# Wireless Communications
# Lecture 12

## Variable-Rate Variable-Power MQAM

[**Adaptive Rate & Power**]

- Maximize average spectral efficiency:

$$\max \int_0^\infty \log_2\left(1 + K\gamma\frac{P(\gamma)}{\overline{P}}\right)p(\gamma)d\gamma$$

where $K > 0$ and

$$\int P(\gamma)p(\gamma)d\gamma = \overline{P}$$

To find the optimum P($\gamma$), we form the Lagrangian

$$L = \int_0^\infty \log_2\left(1 + K\gamma\frac{P(\gamma)}{\overline{P}}\right)p(\gamma)d\gamma - \lambda\int_0^\infty P(\gamma)p(\gamma)d\gamma + \lambda\overline{P}$$

differentiate with respect to P($\gamma$)

$$\frac{\partial L}{\partial P(\gamma)} = 0$$

Then,

$$\frac{P(\gamma)}{\overline{P}} = \begin{cases} \frac{1}{\overline{P}\lambda\ln 2} - \frac{1}{K\gamma} & \gamma > \frac{\overline{P}\lambda\ln 2}{K} \\ 0 & \text{else} \end{cases}$$

- Define $\gamma_0 = \overline{P}\lambda\ln 2$

Compare with capacity achieving power adaptation technique. We had the same $\gamma_0$ ($\max C/B$ not $C$).

$$\gamma_K = \frac{\gamma_0}{K} \Rightarrow K < 1,\ \frac{R}{B} = \int_{\gamma_K}^\infty \log_2\left(\frac{\gamma}{\gamma_K}\right)p(\gamma)d\gamma,\ \frac{C}{B} = \int_{\gamma_0}^\infty \log_2\left(\frac{\gamma}{\gamma_0}\right)p(\gamma)d\gamma$$

- $R < C$ gap can be minimized through coding.

[**Example**] An MQAM is used with $P_{b_1} = 10^{-3}$ & $P_{b_2} = 10^{-6}$. Which one gives a higher spectral efficiency?

**[Channel Inversion with Fixed Rate]**

We know that $\sigma = \frac{1}{\mathbf{E}[1/\gamma]}$.
Then, for MQAM,

$$\frac{R}{B} = \log_2\left(1 + \frac{K}{\mathbf{E}[1/\gamma]}\right), \quad \left(1 + \frac{K}{\mathbf{E}[1/\gamma]}\right) : M \geq 4 \text{ if not, the other bound should be used}$$

**[Truncated channel Inversion]**

$$\sigma = \frac{1}{\int_{\gamma_0}^{\infty} \frac{P(\gamma)}{\gamma} d\gamma}$$

$$\frac{R}{B} = \max_{\gamma_0} \underbrace{\log_2\left(1 + \frac{K}{\int_{\gamma_0}^{\infty} \frac{P(\gamma)}{\gamma} d\gamma}\right)}_{\gamma_0 \uparrow \Rightarrow \uparrow} \underbrace{P(\gamma > \gamma_0)}_{\gamma_0 \uparrow \Rightarrow \downarrow}$$

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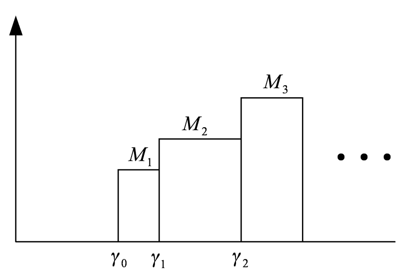
**[Discrete-Power/ Continuous-Power & discrete-rate adaptation]**
Restrict $M$ to $2^{2(j-1)}$ for $2 \leq j \leq N-1$ such that

$$\begin{cases} M_0 = 0 \\ M_1 = 2 \\ \dots \\ M_j = 2^{2(j-1)}, \quad 2 \leq j \leq N-1 \end{cases}$$




Optimum boundaries?

- Sub-optimal solution:

  We know

  $$M(\gamma) = \frac{\gamma}{\gamma_K^*}$$

  Given $\gamma$, $M_j \leq M(\gamma) < M_{j+1}$, choose $M_j$.
  $\Rightarrow$ If $\gamma_K^* M_j \leq \gamma < \gamma_K^* M_{j+1}$, choose $M_j$.

