Binary Consensus with Gaussian Communication Noise: A Probabilistic Approach

Yasamin Mostofi
Department of Electrical and Computer Engineering
University of New Mexico, Albuquerque, NM 87113, USA

Abstract—In this paper we consider the impact of Gaussian communication noise on a network that is trying to reach consensus on the occurrence of an event. We take a probabilistic approach and formulate the consensus problem using Markov chains. We show that the steady state behavior in the presence of any amount of non-zero communication noise is unfavorable as the network loses the memory of the initial state. However, we show that the network can still reach and stay in accurate consensus for a long period of time. In order to characterize this, we derive a close approximation for the second largest eigenvalue of the network and show how it is related to the size of the network and communication noise variance.

I. INTRODUCTION

Cooperative decision-making and control has gotten considerable interest in recent years. Such problems arise in many different areas such as environmental monitoring, surveillance and security, smart homes and factories, target tracking and military systems. Consider a scenario where a network of agents wants to perform a task jointly. Each agent has limited sensing capabilities and has to rely on the group for improving its estimation/detection quality. Consensus problems arise when the agents need to reach an agreement on a value of a parameter. We categorize these problems into two main groups: Estimation Consensus and Detection Consensus.

By estimation consensus we refer to the problems in which each agent has an estimate of the parameter of interest, where the parameter of interest (and therefore the states of the system) can take values over an infinite set or an unknown finite set. Then the agents want to reach a consensus on the value of the parameter. These problems received considerable attention over the past few years. Convergence and equilibrium state of continuous-time and discrete-time consensus protocols have been studied for both time-invariant and time-varying topologies [1], [2], [3], [4]. Furthermore, consensus protocols have been applied to formation problems [5], [6], [7], [8] as well as distributed filtering [9]. The uncertainty in the exchanged information was considered and accounted for in [11] where conditions for achieving consensus were derived. [10] provides a comprehensive survey of literature on such consensus problems.

By detection Consensus, on the other hand, we refer to problems in which the parameter of interest takes values from a finite known set. Then the update protocol that each agent will utilize becomes non-linear. We refer to a subset of detection consensus problems where the network is trying to reach an agreement over a parameter that can only have two values as binary consensus. For instance, networked detection of fire falls into this category. [12] considered and characterized phase transition of a binary consensus problem in the presence of a uniformly distributed communication noise. Since the probability density function of this noise is bounded, there exists a transition point beyond which consensus will be reached in this case [12].

In most consensus applications, the agents will communicate their status wirelessly. On the bit level, receiver noise, which is best modeled as an additive Gaussian noise [13], will corrupt the transmitted bits. Therefore, it becomes important to analyze the performance of consensus problems in the presence of Gaussian communication noise. In this paper we consider a networked binary consensus problem with Gaussian communication noise. Since the noise is not bounded, there is no transition point beyond which consensus is guaranteed. Instead, we utilize a probabilistic approach to characterize and understand the behavior of the network. Introducing Gaussian communication noise, in itself, will result in complex behaviors. Therefore, in this paper we assume that the underlying graph of the network is fully connected in order to focus on the impact of noisy communication links. We characterize the network in terms of probability of reaching and staying in consensus as well as steady-state behavior, by utilizing the properties of binomial coefficients, $Q(.)$ function and the underlying Markov chain. We show that steady-state behavior of such systems is undesirable independent of the amount of communication noise variance as the network loses the memory of the initial state. However, we show that the network can still reach and stay in agreement for a long period of time before it heads towards its steady-state value. Our main contribution is characterizing this probabilistically by deriving a close approximation for the second largest eigenvalue of the underlying transition probability matrix.

The paper is organized as follows. System model and the underlying Markov chain are presented in Section II and III respectively. Steady-state behaviors are characterized in Section IV. Transient behaviors such as probability of having a consensus at $k^{th}$ time step is analyzed in Section V, where our main result is presented: deriving an expression for the second largest eigenvalue of the transition probability matrix. This is followed by conclusions and future directions in Section VI.
Consider $M$ agents that want to reach consensus on the occurrence of an event. Each agent makes a decision on the occurrence of the event based on its one-time local sensor measurement. Let $b_i(0) \in \{0, 1\}$ represent the initial decision of the $i^{th}$ agent, at time step $k = 0$, based on its local measurement. $b_i = 1$ indicates that the $i^{th}$ agent votes that the event occurred whereas $b_i = 0$ denotes otherwise. Each agent sends its binary vote to the rest of the group and revises its vote based on the received information. This process will go on for a while. We say that accurate consensus is achieved if each agent reaches the majority of the votes. In this paper, we assume a fully connected underlying graph to devote our attention to the interesting and unexplored behaviors that result from voting over Gaussian noise channels. This means that every agent can exchange information with every other one. However, the received information will be corrupted by communication noise.

