Wireless Communications
Lecture 10

[SNR per symbol and SNR per bit]

$$\text{SNR} = \frac{P_R}{N_0 B} = \frac{E_s}{N_0 B T_s} = \frac{E_b}{N_0 B T_b}$$

For BPSK:

$$T_b = T_s, \quad E_b = E_s, \quad \text{and} \quad T_s = 1/B.$$

Raised cosine pulse shaper for other pulses. $T_s \sim \frac{1}{B}$
The SNR becomes

$$\text{SNR} = \frac{E_s}{N_0} \text{ or } (\frac{E_b}{N_0} \quad \text{BPSK})$$

The SNR per symbol γ_s and the SNR per bit γ_b are defined as

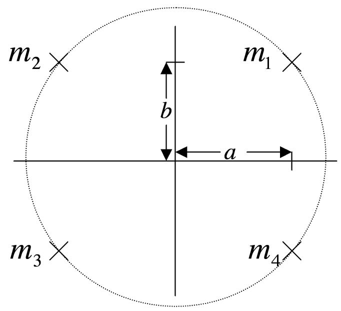
$$\begin{aligned} \gamma_s &= \frac{E_s}{N_0} \\ \gamma_b &= \frac{E_b}{N_0} \end{aligned}$$

[Symbol error rate and bit error rate]

One approximation:

$$\begin{aligned} P_s &\simeq \text{one bit in error} \\ &= P_{b_1}(1-P_{b_2})(1-P_{b_3})\ldots + P_{b_2}(1-P_{b_1})(1-P_{b_3})\ldots + \\ &= MP_{b_1}, \quad M: \text{the number of bits} \end{aligned}$$

[QPSK]



$r_1 = s_{i1} + n_1, \;\; n_1 \sim N(0,\sigma^2)$

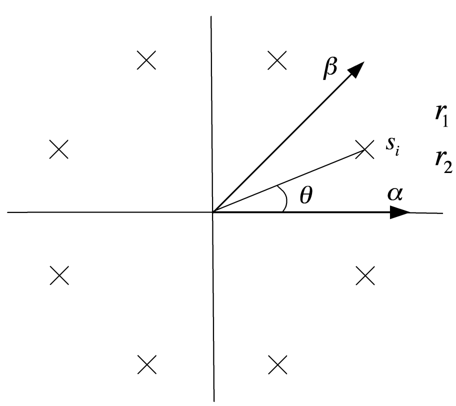
$r_2 = s_{i2} + n_2, \;\; n_2 \sim N(0,\sigma^2)$

$\vec{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \end{bmatrix} \in \{m_1, m_2, m_3, m_4\}$

$$\begin{aligned}
P_s &= P_{s|1 \text{ transmitted}} \times \frac{1}{4} \times 4 = 1 - Q\left(\frac{-s_{i1}}{\sigma}\right) Q\left(\frac{-s_{i2}}{\sigma}\right) \\
&= 1 - (1 - Q(\sqrt{\mathrm{SNR}_b}))^2, \quad \mathrm{SNR}_{\mathrm{symbol}} = \frac{2a^2}{\sigma^2} \\
&\simeq 2Q(\sqrt{\mathrm{SNR}_b})
\end{aligned}$$

$$\Rightarrow P_b \simeq P_s/2$$

[General case] For MPSK



$r_1 = s_{i1} + n_1$

$r_2 = s_{i2} + n_2$

$\Longrightarrow \vec{r} = \vec{s}_i + \vec{n} \Rightarrow$ need angle distribution.

In-phase and quadrature Gaussian $\Rightarrow$ bivariate distribution of $|r|$ and $\theta \Rightarrow$ Integrate for θ distributiion.

$$P_s = 1 - \int_\alpha^\beta P(\theta)d\theta$$

Closed form does not exist. The nearest neighbor approximation

$$P_s \simeq 2Q(\sqrt{2}\sqrt{\mathrm{SNR}_b})\sin(\pi/M)$$

[**Average symbol error probability(general)**] Many systems have : $P_s \simeq \alpha_M Q(\sqrt{\beta_M \gamma_s})$
Finding the average probability of symbol error

$$\begin{aligned}\bar{P}_s &= \int_0^\infty \alpha_M Q(\sqrt{\beta_M \gamma})\lambda e^{-\lambda\gamma} d\gamma, \quad \lambda = \frac{1}{\bar{\gamma}_s} \\ &= \frac{\alpha_M}{2}\left[1 - \sqrt{\frac{0.5\beta_M\bar{\gamma}_s}{1+0.5\beta_M\bar{\gamma}_s}}\right] \\ &\approx \frac{\alpha_M}{2\beta_M\bar{\gamma}_s}\end{aligned}$$

