routing, Symmetric network.

1. Introduction

Owing to their suitability as parallel-processing and communications networks, chordal rings have been studied widely [1-4]. Applications of chordal rings to parallel processing started very early in the history of parallel system design and have continued to date [5-6], although in some cases the interconnection structures include subtle variations and carry different names, thus making it difficult to identify the underlying chordal ring networks. The rich mathematical properties of chordal rings has also attracted numerous mathematical studies, some without explicit or immediate applications. For example, chordal rings figure prominently in many attempts to generate graphs having small diameter or average distance, maximal connectivity, and other graph-theoretic properties [7-11], quests that present many hard (in the sense of complexity theory) and open problems.

One attractive feature of chordal rings is that they have Hamiltonian cycles built-in and readily visible, whereas for other networks, researchers may go to great lengths to establish Hamiltonicity. In fact, multiple edge-disjoint Hamiltonian cycles exist in many chordal rings, making the Hamiltonicity attribute insensitive to a small number of link failures. Other features of chordal rings include symmetry (and thus balance in node message traffic), algorithmic efficiency, and robustness. The bulk of studies of chordal rings in relation to interconnection networks deal with networks of small, fixed node degrees; most commonly, 3-6 in the undirected case (with 4 being the most heavily studied [12-14]), and 2-3 for directed networks.

Some of the advantages of chordal rings persists when special pruning schemes are applied to convert them from completely regular to periodically regular, with a small fixed node degree [15]. This serves as a mechanism for generating fixed-degree networks with desirable properties mirroring those of more densely connected chordal rings. Perfect difference networks [16], a class of densely connected chordal rings with $O(n^{1/2})$ chords per node, chordal-ring structures of other networks [17], and certain related structures [18] have also been studied. On the negative side, determination of diameter and other topological parameters of chordal rings can be difficult. Even for chordal rings with a single skip link type (degree 4), known as double-loop networks, determination of topological properties is nontrivial in general and the problems have not yet been completely solved [19].

In this paper, we study a class of chordal ring networks in which each of the chord lengths s_i is a power of an odd "radix" r, that is, $s_i = r^i$, for $r = 2a + 1 \geq 3$. We show that this class of chordal rings, with nodes indexed by radix-r signed integers using the symmetric digit set $[-(r-1)/2, (r-1)/2]$, or $[-a, a]$, are easy to analyze and offer a number of advantages. The most important of these advantages is the extreme ease of optimal routing, with completely balanced distribution of message traffic, by means of attaching a routing tag to the message (self-routing), which contrasts with the sometimes elaborate shortest-path routing algorithms for chordal rings in general. Although this latter benefit and the previously cited advantages are not unique to chordal rings, not many networks offer all these desirable properties simultaneously.

A preliminary version of this paper, containing some of the results, without proofs, has been published before [20]. Also, peripherally related to the results reported in this paper, are our earlier use of redundant representations to characterize symmetric chordal rings [21], where the similarity is only in linking network properties to a number representation system, and a paper by Beivide et al. [12] devoted to chordal rings of degree 4 having a single odd-length chord. The latter study allows only structures that are similar to 2D torus networks.

The rest of this paper is organized as follows. After some background and basic definitions in the remainder of this section, we study routing problems in our chordal rings, and derive a closed-form expression for their exact diameters as a byproduct, in Section 2. We deal with other topological parameters in Section 3 and with robustness and fault tolerance attributes, as well as some important special cases, in Section 4. We end with our conclusions in Section 5. Key notational conventions used in this paper are summarized in Table 1 for ready reference.

Table 1. List of key notation

Δ	Average internode distance in a network
$\delta(u, v)$	Distance between the nodes u and v
a	Max digit magnitude in symmetric radix-r numbers
B	Bisection width
$CR(n;L)$	Chordal ring with n nodes and skips listed in L
D	Network diameter
d	Node degree
K_n	The complete graph with n nodes
k, l	Number of digits in radix-r node indices
L	List $s_1, \ldots, s_{k-1}$ of skip distances, besides $s_0 = 1$
n	Number of nodes in the network; usually, $n = r^k$
r	Odd radix; $r = 2a + 1 \geq 3$
s_i	The ith skip distance; $s_0 = 1$, $s_{i-1} < s_i$ $(0 < i < k)$
u, v	Arbitrary nodes in a graph or network

An n-node network, with nodes numbered from $-\lfloor n/2 \rfloor$ to $\lceil n/2 \rceil - 1$, is a chordal ring network with chord lengths $1 = s_0, s_1, \ldots, s_{k-1}$ $(s_i < n/2)$ when each node i $(-\lfloor n/2 \rfloor \leq i < \lceil n/2 \rceil)$ is connected to each of the $2k$ nodes $i \pm s_i$ $(0 \leq i < k)$ via an undirected link; all node-index expressions in this paper are evaluated modulo n. However, instead of standard residues in $[0, n-1]$, we use (nearly) symmetric residues in $[-\lfloor n/2 \rfloor, \lceil n/2 \rceil - 1]$. Our focus will be on a class of chordal rings in which the chord lengths s_i are powers of an odd "radix" r, that is, $s_i = r^i$, for some $r = 2a + 1 \geq 3$. We index nodes of the chordal ring $CR(n; r, \ldots, r^{k-1})$ by k-digit radix-r numbers using the symmetric digit set $[-a, a]$. In the bulk of our discussions, we restrict the number of network nodes to the maximal value r^k, an odd number. Thus, we often do away with the floor and ceiling symbols, using instead the interval $[-(n-1)/2, (n-1)/2] = [-(r^k-1)/2, (r^k-1)/2]$ for specifying the range of node indices.