Each transmission gets corrupted by receiver noise, which is best modeled by Additive White Gaussian Noise (AWGN channel) [13]. When a node receives the decisions of other nodes, the receptions can happen in different frequencies or time slots [14]. Then each reception will experience a different (uncorrelated) sample of the receiver thermal noise. Let $n_{j,i}(k)$ represent the noise at the $i^{th}$ node due to the transmission of the information from the $j^{th}$ node to the $i^{th}$ one. $n_{j,i}(k)$ is a zero-mean Gaussian random variable with variance of $\sigma^2$. We take the variance of the noise of all the transmissions to be the same in this paper. This is a fair assumption as variance of this noise is a function of the receiver bandwidth and thermal noise [14]. Let $b_{j,i}(k)$ represent the reception of the $i^{th}$ agent from the transmission of the $j^{th}$ one at $k^{th}$ time step. We will have, $b_{j,i}(k) = b_j(k) + n_{j,i}(k)$ for $1 \leq i, j \leq M$. Each agent will then update its vote based on the received information as follows:

$$b_i(k+1) = \text{Dec}\left(\frac{1}{M} \sum_{j=1}^{M} b_{j,i}(k)\right)$$

$$= \text{Dec}\left(\frac{1}{M} \sum_{j=1}^{M} b_j(k) + \frac{1}{M} \sum_{j=1, j\neq i}^{M} n_{j,i}(k)\right)$$

$$= \text{Dec}\left(\frac{S(k)}{M} + \frac{1}{M} \sum_{j=1, j\neq i}^{M} n_{j,i}(k)\right),$$

(1)

where $\text{Dec}(.)$ represents a decision function for binary 0-1 detection: $\text{Dec}(x) = \begin{cases} 1 & x \geq 0.5 \\ 0 & x < 0.5 \end{cases}$. Let $\Omega_M = \{0, 1, 2, \ldots, M\}$ represent a finite set. Then $S(k) = \sum_{j=1}^{M} b_j(k) \in \Omega_M$ represents the state of the network (sum of 1 votes) at $k^{th}$ time step. $S(k)$ is a measure of the closeness to consensus at time step $k$.

**Definition 1:** We say that the network is in an accurate consensus state in the $k^{th}$ time step if and only if the following holds:

$$\text{if } S(0) \geq \left\lceil \frac{M}{2} \right\rceil \Rightarrow \forall i \quad b_i(k) = 1$$

$$\text{if } S(0) < \left\lceil \frac{M}{2} \right\rceil \Rightarrow \forall i \quad b_i(k) = 0,$$

(2)

where $\lceil . \rceil$ represents the ceiling function. Note that, without loss of generality, we decided that an agent with half votes of zeros and half votes of ones, will vote one.

**Definition 2:** The network can be in a state of consensus that is not related to its initial state. We refer to this state as memoryless consensus. For instance, if 70% of the nodes start by voting 1 but the network ends up in a state where every node is voting 0, the network has reached a memoryless consensus. This is undesirable since the final agreement is not related to the initial state of the system and is merely a function of the communication noise. In this paper, the term consensus will refer to the accurate consensus state unless otherwise is stated.

**Remark 1:** Due to the presence of Gaussian noises, there is no guarantee in reaching or staying in consensus. In other words, there is no transition point beyond which consensus is reached permanently. Rather than that, we are interested in evaluating the probability of reaching and staying in different states of the network, steady state probabilistic behavior of the system as well as relating these parameters to the noise variance and size of the network.