Show P_b for AWGN and fading channel (figure 6.1, 6.2 Goldsmith)

[**Alternative form of Q-function by Craig**]

$$Q(z) = \frac{1}{\pi}\int_0^{\pi/2} e^{\frac{-z^2}{2\sin^2\varphi}} d\varphi, \quad z > 0$$

[Moment Generating Function]

$$\mathcal{M}_\gamma(s) = \int_0^\infty p_\gamma(\gamma) e^{s\gamma} d\gamma$$

Setting $\alpha = \alpha_M$ and $g = 0.5\beta_M$ in $P_s = \alpha_M Q(\sqrt{\beta_M \gamma_s})$

$$\begin{aligned} P_s &= \alpha_M Q(\sqrt{\beta_M \gamma_s})|_{\alpha=\alpha_M, g=0.5\beta_M} \\ P_s &= \alpha Q(\sqrt{2g\gamma_s}) \\ & \quad \text{Using the alternative } Q\text{-function} \\ &= \frac{\alpha}{\pi} \int_0^{\pi/2} e^{\frac{-g\gamma_s}{\sin^2 \psi}} d\varphi \end{aligned}$$

The average error probability in fading for modulation with $P_s = \alpha Q(\sqrt{2g\gamma_s})$ is

$$\begin{aligned} \bar{P}_s &= \frac{\alpha}{\pi} \int_0^\infty \int_0^{\pi/2} e^{\frac{-g\gamma}{\sin^2 \varphi}} d\varphi p_{\gamma_s}(\gamma) d\gamma \\ &= \frac{\alpha}{\pi} \int_0^{\pi/2} \left[\int_0^\infty e^{\frac{-g\gamma}{\sin^2 \varphi}} p_{\gamma_s}(\gamma) d\gamma \right] d\varphi \\ &= \frac{\alpha}{\pi} \int_0^{\pi/2} \mathcal{M}_{\gamma_s} \left(\frac{-g}{\sin^2 \varphi} \right) d\varphi \end{aligned}$$

The MGF $\mathcal{M}_{\gamma_s}(s)$ for Rayleigh distribution is

$$\mathcal{M}_{\gamma_s}(s) = (1 - s\bar{\gamma}_s)^{-1}, \quad s = \frac{-g}{\sin^2 \varphi}$$

[Example] Find outage probability of BPSK at $P_b = 10^{-3}$ for Rayleigh fading with selection diversity for $M = 2$. Equal branch $\overline{\text{SNR}}_s$ of $\bar{\gamma} = 15$ dB.

$$P_b = 10^{-3} \Rightarrow \gamma_b = 10 \text{ dB} \Rightarrow \gamma_0 = 10$$

$$P_{\text{out}} = [1 - e^{-\gamma_0/\bar{\gamma}}]^2 = [1 - e^{-10/10^{1.5}}]^2$$

[Average probability of symbol error(MRC)] The distribution of γ_Σ for MRC

$$p_{\gamma_\Sigma}(\gamma) = \frac{\gamma^{M-1} e^{-\gamma/\bar{\gamma}}}{\bar{\gamma}^M (M-1)!}, \quad \gamma \geq 0.$$

The average probability of symbol error becomes

$$\bar{P}_s = \int_0^\infty \alpha Q(\sqrt{\beta\gamma}) p_{\gamma_\Sigma}(\gamma) d\gamma$$

Capacity of fading channels

- Capacity of AWGN channel: $C = B\log_2(1+\gamma)$
  C: bits/sec, B:Bandwidth, $\gamma = \frac{P}{N_o B}$(received SNR)
- Meaning of capacity
- Fading channels: SNR is a random variable.
- Finding capacity if only channel distribution is known @TX and RX is hard.
- Channel capacity with RX CSI (Channel Side Information)
  - RX knows the channel.
  - TX sends a fixed rate
  - Two types of capacity: Shannon and Outage
  - Shannon capacity: ergodic capacity, max rate with which you can still decode with zero error probability.
  - Capacity with outage: you allow some probability of error.