It is easily shown that each node of $CR(r^k; r, \ldots, r^{k-1})$ has a unique label in the radix-r number system with the symmetric digit set $[-a, a]$. The reasons for our symmetric indexing scheme will become clear when we discuss routing algorithms. Other values of n do not create insurmountable difficulties, but they do lead to needless clutter in presenting the basic ideas in this initial study. We will discuss some implications of the condition $n < r^k$ in Section 4.

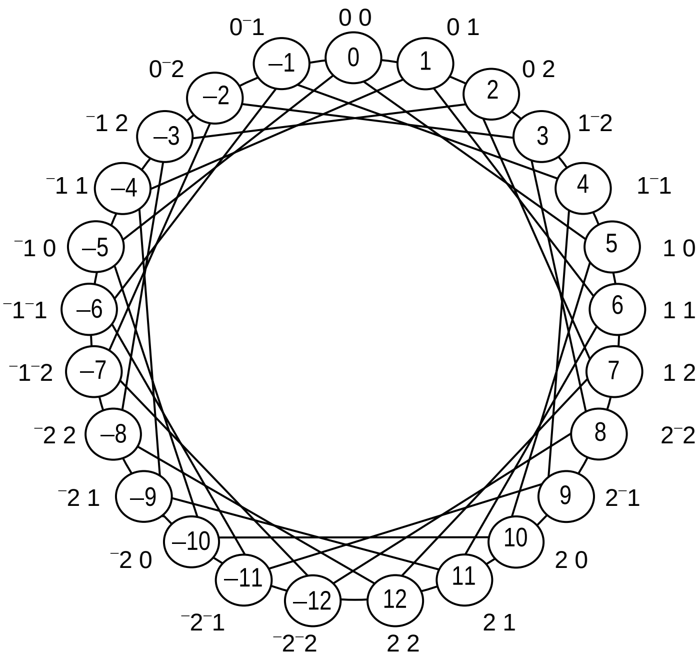

Figure 1. The chordal ring network $CR(25; 5)$ with 25 nodes and chord length 5. Node indices (their symmetric radix-5 representations) appear inside (outside) the circles

Figure 1 depicts a 25-node chordal ring with a single chord length $s_1 = 5$, designated as $CR(25; 5)$, where the first parameter is the number of nodes and those following the semicolon are skip distances besides the mandatory $s_0 = 1$. Nodes 0 to 12 and –12 to –1 can be numbered in the 2-digit symmetric radix-5 number representation system, employing the symmetric digit set $\{\bar{2}, \bar{1}, 0, 1, 2\}$, as $(0\ 0)_{\text{five}}$ to $(2\ 2)_{\text{five}}$ and $(\bar{2}\ \bar{2})_{\text{five}}$ to $(0\ \bar{1})_{\text{five}}$, respectively. The radix-5 node label $(1\ 2)_{\text{five}}$, for example, is a unique label for node 7 and is also indicative of a path from node 0 to node 7; the path consists of one chord of length 5 and two ring links, with the three traversed in any desired order (a total of three paths). The path thus obtained is a shortest path, leading to a simple and elegant shortest-path routing algorithm (to be discussed in Section 3) that is inherently fault-tolerant when the shortest path is not of length 1; a node fault leads to a path with only two extra hops in the latter case. Figure 2 contains a partial representation of *CR*(27; 3, 9), with dark shading used for nodes that are at distance 1 from node 0 and light shading for nodes at distance 2; all other nodes are at distance 3 from node 0.

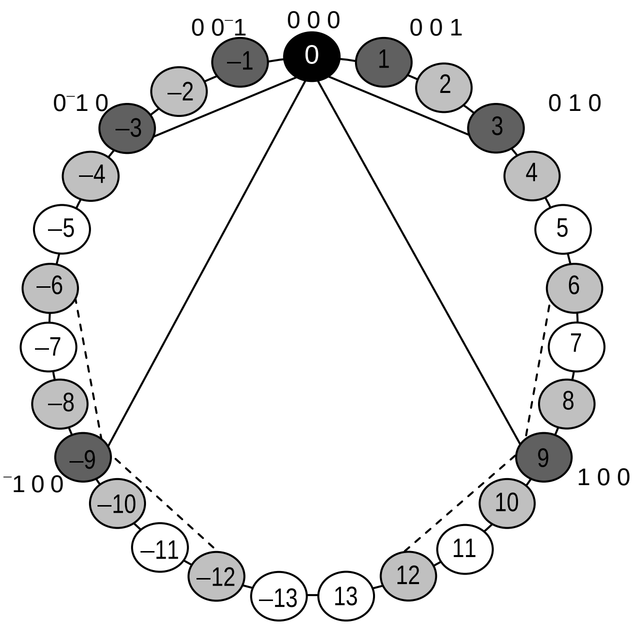

Figure 2. The chordal ring network *CR*(27; 3, 9); only a few of the 81 links are shown

2. Routing and Diameter

Simple and efficient point-to-point and collective communication are prerequisites to the usefulness of any interconnection network. In what follows, we show that our chordal rings allow a very simple self-routing scheme for point-to-point messages based on a routing tag that can be computed from the indices of the source and destination nodes. We prove the algorithm's optimality (i.e., shortest-path routing), use it to derive the network diameter, and discuss collective communication schemes. We also derive the average internode distance and prove that our routing algorithm leads to fully balanced traffic.

Algorithm 1 (point-to-point routing): To route from node $x = x_{k-1} \dots x_1x_0$ to node $y = y_{k-1} \dots y_1y_0$ along a shortest path in $CR(r^k; r, \dots, r^{k-1})$, compute $x - y \bmod r^k$ (symmetric residue, as usual) and attach it to the message as a routing tag $t = t_{k-1} \dots t_1t_0$. In each hop from an intermediate node z along the routing path, pick any nonzero digit in t. Let the chosen nonzero digit be t_i. If $t_i > 0$, route to node $z - r^i$ and decrement t_i. Similarly, if $t_i < 0$, route to node $z + r^i$ and increment t_i. Once every digit of t has become 0, the message is at its destination. ■

Theorem 1: Algorithm 1 is an optimal (shortest-path) routing algorithm for $CR(r^k; r, \dots, r^{k-1})$.