### III. Markov Chain Representation

Let $\Pi_i(k)$ represent the probability that $i \in \Omega_M$ of the agents are voting 1 at $k^{th}$ time step: $\Pi_i(k) = \text{Prob}[S(k) = i], \quad i \in \Omega_M$. Let $P_{i,j}$ represent the probability of the network going from the $i^{th}$ state to the $j^{th}$ one in one time step,

$$P_{i,j} = \text{Prob}[S(k+1) = j|S(k) = i], \quad i, j \in \Omega_M.$$

(3)

Note that, given the current state, $P_{i,j}$ becomes independent of the earlier states: $S(k-1), S(k-2), \ldots, S(0)$. Furthermore, $P_{i,j}$ is independent of $k$, as indicated by Eq. 3. More specifically, $P_{i,j}$ will have a binomial distribution as follows,

$$P_{i,j} = \binom{M}{j} \kappa_i^j (1 - \kappa_i)^{M-j},$$

(4)

where $\kappa_i$ represents the probability that any agent votes 1 in the next time step, given a current state of $i$ ($S(k) = i$). This probability is the same for all the agents. Consider $l^{th}$ agent. We will have:

$$\kappa_i = \text{Prob}\left[\frac{i}{M} + \omega_l(k) > 0.5\right],$$

(5)
where \( w_i(k) = \frac{1}{M} \sum_{j=1, j \neq i}^M n_{i,j}(k) \) is a weighted sum of i.i.d Gaussian random variables, which will in turn have a Gaussian distribution: \( w_i \sim \mathcal{N}(0, \sigma_M) \), where \( \sigma_M^2 = \frac{(M-1) \mu^2}{M} \) and \( \mathcal{N}(\mu, \sigma) \) represents a normal distribution with average of \( \mu \) and standard deviation of \( \sigma \). Therefore,

\[
\kappa_i = Q \left( \frac{5 - i}{\sigma_M} \right),
\]

where \( Q(.) \) represents the Q function: \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2} dz \).

The consensus of \( M \) agents that communicate over noisy links can, therefore, be modeled as a Markov Chain with the sequence of random variables, \( S(0), S(1), S(2), \ldots \), where the possible states for \( S(k) \) form a countable set \( \Omega_M \) with \( M + 1 \) elements. The states represent all the possibilities for the sum of all the votes. \( P_{i,j} \) will then represent the transition probability from state \( i \) to state \( j \) for \( i, j \in \Omega_M \).

We will have the following time update,

\[
\Pi(k+1) = P^T \Pi(k),
\]

where \( \Pi(k) = [ \Pi_0(k) \ \Pi_1(k) \ \ldots \ \Pi_M(k) ]^T \) and \( P = [P_{i,j}] \).

**A. Properties of Matrix \( P \) and the Underlying Markov Chain**

The Markov chain, represented by Eq. 7, is irreducible with aperiodic states, and therefore ergodic for \( \sigma_n \neq 0 \). Matrix \( P \) is a stochastic matrix, i.e. \( \forall i, \sum_{j=0}^M P_{i,j} = 1 \). Therefore, one is an eigenvalue of \( P \). Let \( \lambda_0, \lambda_1, \ldots, \lambda_M \) represent the eigenvalues of matrix \( P \) in a decreasing order: \( |\lambda_0| \geq |\lambda_1| \geq \ldots \geq |\lambda_M| \).

**Remark 2:** Using Gersgorin disk theorem [15], the eigenvalues of \( P \) are located in the following area:

\[
\bigcup_{i=1}^M \{ z \in \mathbb{C} : |z - P_{i,i}| \leq 1 - P_{i,i} \}.
\]

This implies that \( \forall i, |\lambda_i| \leq 1 \).

**Remark 3:** Assume \( \sigma_n \neq 0 \). Then \( P > 0 \) (element-wise). From stochastic property of matrix \( P \), we know that one is an eigenvalue. From Remark 2 and applying Perron’s theorem [15], we will have,

\begin{align*}
& a) \ |\lambda_0| = 1 \text{ as a simple eigenvalue of } P, \\
& b) \ |\lambda_i| < 1 \text{ for } i \neq 0 \text{ and } \\
& c) \ [P^T]^m \to L \text{ as } m \to \infty, \text{ where } L = xy^T, \ x = PT_x, \ y = Py, \text{ and } x^T y = 1.
\end{align*}

**Lemma 1:** The \((M-i)\)th row of matrix \( P \) is a reverse repeated version of the \(i\)th row for \( i \in \Omega_M \), i.e. \( P_{i,j} = P_{M-i,j-M} \) for \( i, j \in \Omega_M \).

