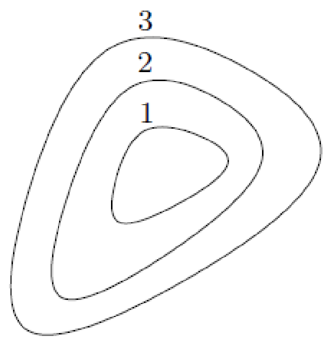
ECE271A HW2

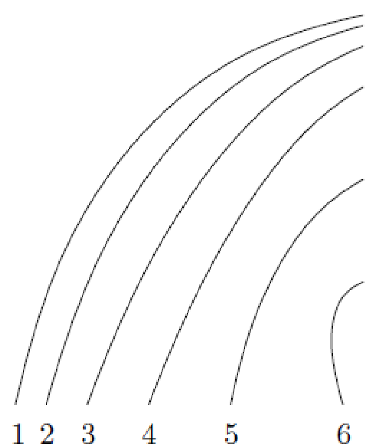
Spring 14

Textbook problems:

3.2 *Level sets of convex, concave, quasiconvex, and quasiconcave functions.* Some level sets of a function f are shown below. The curve labeled 1 shows $\{x \mid f(x) = 1\}$, etc.



Could f be convex (concave, quasiconvex, quasiconcave)? Explain your answer. Repeat for the level curves shown below.



3.15 *A family of concave utility functions.* For $0 < \alpha \le 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha},$$

with $\mathbf{dom}\, u_\alpha = \mathbf{R}_+$. We also define $u_0(x) = \log x$ (with $\mathbf{dom}\, u_0 = \mathbf{R}_{++}$).

(a) Show that for $x > 0$, $u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$.

(b) Show that u_α are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$.

These functions are often used in economics to model the benefit or utility of some quantity of goods or money. Concavity of u_α means that the marginal utility (*i.e.*, the increase in utility obtained for a fixed increase in the goods) decreases as the amount of goods increases. In other words, concavity models the effect of *satiation*.

3.16 For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) $f(x) = e^x - 1$ on $\mathbf{R}$.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{R} \times \mathbf{R}_{++}$.

(f) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbf{R}^2_{++}$.

4.1 Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{array}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

(a) $f_0(x_1, x_2) = x_1 + x_2$.

(b) $f_0(x_1, x_2) = -x_1 - x_2$.

(c) $f_0(x_1, x_2) = x_1$.

(d) $f_0(x_1, x_2) = \max\{x_1, x_2\}$.

(e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$.

Additional exercise problems:

1.7 *Dual cones in* $\mathbf{R}^2$. Describe the dual cone for each of the following cones.

(a) $K = \{0\}$.

(b) $K = \mathbf{R}^2$.

(c) $K = \{(x_1, x_2) \mid |x_1| \le x_2\}$.

(d) $K = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$.

2.2 *A general vector composition rule.* Suppose

$$f(x) = h(g_1(x), g_2(x), \dots, g_k(x))$$

where $h : \mathbf{R}^k \to \mathbf{R}$ is convex, and $g_i : \mathbf{R}^n \to \mathbf{R}$. Suppose that for each i, one of the following holds:

- h is nondecreasing in the ith argument, and g_i is convex
- h is nonincreasing in the ith argument, and g_i is concave
- g_i is affine.

Show that f is convex. (This composition rule subsumes all the ones given in the book, and is the one used in software systems such as CVX.) You can assume that $\mathbf{dom}\, h = \mathbf{R}^k$; the result also holds in the general case when the monotonicity conditions listed above are imposed on $\tilde{h}$, the extended-valued extension of h.

2.12 *Continued fraction function.* Show that the function

$$f(x) = \cfrac{1}{x_1 - \cfrac{1}{x_2 - \cfrac{1}{x_3 - \cfrac{1}{x_4}}}}$$

defined where every denominator is positive, is convex and decreasing. (There is nothing special about $n = 4$ here; the same holds for any number of variables.)

3.2 *'Hello World' in CVX.* Use CVX to verify the optimal values you obtained (analytically) for exercise 4.1 in *Convex Optimization.*