Proof: Owing to node symmetry, we only need to show that the route chosen between source node $x = x_{k-1} \dots x_1x_0$ and destination node $y = 0$ is optimal. The route chosen by Algorithm 1 is of length $\Sigma_{i=0 \text{ to } k-1} \mid x_i \mid$. We prove this to be a shortest path by contradiction. Suppose there is a shorter path consisting of w_i skips of length r^i, $0 \le i < k$; i.e., $\Sigma_{i=0 \text{ to } k-1} \mid w_i \mid < \Sigma_{i=0 \text{ to } k-1} \mid x_i \mid$. Clearly, the residues of $w = (w_{k-1} \dots w_1w_0)_r$ and x modulo r^k must be the same, given that both paths lead to the same destination node. In other words, w is either the same as x or differs from it by a multiple of r^k. The case $w = x$ is immediately ruled out by the fact that nonredundant radix-r representations are unique, forcing $w_i = x_i$ and contradicting our assumption. If $w = x + mr^k$, where m is a nonzero (positive or negative) integer, then the k-digit radix-r representation of w would be impossible using only digits in the range $[-(r-1)/2, (r-1)/2]$; thus, there must exist at least one of the w digits whose magnitude is greater than $(r-1)/2$. Let $w_i > (r-1)/2$; the case $w_i < -(r-1)/2$ is similar. Replace w_i with $w_i - r$, which has a magnitude of at most $(r-1)/2$, and increment w_{i+1} by 1. Note that for $i = k - 1$, there will be no next digit to increment, but this is appropriate, given that we are concerned with values modulo r^k. This transformation does not increase the total weight of the number (sum of digit magnitudes), and thus produces a path of length no greater than that defined by w. Repeated application of the preceding step eventually leads to a number with all digits in the range $[-(r-1)/2, (r-1)/2]$, and this latter number must equal x. Thus, we were led from w, through a succession of paths of the same or shorter length, to x, implying that x does indeed define a shortest path. ■

Note that greedy routing (forwarding along a link that takes the message closest to its destination) is a special case of Algorithm 1 and corresponds to picking the leftmost nonzero t_i in each hop. As a corollary, greedy routing also leads to the selection of a shortest path. A greedy routing algorithm occasionally has more than one choice. For example, to route from node 3 to node 0 in the chordal ring of figure 1, either node 2 or node –2 can be used as the first hop. Hence, greedy routing is a special case of Algorithm 1 only if ties are broken by favoring longer skip links.

Example 1: To better grasp the proof of Theorem 1, consider routing a message form node 11 to node 0 in the chordal ring of figure 2. Algorithm 1 prescribes the path $11 \to 2 \to -1 \to 0$, which is of length 3 and is derived directly from the node index $11 = (1\ \ 1\ \ {}^{-}1)_{\text{three}}$. Consider the alternate path defined by $11 = (0\ 3\ 2)_{\text{three}}$. This path of length 5 is not a shortest path, because the transformation used in the proof of Theorem 1 can convert $(0\ 3\ 2)_{\text{three}}$ to the shorter path defined by $(1\ \ 0\ \ 2)_{\text{three}}$. However, the path defined by $11 = (1\ \ 0\ \ 2)_{\text{three}}$ is a shortest path; it too can be transformed to $(1\ \ 1\ \ {}^{-}1)_{\text{three}}$. So, when there are multiple shortest paths, Algorithm 1 favors a particular (canonical) path which corresponds to the unique representation of distance in the radix-r number system using the symmetric digit set $[-(r-1)/2, (r-1)/2]$.

Theorem 2: The diameter of $CR(r^k; r, \dots, r^{k-1})$ is $D = k(r-1)/2$.

Proof: Immediate from the fact that Algorithm 1 is a shortest-path routing algorithm and its chosen path is longest when all k digits of the tag have magnitudes of $(r-1)/2$. ■

Figure 3 depicts a representation of the chordal ring network *CR*(25; 5) of figure 1 as a set of points on the infinite grid $G_{25,5}$ [19]. A parallelogram that has its corners at grid points labeled 0 in figure 3, or its "digitized" version, tessellates the plane and allows the visualization and derivation of network diameter as the rectilinear or grid distance from a node inside the parallelogram to the closest of its four corners, with the interior node chosen to maximize this distance. In the case of our chordal rings, using the grid points labeled 0 as centers, rather than corners, of the parallelograms tessellating the plane reveals the symmetry of distances (figure 4) that facilitates the derivation of the average internode distance, and also leads to a simple and elegant optimal routing algorithm.

Theorem 3: The average internode distance of the network $CR(r^k; r, \ldots, r^{k-1})$ is $\Delta = k(r^2 - 1)/(4r)$.

Proof: Immediate from the fact that Algorithm 1 is a shortest-path routing algorithm and its chosen path includes an average of $2[1 + 2 + \ldots + (r - 1)/2]/r = (r^2 - 1)/(4r)$ hops for each of the k digit positions in the routing tag. ■

Note that the average internode distance Δ of the network $CR(r^k; r, \ldots, r^{k-1})$ is related to its diameter D by the formula $\Delta/D = 1/2 + 1/(2r)$, that is, Δ is slightly greater than $D/2$, with the difference smaller for larger values of r.

