ECE271A HW4

Spring 14

Textbook problems

6.2 *ℓ_1-, ℓ_2-, and ℓ_∞-norm approximation by a constant vector.* What is the solution of the norm approximation problem with one scalar variable $x \in \mathbf{R}$,

$$\text{minimize} \quad \|x\mathbf{1} - b\|,$$

for the ℓ_1-, ℓ_2-, and ℓ_∞-norms?

6.9 *Minimax rational function fitting.* Show that the following problem is quasiconvex:

$$\text{minimize} \quad \max_{i=1,\ldots,k} \left| \frac{p(t_i)}{q(t_i)} - y_i \right|$$

where

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m, \qquad q(t) = 1 + b_1 t + \cdots + b_n t^n,$$

and the domain of the objective function is defined as

$$D = \{(a, b) \in \mathbf{R}^{m+1} \times \mathbf{R}^n \mid q(t) > 0, \ \alpha \le t \le \beta\}.$$

In this problem we fit a rational function $p(t)/q(t)$ to given data, while constraining the denominator polynomial to be positive on the interval $[\alpha, \beta]$. The optimization variables are the numerator and denominator coefficients a_i, b_i. The interpolation points $t_i \in [\alpha, \beta]$, and desired function values y_i, $i = 1, \ldots, k$, are given.

<u>Additional problems:</u>

3.9 *Complex least-norm problem.* We consider the complex least ℓ_p-norm problem

$$\begin{array}{ll} \text{minimize} & \|x\|_p \\ \text{subject to} & Ax = b, \end{array}$$

where $A \in \mathbf{C}^{m\times n}$, $b \in \mathbf{C}^m$, and the variable is $x \in \mathbf{C}^n$. Here $\|\cdot\|_p$ denotes the ℓ_p-norm on $\mathbf{C}^n$, defined as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

for $p \geq 1$, and $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$. We assume A is full rank, and $m < n$.

(a) Formulate the complex least ℓ_2-norm problem as a least ℓ_2-norm problem with real problem data and variable. *Hint.* Use $z = (\Re x, \Im x) \in \mathbf{R}^{2n}$ as the variable.

(b) Formulate the complex least ℓ_∞-norm problem as an SOCP.

(c) Solve a random instance of both problems with $m = 30$ and $n = 100$. To generate the matrix A, you can use the Matlab command `A = randn(m,n) + i*randn(m,n)`. Similarly, use `b = randn(m,1) + i*randn(m,1)` to generate the vector b. Use the Matlab command `scatter` to plot the optimal solutions of the two problems on the complex plane, and comment (briefly) on what you observe. You can solve the problems using the CVX functions `norm(x,2)` and `norm(x,inf)`, which are overloaded to handle complex arguments. To utilize this feature, you will need to declare variables to be `complex` in the `variable` statement. (In particular, you do not have to manually form or solve the SOCP from part (b).)

5.2 *Minimax rational fit to the exponential.* (See exercise 6.9 of *Convex Optimization.*) We consider the specific problem instance with data

$$t_i = -3 + 6(i-1)/(k-1), \quad y_i = e^{t_i}, \quad i = 1, \ldots, k,$$

where $k = 201$. (In other words, the data are obtained by uniformly sampling the exponential function over the interval $[-3, 3]$.) Find a function of the form

$$f(t) = \frac{a_0 + a_1 t + a_2 t^2}{1 + b_1 t + b_2 t^2}$$

that minimizes $\max_{i=1,\ldots,k} |f(t_i) - y_i|$. (We require that $1 + b_1 t_i + b_2 t_i^2 > 0$ for $i = 1, \ldots, k$.)

Find optimal values of a_0, a_1, a_2, b_1, b_2, and give the optimal objective value, computed to an accuracy of 0.001. Plot the data and the optimal rational function fit on the same plot. On a different plot, give the fitting error, *i.e.*, $f(t_i) - y_i$.

Hint. You can use `strcmp(cvx_status,'Solved')`, after `cvx_end`, to check if a feasibility problem is feasible.

5.13 *Fitting with censored data.* In some experiments there are two kinds of measurements or data available: The usual ones, in which you get a number (say), and *censored data*, in which you don't get the specific number, but are told something about it, such as a lower bound. A classic example is a study of lifetimes of a set of subjects (say, laboratory mice). For those who have died by the end of data collection, we get the lifetime. For those who have not died by the end of data collection, we do not have the lifetime, but we do have a lower bound, *i.e.*, the length of the study. These are the censored data values.

We wish to fit a set of data points,

minimize

$$J = \sum_{k=1}^{K} \left(y^{(k)} - c^T x^{(k)} \right)^2 .$$

Here is the tricky part: some of the values of $y^{(k)}$ are censored; for these entries, we have only a (given) lower bound. We will re-order the data so that $y^{(1)}, \ldots, y^{(M)}$ are given (*i.e.*, uncensored), while $y^{(M+1)}, \ldots, y^{(K)}$ are all censored, *i.e.*, unknown, but larger than D, a given number. All the values of $x^{(k)}$ are known.

(a) Explain how to find c (the model parameter) and $y^{(M+1)}, \ldots, y^{(K)}$ (the censored data values) that minimize J.

(b) Carry out the method of part (a) on the data values in `cens_fit_data.m`. Report $\hat{c}$, the value of c found using this method.

Also find $\hat{c}_{\text{ls}}$, the least-squares estimate of c obtained by simply ignoring the censored data samples, *i.e.*, the least-squares estimate based on the data

$$(x^{(1)}, y^{(1)}), \ldots, (x^{(M)}, y^{(M)}).$$

The data file contains c_{true}, the true value of c, in the vector `c_true`. Use this to give the two relative errors

$$\frac{\|c_{\text{true}} - \hat{c}\|_2}{\|c_{\text{true}}\|_2}, \qquad \frac{\|c_{\text{true}} - \hat{c}_{\text{ls}}\|_2}{\|c_{\text{true}}\|_2}.$$

5.15 *Learning a quadratic pseudo-metric from distance measurements.* We are given a set of N pairs of points in $\mathbf{R}^n$, $x_1, \ldots, x_N$, and $y_1, \ldots, y_N$, together with a set of distances $d_1, \ldots, d_N > 0$.

The goal is to find (or estimate or learn) a quadratic pseudo-metric d,

$$d(x,y) = \left((x-y)^T P (x-y)\right)^{1/2},$$

with $P \in \mathbf{S}^n_+$, which approximates the given distances, *i.e.*, $d(x_i, y_i) \approx d_i$. (The pseudo-metric d is a metric only when $P \succ 0$; when $P \succeq 0$ is singular, it is a pseudo-metric.)

To do this, we will choose $P \in \mathbf{S}^n_+$ that minimizes the mean squared error objective

$$\frac{1}{N}\sum_{i=1}^{N}(d_i - d(x_i, y_i))^2.$$

(a) Explain how to find P using convex or quasiconvex optimization. If you cannot find an exact formulation (*i.e.*, one that is guaranteed to minimize the total squared error objective), give a formulation that approximately minimizes the given objective, subject to the constraints.

(b) Carry out the method of part (a) with the data given in `quad_metric_data.m`. The columns of the matrices `X` and `Y` are the points x_i and y_i; the row vector `d` gives the distances d_i. Give the optimal mean squared distance error.

We also provide a test set, with data `X_test`, `Y_test`, and `d_test`. Report the mean squared distance error on the test set (using the metric found using the data set above).