**Proof:** \( \kappa_{M-i} = Q \left( \frac{5 - M - i}{\sigma_M} \right) = Q \left( \frac{5 - \pi_i}{\sigma_M} \right) = 1 - \kappa_i \).

Then from the expression of \( P_{i,j} \) of Eq. 4, it can be easily seen that \( P_{i,j} = P_{M-i,j-M} \).

**IV. Steady-state Behavior**

In this section, we are interested in steady state analysis of the aforementioned networked consensus problem. Using Eq. 7, we will have

\[
\Pi(k) = [P^T]^k \Pi(0).
\]
Lemma 3: \( \lim_{k \to \infty} \lim_{\sigma_n \to \infty} \Pi(k) = B \), where \( B = 0.5^M \left[ \begin{array}{ccc} M & M & \ldots & M \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ \end{array} \right]^T \).

Proof: In this case, we will have, \( \lim_{\sigma_n \to \infty} \kappa_i = 0.5 \) and \( \lim_{\sigma_n \to \infty} P_{i,j} = \left( \begin{array}{c} M \\ j \end{array} \right) 0.5^M \). Therefore, we will have,

\[
P_{\sigma_n \to \infty}^T = \lim_{\sigma_n \to \infty} P_{\sigma_n \to \infty}^T = \left[ B \ B \ \ldots \ B \right].
\] (9)

Then \( P_T^\sigma \) becomes a rank one matrix with \( \lambda_0 = 1 \) and \( \lambda_i = 0 \) for \( 1 \leq i \leq M \). By noting that \( \sum(B) = 0.5^M \sum_{i=0}^{M} \left( \begin{array}{c} M \\ i \end{array} \right) = 1 \), it can be easily confirmed that \( P_{\sigma_n \to \infty}^T \) for the case, \( B \) is the eigenvector of \( P_{\sigma_n \to \infty}^T \) that corresponds to \( \lambda_0 = 1 \) and is the steady-state vector.

2) Special Case: \( M \to \infty \): This case has similar properties to the case of \( \sigma_n = 0 \). It can be easily shown that \( \kappa_{i,M \to \infty} = \kappa_{i,\sigma_n=0} \) and \( P_{1,3,M \to \infty} = P_{1,3,\sigma_n=0} \). This means that as \( M \to \infty \), the network will reach consensus in one step and will stay there.

V. PROBABILITY OF BEING IN CONSENSUS AFTER \( k \) TIME STEPS: TRANSIENT BEHAVIORS

We observed in the previous section that for any amount of communication noise, accurate consensus can not be achieved in an intelligent way asymptotically. However, this does not mean that networked consensus in the presence of Gaussian noise will never work. Fig. 1 shows \( \Pi(k) \) as a function of time step, \( k \), for \( M = 4 \) agents. The initial condition in this example is \( \Pi[0] = \left[ \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \end{array} \right]^T \), meaning that 3 out of 4 members are voting 1. Then the system is in accurate consensus at time step \( k \) if \( \Pi[k] = \left[ \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \end{array} \right]^T \), i.e. all members voting one. The solid line is showing the probability of having an accurate consensus (\( \Pi_1(k) \)). It can be seen that for a good amount of time (enough in practical situations), the system will be in accurate consensus with high probability. Another interesting feature is the steady-state behavior. In steady-state, for this case, the system will be in consensus with high probability (\( \Pi_0 \) and \( \Pi_4 \) close to .5). However, it will be a memoryless consensus. In general, the smaller the communication noise variance is, the higher the probability of reaching and maintaining an accurate consensus state will be. The second largest eigenvalue, \( \lambda_1 \), plays a key role in determining how fast the network is approaching its steady-state. The closer the second eigenvalue is to the unit circle, the network will be in consensus for a longer period of time. We will then derive an expression for the second largest eigenvalue of matrix P next.

A. Second Largest Eigenvalue

In this part, we will derive an expression for the second largest eigenvalue of \( P \) as it becomes a crucial factor in determining the transient probability of the network being in consensus before it aims towards its steady-state undesirable behavior. Before deriving an expression for the general \( P \) matrix, we will consider a \( 3 \times 3 \) case to gain some insight.