Because $k = \log_r n$, the diameter of $CR(r^k; r, \ldots, r^{k-1})$, as derived in Theorem 1, can be rewritten as $D = k(r - 1)/2 = (\log_r n)(r - 1)/2$. This is identical to the diameter of an r-ary k-cube, which also has the same number r^k of nodes. Of practical interest is the choice of the odd radix r that would minimize the diameter. To obtain this optimal radix, we equate dD/dr with 0, which leads to the optimality condition $\ln r = (r - 1)/r$. Of all possible odd radices, $r = 3$ comes closest to satisfying this condition. With this optimal choice, $d \cong 1.26 \log_2 n$ and $D \cong 0.63 \log_2 n$.

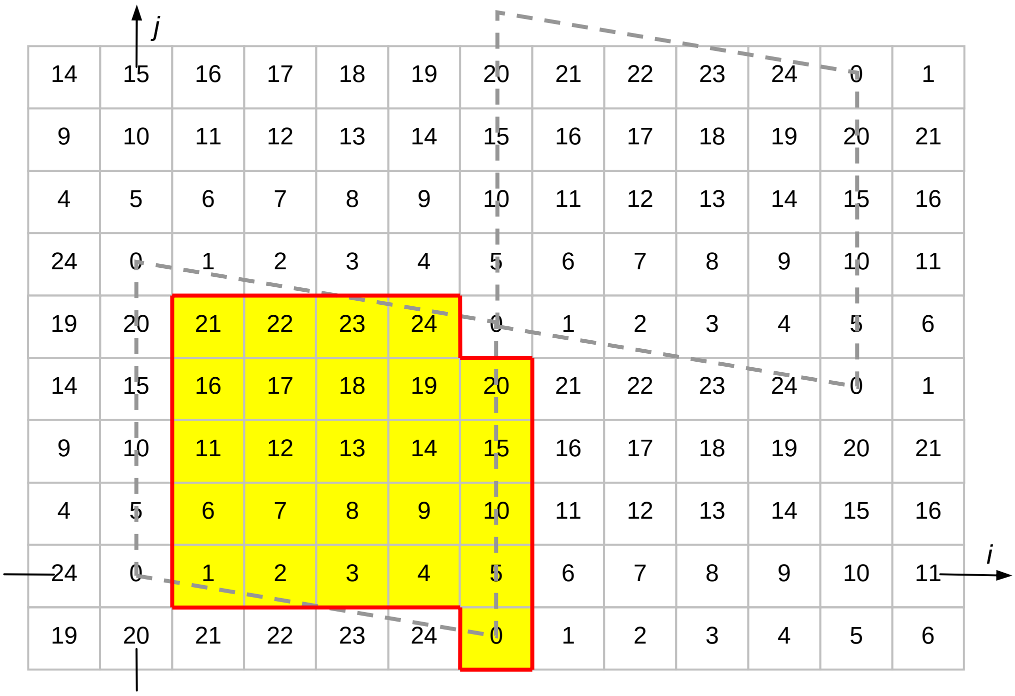

Figure 3. A grid representation of the chordal ring *CR*(25; 5)

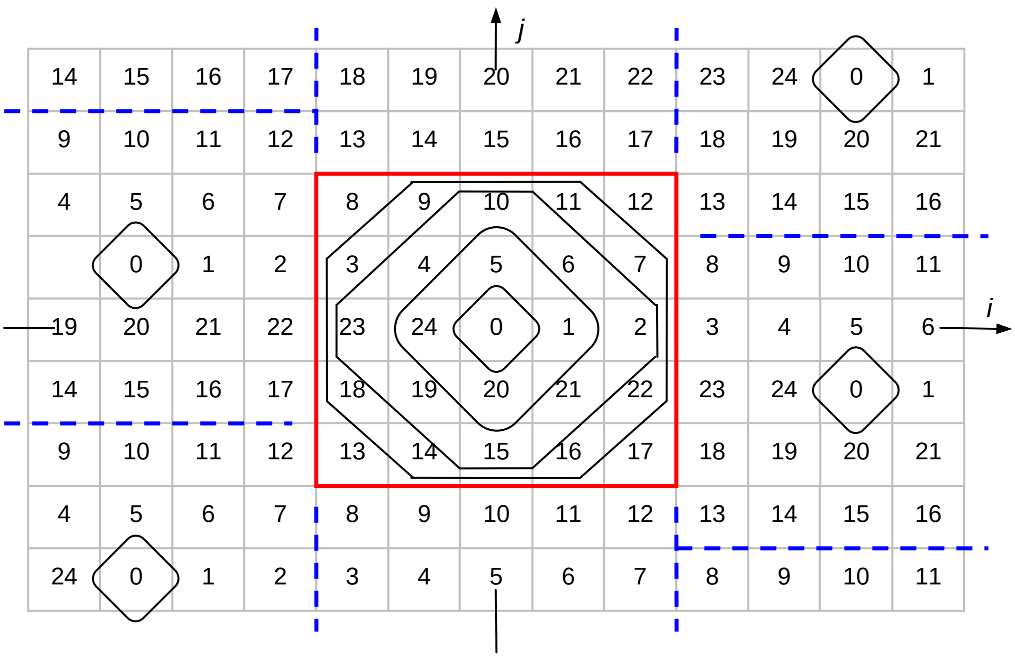

Figure 4. An alternate grid representation of the chordal ring *CR*(25; 5)

We see that a diameter which is better than the diameter of an *n*-node hypercube is achieved, but at a greater cost in terms of node degree. More on this comparison will be offered later. Note that the diameter formula above is approximate when *n* is not a power of *r*. Moreover, optimizing a discrete parameter using continuous analysis is riddled with pitfalls. For these two reasons, we need a rigorous proof that $r = 3$ is optimal. This is provided in Theorem 4.

