ECE271A HW6

Spring 2014

6.4 *Maximum likelihood prediction of team ability.* A set of n teams compete in a tournament. We model each team's ability by a number $a_j \in [0,1]$, $j = 1, \ldots, n$. When teams j and k play each other, the probability that team j wins is equal to $\mathbf{prob}(a_j - a_k + v > 0)$, where $v \sim \mathcal{N}(0, \sigma^2)$.

You are given the outcome of m past games. These are organized as

$$(j^{(i)}, k^{(i)}, y^{(i)}), \quad i = 1, \ldots, m,$$

meaning that game i was played between teams $j^{(i)}$ and $k^{(i)}$; $y^{(i)} = 1$ means that team $j^{(i)}$ won, while $y^{(i)} = -1$ means that team $k^{(i)}$ won. (We assume there are no ties.)

(a) Formulate the problem of finding the maximum likelihood estimate of team abilities, $\hat{a} \in \mathbf{R}^n$, given the outcomes, as a convex optimization problem. You will find the *game incidence matrix* $A \in \mathbf{R}^{m \times n}$, defined as

$$A_{il} = \begin{cases} y^{(i)} & l = j^{(i)} \\ -y^{(i)} & l = k^{(i)} \\ 0 & \text{otherwise,} \end{cases}$$

useful.

The prior constraints $\hat{a}_i \in [0,1]$ should be included in the problem formulation. Also, we note that if a constant is added to all team abilities, there is no change in the probabilities of game outcomes. This means that $\hat{a}$ is determined only up to a constant, like a potential. But this doesn't affect the ML estimation problem, or any subsequent predictions made using the estimated parameters.

(b) Find $\hat{a}$ for the team data given in `team_data.m`, in the matrix `train`. (This matrix gives the outcomes for a tournament in which each team plays each other team once.) You may find the CVX function `log_normcdf` helpful for this problem.

You can form A using the commands

```
A = sparse(1:m,train(:,1),train(:,3),m,n) + ...
    sparse(1:m,train(:,2),-train(:,3),m,n);
```

(c) Use the maximum likelihood estimate $\hat{a}$ found in part (b) to predict the outcomes of next year's tournament games, given in the matrix `test`, using $\hat{y}^{(i)} = \mathbf{sign}(\hat{a}_{j^{(i)}} - \hat{a}_{k^{(i)}})$. Compare these predictions with the actual outcomes, given in the third column of `test`. Give the fraction of correctly predicted outcomes.

The games played in `train` and `test` are the same, so another, simpler method for predicting the outcomes in `test` it to just assume the team that won last year's match will also win this year's match. Give the percentage of correctly predicted outcomes using this simple method.