- Using Power policy:

  $$M = 1 + K\gamma \frac{P(\gamma)}{\overline{P}}$$

  We will have

  $$\begin{aligned} \frac{P_j(\gamma)}{\overline{P}} &= \begin{cases} \frac{(M_j - 1)}{K\gamma} & M_j \leq \frac{\gamma}{\gamma_K^*} \leq M_{j+1} \\ 0 & M_j = 0 \end{cases} \\ \text{Received SNR} &= \frac{M_j - 1}{K} \end{aligned}$$

- Average spectral efficiency:

  $$\frac{R}{B} = \sum_{j=1}^{N-1} \log_2(M_j) prob\{M_j \leq \frac{\gamma}{\gamma_K^*} < M_{j+1}\}$$

  Find $\gamma_K^*$ to maximize $R/B$ subject to

  $$\sum_{j=1}^{N-1} P_j(\gamma) prob\{M_j \leq \frac{\gamma}{\gamma_K^*} < M_{j+1}\} = \bar{P}$$

  Need to solve with numerical methods.

  [**Note**] we can also change $N$.

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**[Discrete Rate Channel Inversion case]**

$$\frac{R}{B} = \log_2 \left\lfloor 1 + \frac{K}{\mathbf{E}[1/\gamma]} \right\rfloor_\mu$$

where $\mu = \{M_0, M_1, \ldots, M_{N-1}\}$

Truncated cases:

$$\frac{R}{B} = \max_{\gamma_0} \log_2 \left[ 1 + \frac{K}{\int_{\gamma_0}^{\infty} \frac{P(\gamma)}{\gamma}} d\gamma \right] P(\gamma > \gamma_0)$$

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**[Example]** Compare adaptive & non-adaptive schemes

(a) using $P_b$ approximation for MQAM, find an expression for average $P_b$ in Rayleigh fading.

$$\begin{aligned} P_b &\approx 0.2 e^{-1.5\gamma/(M-1)} \\ \overline{P}_b &= 0.2 \int_0^{\infty} \lambda e^{-\lambda\gamma - \frac{1.5\gamma}{M-1}} d\gamma = \frac{0.2\lambda}{\lambda + 1.5/(M-1)} \end{aligned}$$

(b) Max constellation size for $\overline{P}_b \leq 10^{-3}$ & $\lambda = \frac{1}{100}$.

$\overline{P}_b \leq 10^{-3} \Rightarrow M \leq 1.75 \Rightarrow$ spectral efficiency $= 0.81 \Rightarrow$ integer MQAM spectral efficiency $= 0$

Compare with adaptive case of Figure 9.3, $\overline{\gamma} = 20$ dB, then spectral efficiency $= 4$ bps/Hz.

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**[Example]** Rayleigh fading channel with average SNR of 20 dB and target $P_b = 10^{-4}$.

(a) Find optimal rate & power for adaptive rate & power scheme.

$$\frac{P(\gamma)}{\overline{P}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma_K} & \gamma > \frac{\gamma_0}{K} \\ 0 & \text{else} \end{cases}$$

where $\gamma_K = \frac{\gamma_0}{K}$ and $M(\gamma) = \frac{\gamma}{\gamma_K}$

$$K = \frac{-1.5}{\log(5P_b)} = 0.1973, \quad \int_0^{\infty} P(\gamma)p(\gamma)d\gamma = \overline{P}, \quad \int_{\frac{\gamma_0}{K}}^{\infty} \left( \frac{1}{\gamma_0} - \frac{1}{\gamma_K} \right) \lambda e^{-\lambda\gamma} d\gamma = 1$$

$\lambda = 1/100$, using Matlab $\gamma_0 = 0.8486$.

Send only if $\gamma \geq \frac{0.8486}{0.1973} = 4.3$

(b) Average spectral efficiency

$$\begin{aligned}\mathbf{E}\left[\frac{R}{B}\right] &= \mathbf{E}\left[\log_2(M)\right] \\ &= \int_{\gamma_K}^{\infty} \log_2\left(\frac{\gamma}{\gamma_K}\right)\lambda e^{-\lambda\gamma}d\gamma\end{aligned}$$

with Matlab, $\mathbf{E}\left[\frac{R}{B}\right] = 3.7681$ bits/sec/Hz

(c) Compare (b) with truncated channel inversion.

Truncated inversion is $\frac{P(\gamma)}{P}\gamma = \sigma, \gamma \geq \gamma_0$.

$$\begin{aligned}\sigma &= \frac{1}{\int_{\gamma_0}^{\infty} P(\gamma)/\gamma d\gamma} \\ \frac{R}{B} &= \max_{\gamma_0} \log_2(1 + K\sigma) prob\{\gamma > \gamma_0\}\end{aligned}$$

using Matlab, $\gamma_0 = 23.18$, $\frac{R}{B} = 3.3628$.