1) Case of \( M = 2 \): In this part, we will consider a case where two agents are trying to achieve consensus. We are interested in finding an analytical expression for the second largest eigenvalue of matrix \( P \).

Lemma 4: Let \( P_{3 \times 3} \) represent \( P \) matrix for \( M = 2 \). Let \( \lambda_{1,3 \times 3} \) represent the second largest eigenvalue of \( P_{3 \times 3} \). Then, the second largest eigenvalue of \( P_{3 \times 3} \) will be as follows: \( \lambda_{1,3 \times 3} = 1 - 2Q \left( \frac{1}{2\sigma_M} \right) \).

Proof: For \( M = 2 \), we will have the following utilizing \( P_{1,3 \times 3} \), \( P_{3 \times 3} \), \( P_{2,0} \), \( P_{1,0} \), \( P_{2,1} \), \( P_{2,0} \), \( P_{0,0} \), \( P_{0,1} \), \( P_{0,2} \), \( P_{0,0} \), \( P_{0,1} \), \( P_{0,2} \). Let \( \lambda \) represent a general eigenvalue of \( P_{3 \times 3} \). Then \( P_{3 \times 3} \beta = \lambda \beta \),

\[
P_{3 \times 3}^T \beta = \beta \left[ \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \end{array} \right] \text{ representing the corresponding eigenvector of } P_{3 \times 3}^T. \] By subtracting the first and third equations of Eq. 10, we will have, \( \beta_1(P_{0,0} - P_{2,0}) - \beta_3(P_{0,0} - P_{2,0}) = \lambda(\beta_2 - \beta_3) \). Therefore, \( \lambda = P_{0,0} - P_{2,0} = 1 - 2k_0 = 1 - 2Q \left( \frac{1}{2\sigma_M} \right) \). Since \( \lim_{\sigma_n \to 0} \lambda = 1 \), \( \lambda \) represents the second largest eigenvalue of \( P_{3 \times 3} \). The smaller eigenvalues will go to zero as \( \sigma_n \to 0 \) (see Section IV-A). Therefore, \( \lambda_{1,3 \times 3} = 1 - 2Q \left( \frac{1}{2\sigma_M} \right) \).

Note that \( Q \left( \frac{1}{2\sigma_M} \right) \leq .5 \) since \( \sigma_M \geq 0 \).

2) General Case: In general, finding an expression for the second largest eigenvalue of the \( M \times M \) case is extremely difficult without approximation. In this part, we will find an approximation for the second largest eigenvalue of matrix \( P \), that matches well with its true value. Without loss of generality, we derive the results of this part for \( M \) even.

Lemma 5: Let \( P \) represent a transition probability matrix generated using \( \kappa_i \), as indicated by Eq. 4. We will have,

\[
\sum_{j=0}^{M-1} \left( \frac{M}{2} - j \right) (P_{i,j} - P_{M-i,j}) = \frac{M}{2}(1 - 2\kappa_i). \] (11)
Proof: We have, 
\[ \sum_{j=0}^{M-1} \left( \frac{M}{j} - j \right) (P_{i,j} - P_{M-i,j}) = \sum_{j=0}^{M-1} \left( \frac{M}{j} - j \right) \left( m_{i,j} \kappa_{i} \right) (1 - \kappa_{i})^{M-j} \] 
- \sum_{j=0}^{M-1} \left( \frac{M}{j} - j \right) \left( \frac{M}{j} \right) (1 - \kappa_{i})^{j} \kappa_{i}^{M-j}.