Theorem 4: The diameter *D* of $CR(n; r, \ldots, r^{k-1})$, satisfying $2r^{k-1} < n \le r^k$, is minimized for $r = 3$.

Proof: In comparing the diameter of $CR(n; r, \ldots, r^{k-1})$ for $r \ge 5$ with that of $CR(n; 3, \ldots, 3^{l-1})$, we assume $n = r^k$. This maximizes the number of nodes in $CR(n; r, \ldots, r^{k-1})$ for a given diameter, thus making the comparison with $r = 3$ most favorable for the alternate radix $r \ge 5$. To simplify the comparison, we require that the number of nodes in the competing radix-3 chordal ring be 3^l, with $3^l \ge r^k$, or, equivalently, $l \ge \lceil k \log_3 r \rceil$. Setting $l = k \log_3 r + 1$ further skews the comparison in favor of the chordal ring with $r > 3$. Based on Theorem 2, the diameters of the two networks are $k(r-1)/2$ and $l(3-1)/2 = l$. If we prove that under the conditions outlined above, $k(r-1)/2 > k \log_3 r + 1$ for $r \ge 5$, the optimality of $r = 3$ follows. The latter inequality simplifies to $r > 2 \log_3 r + 1 + 2/k$, which holds for $r \ge 5$, given that $k \ge 2$. ∎

Example 2: The argument in the proof of Theorem 4 becomes more clear if we note that for $n = 25$, the diameter of $CR(25; 5)$ is $2(5-1)/2 = 4$, whereas $CR(27; 3)$ with 2 additional nodes has a diameter of $3(3-1)/2 = 3$. These are the networks depicted in figures 1 and 2, respectively.

The optimality of $r = 3$ (Theorem 4) is not surprising. The idea of odd-radix chordal rings came to the author as he was looking at a mathematical puzzle dealing with weighing. Suppose that you have a balance and want to choose an optimal set of 4 fixed weights that would allow you the widest possible range of measurement in increments of 1 gram. The solution is 1, 2, 4, 8 (offering a measurement range of 1-15 grams), if weights must be placed on one side and the material or items to be weighed on the other. If, however, fixed weights can go on both sides of the scale, the optimal becomes 1, 3, 9, 27, offering a much wider range of measurements (1-40 grams). The placement of fixed weights for any desired weight *x* is found from the symmetric radix-3 representation of *x* using the digit set $\{\bar{1}, 0, 1\}$; for example, $x = 14 = (1\ \bar{1}\ \bar{1}\ \bar{1})_{\text{three}}$ requires that the 27-gram weight go on one side and the three other weights on the side of the material/items being weighed. The corresponding notion in chordal rings is traversing some links backwards along the shortest path. Furthermore, the number of fixed weights stands for the network diameter and the range of measurements represents the maximal number of nodes.

3. Other Structural Properties

An important topological parameter of a network is its bisection width *B*, an indicator of communication performance under heavy random traffic. The parameter *B* is quite difficult to obtain for an arbitrary interconnection network.

Theorem 5: The bisection width *B* of $CR(r^k; r, \ldots, r^{k-1})$ is between the lower bound $(n - 1/n)/(r - 1/r)$ and the upper bound $2(n-1)/(r-1)$. In particular, for any fixed radix *r*, we have $B = O(n)$, with the coefficient of the leading term in the approximate range of $[1/r, 2/r]$.

Proof: The upper bound on the bisection width is easily established by noting that it corresponds to cuts on the diametrically opposite sides of a drawing of the chordal ring network (see, e.g., figure 1). Given any boundary between two consecutive nodes, 1 ring link, *r* chords of length *r*, r^2 chords of length r^2, . . . , and r^{k-1} chords of length r^{k-1} cross it. Doubling to account for the links cut at the other side of the chordal ring, we readily obtain $B \le 2(1 + r + r^2 + \ldots + r^{k-1}) = 2(r^k - 1)/(r-1)$. The lower bound can be established by visualizing an embedding of the complete graph K_n into our network and bounding the maximum congestion *C* of the embedding based on a balanced distribution of paths, that is, dividing the $n^2/2$ paths of average length $\Delta = (k/2)(r - 1/r)$ over the *kn* available links equally. From the two inequalities $B \ge [(n-1)(n+1)/4]/C$ and $C \ge [n(n-1)/2]/[(k/2)(r - 1/r)]$, we get $B \ge (n - 1/n)/(r - 1/r)$. ∎

Based on Theorem 5, we know the bisection width *B* of $CR(r^k; r, \ldots, r^{k-1})$ to within a multiplicative factor of approximately 2. For radix $r = 3$ that minimizes the diameter, the bisection width *B*, which is in the approximate range $[3n/8, n]$, can be seen to be comparable to that of an *n*-node binary hypercube (having $B = n/2$). For the cost-optimal choice of $r = 5$, as derived in Theorem 6 below, the approximate range of *B* is $[5n/24, n/2]$, somewhat lower, but still not far from that of a hypercube of comparable size. Note that an r^k-node *r*-ary *k*-cube, with $k > 2$, has a bisection width of $2n/r$. As discussed in Section 2, the latter network has a diameter of $k(r-1)/2$, assuming that *r* is odd. We see that our odd-radix chordal rings and *r*-ary *k*-cube networks are very similar in terms of the two key topological parameters of network diameter and bisection width.

