

ECE271A HW5

Spring 2014

Textbook problems:

5.1 *A simple example.* Consider the optimization problem

$$\begin{aligned} &\text{minimize} && x^2 + 1 \\ &\text{subject to} && (x - 2)(x - 4) \leq 0, \end{aligned}$$

with variable $x \in \mathbf{R}$.

- (a) *Analysis of primal problem.* Give the feasible set, the optimal value, and the optimal solution.
- (b) *Lagrangian and dual function.* Plot the objective $x^2 + 1$ versus x . On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ . Verify the lower bound property ($p^* \geq \inf_x L(x, \lambda)$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g .
- (c) *Lagrange dual problem.* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^* . Does strong duality hold?
- (d) *Sensitivity analysis.* Let $p^*(u)$ denote the optimal value of the problem

$$\begin{aligned} &\text{minimize} && x^2 + 1 \\ &\text{subject to} && (x - 2)(x - 4) \leq u, \end{aligned}$$

as a function of the parameter u . Plot $p^*(u)$. Verify that $dp^*(0)/du = -\lambda^*$.

5.5 *Dual of general LP.* Find the dual function of the LP

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Gx \preceq h \\ &&& Ax = b. \end{aligned}$$

Give the dual problem, and make the implicit equality constraints explicit.

5.13 *Lagrangian relaxation of Boolean LP.* A *Boolean linear program* is an optimization problem of the form

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && Ax \preceq b \\ & && x_i \in \{0, 1\}, \quad i = 1, \dots, n, \end{aligned}$$

and is, in general, very difficult to solve. In exercise 4.15 we studied the LP relaxation of this problem,

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && Ax \preceq b \\ & && 0 \leq x_i \leq 1, \quad i = 1, \dots, n, \end{aligned} \tag{5.107}$$

which is far easier to solve, and gives a lower bound on the optimal value of the Boolean LP. In this problem we derive another lower bound for the Boolean LP, and work out the relation between the two lower bounds.

(a) *Lagrangian relaxation.* The Boolean LP can be reformulated as the problem

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && Ax \preceq b \\ & && x_i(1 - x_i) = 0, \quad i = 1, \dots, n, \end{aligned}$$

which has quadratic equality constraints. Find the Lagrange dual of this problem. The optimal value of the dual problem (which is convex) gives a lower bound on the optimal value of the Boolean LP. This method of finding a lower bound on the optimal value is called *Lagrangian relaxation*.

(b) Show that the lower bound obtained via Lagrangian relaxation, and via the LP relaxation (5.107), are the same. *Hint.* Derive the dual of the LP relaxation (5.107).

5.17 *Robust linear programming with polyhedral uncertainty.* Consider the robust LP

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && \sup_{a \in \mathcal{P}_i} a^T x \leq b_i, \quad i = 1, \dots, m, \end{aligned}$$

with variable $x \in \mathbf{R}^n$, where $\mathcal{P}_i = \{a \mid C_i a \preceq d_i\}$. The problem data are $c \in \mathbf{R}^n$, $C_i \in \mathbf{R}^{m_i \times n}$, $d_i \in \mathbf{R}^{m_i}$, and $b \in \mathbf{R}^m$. We assume the polyhedra \mathcal{P}_i are nonempty.

Show that this problem is equivalent to the LP

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && d_i^T z_i \leq b_i, \quad i = 1, \dots, m \\ & && C_i^T z_i = x, \quad i = 1, \dots, m \\ & && z_i \succeq 0, \quad i = 1, \dots, m \end{aligned}$$

with variables $x \in \mathbf{R}^n$ and $z_i \in \mathbf{R}^{m_i}$, $i = 1, \dots, m$. *Hint.* Find the dual of the problem of maximizing $a_i^T x$ over $a_i \in \mathcal{P}_i$ (with variable a_i).

Additional problems:

3.8 *Schur complements and LMI representation.* Recognizing Schur complements (see §A5.5) often helps to represent nonlinear convex constraints as linear matrix inequalities (LMIs). Consider the function

$$f(x) = (Ax + b)^T (P_0 + x_1 P_1 + \cdots + x_n P_n)^{-1} (Ax + b)$$

where $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, and $P_i = P_i^T \in \mathbf{R}^{m \times m}$, with domain

$$\mathbf{dom} f = \{x \in \mathbf{R}^n \mid P_0 + x_1 P_1 + \cdots + x_n P_n \succ 0\}.$$

This is the composition of the matrix fractional function and an affine mapping, and so is convex. Give an LMI representation of $\mathbf{epi} f$. That is, find a symmetric matrix $F(x, t)$, affine in (x, t) , for which

$$x \in \mathbf{dom} f, \quad f(x) \leq t \quad \iff \quad F(x, t) \succeq 0.$$

Remark. LMI representations, such as the one you found in this exercise, can be directly used in software systems such as CVX.

4.1 *Numerical perturbation analysis example.* Consider the quadratic program

$$\begin{aligned} \text{minimize} \quad & x_1^2 + 2x_2^2 - x_1x_2 - x_1 \\ \text{subject to} \quad & x_1 + 2x_2 \leq u_1 \\ & x_1 - 4x_2 \leq u_2, \\ & 5x_1 + 76x_2 \leq 1, \end{aligned}$$

with variables x_1 , x_2 , and parameters u_1 , u_2 .

- (a) Solve this QP, for parameter values $u_1 = -2$, $u_2 = -3$, to find optimal primal variable values x_1^* and x_2^* , and optimal dual variable values λ_1^* , λ_2^* and λ_3^* . Let p^* denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found (within reasonable numerical accuracy).

Hint: See §3.7 of the CVX users' guide to find out how to retrieve optimal dual variables. To specify the quadratic objective, use `quad_form()`.

- (b) We will now solve some perturbed versions of the QP, with

$$u_1 = -2 + \delta_1, \quad u_2 = -3 + \delta_2,$$

where δ_1 and δ_2 each take values from $\{-0.1, 0, 0.1\}$. (There are a total of nine such combinations, including the original problem with $\delta_1 = \delta_2 = 0$.) For each combination of δ_1 and δ_2 , make a prediction p_{pred}^* of the optimal value of the perturbed QP, and compare it to p_{exact}^* , the exact optimal value of the perturbed QP (obtained by solving the perturbed QP). Put your results in the two righthand columns in a table with the form shown below. Check that the inequality $p_{\text{pred}}^* \leq p_{\text{exact}}^*$ holds.

δ_1	δ_2	p_{pred}^*	p_{exact}^*
0	0		
0	-0.1		
0	0.1		
-0.1	0		
-0.1	-0.1		
-0.1	0.1		
0.1	0		
0.1	-0.1		
0.1	0.1		