By noting that \[ \sum_{j=0}^{M} \left( \frac{M}{j} - j \right) (M) \kappa_{i} (1 - \kappa_{i})^{M-j} = \] 
- \sum_{j=0}^{M-1} \left( \frac{M}{j} - j \right) \left( \frac{M}{j} \right) \kappa_{i} (1 - \kappa_{i})^{j} \kappa_{i}^{M-j}, \] 
we have, 
\[ \sum_{j=0}^{M} \left( \frac{M}{j} - j \right) \left( \frac{M}{j} \right) \kappa_{i} (1 - \kappa_{i})^{M-j} = \] 
\[ \sum_{j=0}^{M} \left( \frac{M}{j} - j \right) \left( \frac{M}{j} \right) \kappa_{i} (1 - \kappa_{i})^{j} \kappa_{i}^{M-j} \]
which will be as follows using Lemma 5 (note that Lemma 5 is valid as long as \( \kappa_{i} \) represents a probability value):
\[ \chi(i) = \frac{M}{2} (1 - 2 \kappa_{i,\text{approx}}) = \frac{M}{2} (1 - 2 \kappa_{i}) (1 - 2 \kappa_{0}), \quad \forall i \in \Omega_{M}. \]

This means that \[ \left[ \frac{M}{2}, \frac{M}{2}, \ldots, 2, 1 \right]^{T} \] will be a right eigenvector of \( \Gamma^{T} \) with \( 1 - 2 \kappa_{0} \) representing the corresponding eigenvalue.

Remark 5: Let \( P_{\text{approx}} \) represent an approximation of matrix \( P \) under Assumption 1. It can be easily confirmed that \( P_{\text{approx}} \) is stochastic. Furthermore, by applying Perron’s theorem, it can be seen that \( P_{\text{approx}} \) has a simple eigenvalue at 1 and all the rest within the unit circle for \( \sigma_{n} \neq 0 \).

Lemma 7: The first and second largest eigenvalues of \( P_{\text{approx}, \sigma_{n}=0} \) are one while the rest are within the unit circle.
Proof: We have \( \lim_{\sigma_{n} \to 0} \kappa_{i,\text{approx}} = \frac{M}{2} \). From stochastic property of \( P_{\text{approx}, \sigma_{n}=0} \) we know that one is one of its eigenvalues. It can be easily confirmed that \( P_{\text{approx}, \sigma_{n}=0} \times Z_{1} = Z_{1} \) and \( P_{\text{approx}, \sigma_{n}=0} \times Z_{2} = Z_{2} \), where \( Z_{1} = [1 \ 0 \ 0 \ldots 0]^{T} \) and \( Z_{2} = [0 \ 0 \ldots 1]^{T} \). This suggests that \( P_{\text{approx}, \sigma_{n}=0} \) has one as its eigenvalue with multiplicity of at least two. Let \( \Phi_{\lambda}(A) = \det(A - \lambda I) \), where \( I \) is the unitary matrix. Then \( \Phi_{\lambda}(P_{\text{approx}, \sigma_{n}=0}) = (1 - \lambda) \Phi_{\lambda}(P_{\text{approx}, \sigma_{n}=0}[1,1]) \), where \( A[1,1] \) represents a matrix that results from omitting the first row and column of \( A \). Similarly, let \( P_{\text{approx}, \sigma_{n}=0} \) represent the matrix that results from omitting the last row and column of \( P_{\text{approx}, \sigma_{n}=0} \). Then, \( \Phi_{\lambda}(P_{\text{approx}, \sigma_{n}=0}) = (1 - \lambda)^{2} \Phi_{\lambda}(P_{\text{approx}, \sigma_{n}=0}) \). By applying Gersgorin theorem, it can be easily confirmed that all the eigenvalues of \( P_{\text{approx}, \sigma_{n}=0} \) are inside the unit circle. Therefore, the first and second largest eigenvalues of \( P_{\text{approx}, \sigma_{n}=0} \) are one while the rest are inside the unit circle.

Theorem 1: Let \( P_{\text{approx}} \) represent an approximation of matrix \( P \) under Assumption 1. Let \( \lambda_{1,\text{approx}} \) represent the second largest eigenvalue of \( P_{\text{approx}} \). Then we will have, \( \lambda_{1,\text{approx}} = 1 - 2 \kappa_{0} = 1 - 2Q \left( \frac{1}{2M} \right). \)