We now turn to structural properties of odd-radix chordal rings that directly influence their realizability and implementation cost. One way to take the network cost into account in determining the best radix is to minimize the degree-diameter product $dD = (r-1)(\ln^2 n / \ln^2 r)$. This is tantamount to assuming, rather simplistically, that cost varies linearly with *d* and that performance is proportional to $1/D$. Differentiating the formula for the degree-diameter product *dD* with respect to *r* and equating the result with 0 yields the condition $\ln r = 2(r-1)/r$. This condition is satisfied, approximately, for $r = 5$. With the optimal choice $r = 5$, we have $d = D \cong 0.86 \log_2 n$ and $dD \cong 0.74(\log_2 n)^2$. These values compare favorably with the respective parameters of the *n*-node hypercube, which has $d = D = \log_2 n$. Just as was the case with diameter optimization in Section 2 (Theorem 3 and the discussion that precedes it), we need a rigorous proof that $r = 5$ is optimal with respect to the degree-diameter product. This is supplied in Theorem 7. However, we first need the following result on the diameter of $CR(n; r, \ldots, r^{k-1})$ for arbitrary number *n* of nodes.

Theorem 6: The diameter of $CR(n; r, \ldots, r^{k-1})$, with the number *n* of nodes satisfying $n > 2r^{k-1}$, is $D = (k-1)(r-1)/2 + \lceil (n - r^{k-1}) / (2r^{k-1}) \rceil$.

Proof: The proof follows from the fact that Algorithm 1 is a shortest-path routing algorithm and its chosen path is longest when the lower $k - 1$ digits of the routing tag all have the magnitude $(r - 1)/2$ and the most-significant digit has the magnitude $\lceil (n - r^{k-1}) / (2r^{k-1}) \rceil$. ■

Note that, with an arbitrary number n of nodes, the magnitude of the most-significant digit in a node index is unbounded. For example, in $CR(144; 3, 9)$, the magnitude of the MSD can be as large as 8.

Lemma 1: The degree-diameter product dD of the network $CR(r^k; r, \ldots, r^{k-1})$, $r \neq 5$, is asymptotically larger than dD for $CR(r^k; 5, \ldots, 5^{l-1})$, where l is the smallest number satisfying $5^l > r^k$.

Proof: The degree-diameter product is $2k \times k(r - 1)/2 = k^2(r - 1)$ for $CR(r^k; r, \ldots, r^{k-1})$ and no greater than $2l \times 2l = 4l^2 = 4\lceil k \log_5 r \rceil^2$ for $CR(r^k; 5, \ldots, 5^{l-1})$. For $r = 3$, $\log_5 r$ is less than 1, and the result follows immediately. For $r > 5$, the latter function has a smaller rate of growth than the former, making the statement true for values of k (network sizes) that are large enough. ■

Even though Lemma 1 shows the asymptotic optimality of $r = 5$ with respect to the degree-diameter product, the advantage may occur for extremely large networks that are of no practical interest at present or in the foreseeable future. To complete the picture with regard to networks of moderate sizes, we prove the following result.

Theorem 7: The degree-diameter product dD of the network $CR(n; r, \ldots, r^{k-1})$, with $n > 2r^{k-1}$, is minimized for some r in $\{3, 5, 7, 9\}$ and $k \geq \log_r n$.

Proof: The proof is accomplished by an exhaustive examination of a finite set of alternatives, as described below. For each radix r, $11 \leq r \leq 25$, we consider networks of size r^k in order of increasing size. These sizes are chosen because they maximize the number of nodes for a given diameter and thus provide an advantage for the radix r in comparison with the radices in $\{3, 5, 7, 9\}$. In each case, we derive a network of the same size with one of the alternate radices that has a smaller or equal degree-diameter product. For each value of r, we proceed with the enumeration until the asymptotic advantage proven in Lemma 1 takes hold. According to Lemma 1, we can end our enumeration process at a value of r beyond which the inequality $4\lceil k \log_5 r \rceil^2 \leq k^2(r - 1)$ is satisfied, given that in the latter case, $r = 5$ is a better choice of radix. A sufficient condition for the latter inequality to hold is to have $4(k \log_5 r + 1)^2 \leq k^2(r - 1)$ or, alternatively, $(r - 1)^{1/2}/2 - \log_5 r \geq 1/k$. It is readily established that this condition holds for $r \geq 27$, regardless of the value of $k \geq 2$. The proof is complete upon showing example odd-radix chordal ring networks where each of the radices 3, 5, 7, and 9 is optimal. Radices 3 and 9 are optimal for $n = 81$, because they lead to the degree-diameter product $8 \times 4 = 4 \times 8 = 32$, whereas other radices produce $dD \geq 36$. Radix 7 is optimal for $n = 49$ (dD of 24, versus 28 or more for other radices). Radix 5 is optimal for $n = 125$ (dD of 36, versus 40 or more for other radices). ■

Chordal rings $CR(n; r, \ldots, r^{k-1})$, based on our radix-$r$ construction, are quite efficient with regard to VLSI layout and packaging. In fact, the examples in figures 3 and 4 indicate that the VLSI layouts of these networks are rather similar to those of kD tori. The same number of wraparound links are needed as in torus networks of equal sizes, although the rules for the connectivity of the wraparound links are different in the two networks. This difference, however, does not increase the layout area requirement. Folding techniques that are applied to the VLSI layout of torus networks can also be used to remove the need for long wires between neighboring nodes in the layout of odd-radix chordal rings.

An interesting, and potentially useful, property of odd-radix chordal rings is that they are hierarchically structured. The chordal ring $CR(r^k; r, \ldots, r^{k-1})$ is built of r copies of $CR(r^{k-1}; r, \ldots, r^{k-2})$, which in turn is formed by r copies of $CR(r^{k-2}; r, \ldots, r^{k-3})$, and so on. At the end of this recursion, we reach r-node rings, which form the basic degree-2 components of the hierarchical construction. Going in the opposite direction, the basic components are interconnected by means of 2 external links per node to form second-level components. At the kth level, each node will have $2k$ links (two per level of recursion).