[**Shannon (Ergodic) Capacity**] For RX CSI, we will have

$$C = \int_0^\infty B\log_2(1+\gamma)p(r)dr$$

log is a concave function ⇒ applying Jenson's Inequality.
For concave $f(x)$,

$$\mathbf{E}(f(x)) \le f(\mathbf{E}(x)) \Rightarrow C = \int B\log_2(1+\gamma)p(\gamma)d\gamma \le B\log_2(1+\bar{\gamma})$$

It is worse than AWGN with $\bar{\gamma}$.

[Capacity with outage] RX CSI, outage case:
RX assumes γ_{min} for received SNR & sends

$$\begin{aligned} C &= B\log_2(1+\gamma_{min}) \Rightarrow \text{if } \gamma < \gamma_{min} \Rightarrow \text{bits cannot be decoded without error} \\ P_{out} &= \text{Probability of not decoding correctly} \\ &= P(\gamma < \gamma_{min}) \\ C_{out} &= \text{number of received correctly } = (1-P_{out})B\log_2(1+\gamma_{min}) \end{aligned}$$

$\gamma_{min} \uparrow \rightarrow C \uparrow$ but

$$\gamma_{min} \uparrow \Rightarrow \begin{cases} 1+\gamma_{min} & \uparrow \\ 1-P_{out} & \downarrow \end{cases}$$

Choose γ_{min} that gives the optimum.

Channel side information at Transmitter and Receiver

[Shannon capacity, TX & RX CSI:]

- Adapt the rate to SNR.

$$C = \max \int_0^\infty B\log_2\left(1+\frac{P(\gamma)\gamma}{\bar{P}}\right)p(\gamma)d\gamma$$

$$\int_0^\infty P(\gamma)p(\gamma)d\gamma \le \bar{P}$$

- Optimum $P(\gamma)$: Form Lagrangian :

$$J = \int_0^\infty B\log_2\left(1+\frac{P(\gamma)\gamma}{\bar{P}}\right)p(\gamma)d\gamma - \lambda\left(\int_0^\infty P(\gamma)p(\gamma)d\gamma - \bar{P}\right)$$

differentiate with respect to $P(\gamma)$,

$$\frac{\partial J}{\partial P(\gamma)} = \frac{(B/\ln 2)(\gamma/\overline{P})}{1+P(\gamma)\gamma/\bar{P}}p(\gamma) - \lambda p(\gamma) = 0$$

then,

$$P(\gamma) = \frac{(B/\ln 2)(\gamma/\overline{P}) - \lambda}{\gamma(\lambda/\overline{P})} > 0$$

$$\gamma > \frac{\overline{P}\lambda}{B}\ln 2 = \gamma_0 \Rightarrow P(\gamma) = \begin{cases} \frac{\overline{P}}{\gamma_0} - \frac{\overline{P}}{\gamma} & \gamma \ge \gamma_0 \\ 0 & \text{else} \end{cases}$$

$$C = \int_{\gamma_0}^{\infty} B \log_2(\gamma/\gamma_0) p(\gamma) d\gamma$$

Finding γ_0:

$$\int_0^{\infty} \frac{P(\gamma)}{\overline{P}} p(\gamma) d\gamma = 1$$
$$\int_{\gamma_0}^{\infty} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right) p(\gamma) d\gamma = 1$$

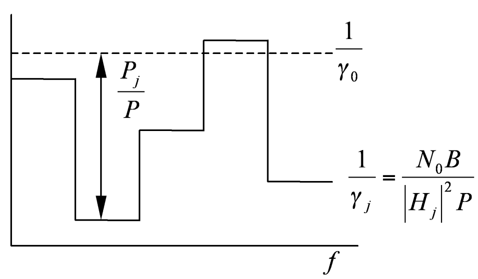
[Wideband Channels:] want to distribute total TX power over the frequency bins:

$$C = \sum B \log_2\left(1 + \frac{|H_j|^2 P_j}{N_o B}\right)$$
$$\sum P_j \leq P$$

Similar Lagrangian method:

$$\frac{P_j}{P} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma_j} & \gamma_j \geq \gamma_0 \\ 0 & \text{else} \end{cases}$$

Find $\gamma_0 \Rightarrow \sum_j \frac{1}{\gamma_0} - \frac{1}{\gamma_j} = 1$ where $\gamma_j = \frac{|H_j|^2 P_j}{N_o B}$



Water-filling (Multicarrier Modulation) : DSL