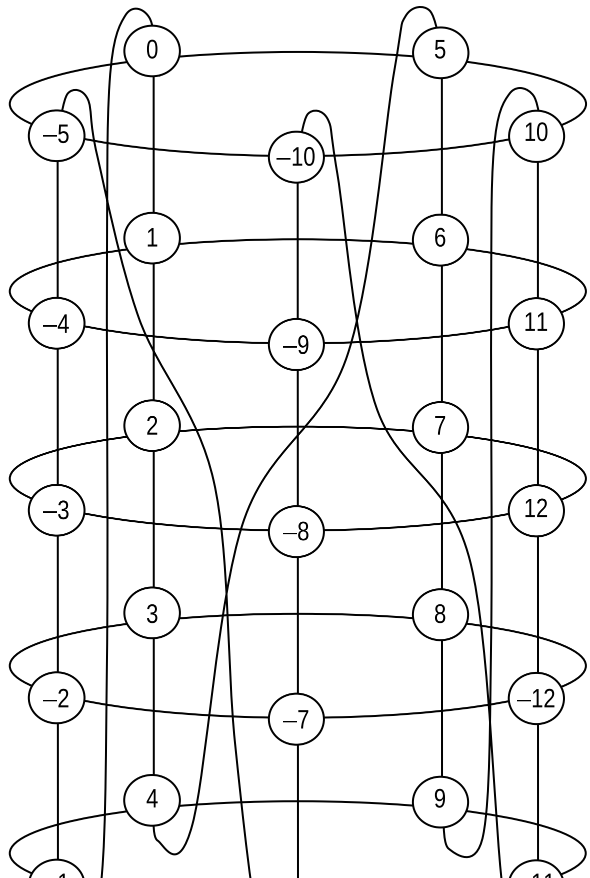

Figure 5. The hierarchical structure of $CR(25; 5)$ as five interconnected 5-cycles

4. Fault Tolerance and Extensions

Inspection of figures 3 and 4 indicates that there are often multiple node- and edge-disjoint shortest paths between a given pair of nodes in $CR(n; r, \ldots, r^{k-1})$. For example, from node 0 to node 7, with the radix-5 node index 1 2, we have the paths 5 + 1 + 1 (through intermediate nodes 5 and 6) and 1 + 1 + 5 (via 1 and 2). Of course, network robustness does not require that alternate shortest paths exist in all cases. It suffices that in the unlikely event of failures, some near-shortest path be available between any pair of nodes. Our chordal rings are robust in this latter sense.

It is well-known that connected circulant graphs are maximally connected [22]. Our odd-radix chordal rings clearly satisfy the connectivity requirement, thus leading to the following result.

Theorem 8: An arbitrary pair of nodes, *u* and *v*, in the odd-radix chordal ring network $CR(r^k; r, \ldots, r^{k-1})$ are connected by $2k$ node/edge-disjoint paths, giving our chordal rings the maximum possible connectivity of $2k$. ■

Of course, the existence of alternate paths, that can be used in the event of the unavailability of nodes or links that are on the shortest path chosen by Algorithm 1, is only a necessary requirement for robustness. A complementary requirement is that the alternate paths not be much longer than the shortest path. We conjecture that a fairly small upper bound on the difference between the length of the longest of these alternate paths and the shortest path between the same two nodes can be derived, but have been unable to establish this bound thus far.

Even though we have been unable to bound the length of the alternate paths in general, we do have a bound for the worst case of diametral paths. Using a proof method very similar to that used in establishing the fault diameter of *k*-ary *n*-cubes [23], or *r*-ary *k*-cubes with our notation, we can derive the corresponding result for our chordal ring networks. This is stated as Theorem 9 below.

Theorem 9: The fault diameter of $CR(r^k; r, \ldots, r^{k-1})$, that is, the diameter of the surviving part of the network with $2k - 1$ worst-case faults (guaranteed to leave the network connected), is no greater than $D + 1$. ■

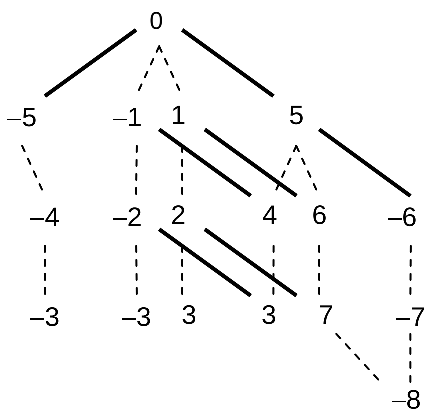

Figure 6. A graphical depiction of some shortest paths from node 0 to other nodes in the chordal ring network $CR(16; 5)$. Solid and dashed lines represent chords and ring links, respectively

A fault-tolerant routing algorithm for the chordal ring $CR(n; r, \ldots, r^{k-1})$ can be readily devised. Figure 6 illustrates the availability of several shortest paths between some pairs of nodes that can be exploited for efficient fault-tolerant routing. We have devised three versions of our fault-tolerant routing algorithm (assuming global knowledge about faults and their locations, global knowledge about number of faults but not their locations, or only local knowledge about faulty neighbors) and will report on them in a the near future.

Given that networks whose size is a power of 2 are of practical interest, we discuss the case of $n = 2^q$ in the following paragraphs. Figure 7 shows $CR(16; 5)$ as an example of such a network, and figure 8 depicts its grid representation (in a manner similar to figure 3).

Because 2^q is relatively prime with respect to any odd radix *r*, beginning from a node and taking the same r^i chord type throughout will eventually lead us back to the origin, having visited every other node exactly once. Thus, the following result follows immediately.

Theorem 10: When $n = 2^q > 2r^{k-1}$, the odd-radix chordal ring $CR(2^q; r, \ldots, r^{k-1})$ contains *k* different edge-disjoint Hamiltonian cycles. ■

Theorem 10 implies that any algorithm, such as all-to-all broadcasting or total exchange, that relies on a Hamiltonian cycle for its efficient execution, is resilient to up to $k - 1$ edge failures without losing any performance. Note that, given the network's node degree of $2k$, the number of edge-disjoint Hamiltonian cycles postulated by Theorem 10 is the maximum possible.

For $n = 2^q$, a particular class of algorithms, known as ascend/descend algorithms [15], become attractive. In this class of algorithms, the communication pattern requires that node *u* communicate with node $u + 2^i$, for $i = 0, 1, \ldots, q - 1$, in *q* phases. It is an easy matter to emulate an ascend or descend algorithm on an odd-radix chordal ring using the appropriate hops in each phase, with $CR(2^q; 3, \ldots, 3^{k-1})$ being particularly efficient in this regard.

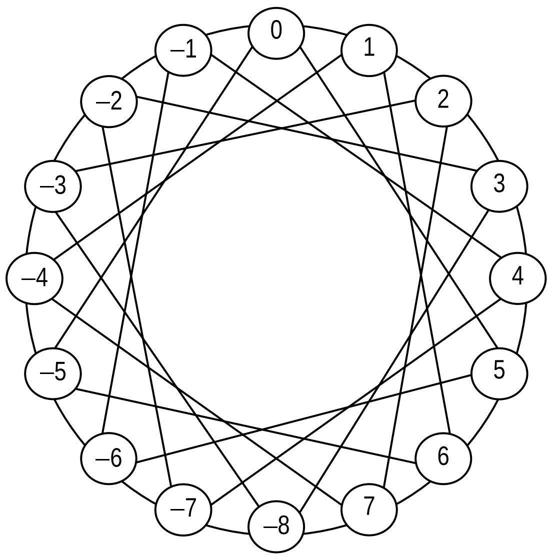

Figure 7. The chordal ring network $CR(16; 5)$ with 16 nodes and chord length 5

Theorem 11: The chordal ring $CR(2^q; 3, \ldots, 3^{k-1})$, where $2^{q-1} > 3^{k-1}$, can emulate an ascend or descend algorithm with no more than $q(q + 1)/2$ conflict-free communication steps.

Proof: The proof is immediate if we show that communication between node *u* and node $u + 2^i$ for all *u* in $[0, n - 1]$ needs no more than $i + 1$ conflict-free routing steps. We prove this by induction on *i*. It is certainly true for $i = 0$, as we simply use the ring links between *u* and $u + 1$ in a single conflict-free routing step. To route from node *u* to node $v = u + 2^i$, we first route to a node *w* whose index is no more than 2^{i-1} away from that of *v*. This is always possible in one conflict-free routing step, given that there is a power of 3 between $2^i - 2^{i-1} = 2^{i-1}$ and $2^i + 2^{i-1} = 3 \times 2^{i-1}$. ■

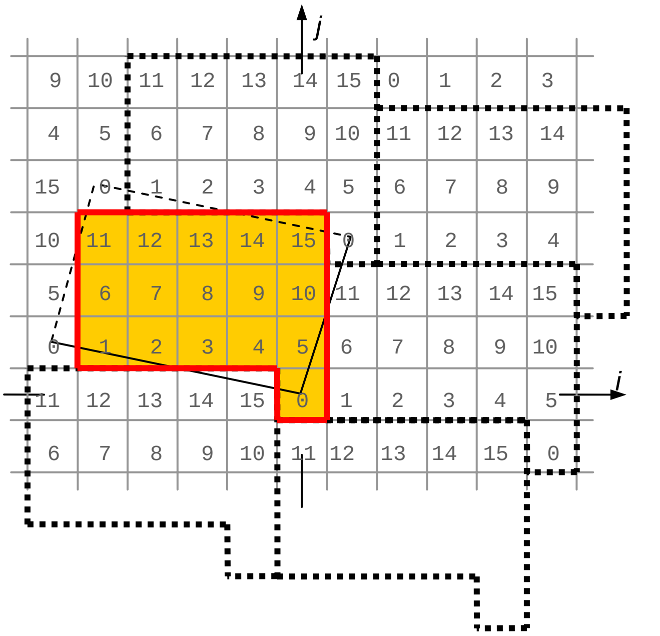

Figure 8. Part of the infinite grid $G_{16,5}$ associated with the chordal ring $CR(16; 5)$

5. Conclusion

We have introduced a class of chordal ring networks and shown them to possess interesting properties with respect to static parameters and dynamic performance under fault-free and faulty conditions. A prominent feature of our networks is the extreme ease of optimal routing, with very little computational requirements. By contrast, finding the shortest path in an arbitrary chordal ring network in nontrivial, sometimes even in the simple case of one skip link [24, 25]. Similarly, the diameter of our graphs is readily obtained in a closed-form expression, which is far from being the case for arbitrary chordal ring networks [19, 26, 27].

Further research is needed to generalize some of our results that pertain only to particular network sizes to arbitrary n in an attempt to improve system scalability. Determining the exact bisection width, obtaining additional results on fault tolerance (including proving or disproving some of our conjectures), and devising emulation schemes for other networks are also desirable. Constructing periodically regular chordal rings [15], in which any node v has only one chord of length $r^{k-1 - v \bmod (k-1)}$, is also of some interest. Such periodic regularity allows us to reduce node degree while preserving certain desirable topological and algorithmic properties.

Finally, two other research problems may be pursued. First, the relationship of odd-radix chordal rings to small-world networks, that have been widely studied by researchers in several disciplines in recent years [28], is quite interesting. In other words, the extent to which radix-based chordal ring networks can serve as suitable deterministic models for small-world networks may be studied [29, 30]. Second, adapability to change, as measured by the recently proposed “topology lifetime” measure [31] may be investigated.

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