Proof: Let \( P \) be partitioned as follows: 
\[ P^{T} = \begin{bmatrix} P_{1} \\ P_{2} \\ P_{3} \end{bmatrix}, \]
where \( P_{1} \) is the matrix with the first \( \frac{M}{2} \) rows of \( P^{T} \), \( P_{2} \) is the \( \frac{M}{2} \)th row of \( P^{T} \) and \( P_{3} \) is the matrix with the last \( \frac{M}{2} \) rows of \( P^{T} \) (note that we assumed an even \( M \), without loss of generality, in this part). Let \( \lambda \) represent an eigenvalue of \( P \) with \( \beta = \begin{bmatrix} \beta_{0} & \beta_{1} & \ldots & \beta_{M} \end{bmatrix}^{T} \) representing the corresponding eigenvector. We will have,
\[ \begin{bmatrix} P_{1} \\ P_{2} \\ P_{3} \end{bmatrix} \beta = \lambda \begin{bmatrix} \beta_{0} & \beta_{1} & \ldots & \beta_{M} \end{bmatrix}^{T} \]
resulting in \( P_{1} \beta = \lambda \begin{bmatrix} \beta_{0} & \beta_{1} & \ldots & \beta_{M-1} \end{bmatrix}^{T} \) and \( P_{3} \beta = \lambda \begin{bmatrix} \beta_{M+1} & \beta_{M+2} & \ldots & \beta_{M} \end{bmatrix}^{T} \). Let \( D \) represent the backward identity matrix: \( D(i,j) = 1 \) only if \( j = M - i \) for \( i,j \in \Omega_{M} \). Then \( DP_{3} \beta = \lambda \begin{bmatrix} \beta_{M} & \beta_{M-1} & \ldots & \beta_{M+1} \end{bmatrix}^{T} \). By utilizing the special structure of matrix \( P \) denoted in Lemma 1, we have,
\[ P_\beta \beta - DP_\beta \beta = \lambda \begin{bmatrix} \beta_0 - \beta_M \\ \beta_1 - \beta_{M-1} \\ \vdots \\ \beta_{M-1} - \beta_{M+1} \end{bmatrix}, \]

which results in \[ \Sigma \zeta = \lambda \zeta, \] where \( \zeta = \begin{bmatrix} \zeta_0 & \zeta_1 & \cdots & \zeta_{M-1} \end{bmatrix}^T \) with \( \zeta_i = \beta_i - \beta_{M-i} \) for \( 0 \leq i \leq \frac{M}{2} - 1 \) and \( \Sigma_{j,i} = P_{\beta_{j-i}} - P_{\beta_{M-i-j}} \) for \( 0 \leq i,j \leq \frac{M}{2} - 1 \). Therefore, the eigenvalues of \( \Sigma \) are eigenvalues of \( P \). Let \( \Sigma_{\text{approx}} \) represent the \( \Sigma \) generated from \( P_{\text{approx}} \). From Lemma 6, we will have \( \Sigma_{\text{approx}} = \Gamma \), with \( 1 - 2\kappa_0 \) as one of its eigenvalues. Therefore, \( 1 - 2\kappa_0 \) is one of the eigenvalues of matrix \( P_{\text{approx}} \). Furthermore, it is the second largest eigenvalue since as \( \sigma_n \) goes to zero, this eigenvalue goes towards one, which only happens to the second largest eigenvalue (see Lemma 7). Therefore, \( \lambda_{1,\text{approx}} = 1 - 2\kappa_0 = 1 - 2Q \left( \frac{1}{25M} \right) \).

To see how well this approximation works, Fig. 2 shows the second largest eigenvalue and its approximation as a function of \( \sigma_n \) and for \( M = 4, M = 7 \) and \( M = 16 \). As can be seen, the approximation works well especially for smaller \( M \) and larger \( \sigma_n \). For instance for \( M = 16 \), we see a small mismatch for smaller values of \( \sigma_n \). This is as expected since the \( Q(\cdot) \) function approximation of Assumption 1 works better the larger \( \sigma_M \) is. Nevertheless, the proposed approximation can be considerably useful in understanding the behavior of group consensus in terms of probability of reaching and staying in consensus.

![Fig. 2. Approximation of the second largest eigenvalue](image)

**VI. CONCLUSIONS AND FUTURE DIRECTIONS**

In this paper, we considered a binary consensus problem in the presence of Gaussian communication noise. We took a probabilistic approach since in the presence of Gaussian communication noise, there will be no transition point beyond which convergence is guaranteed. We showed that for any amount of communication noise, the network’s steady-state behavior is undesirable since it can only reach a memoryless consensus, i.e. a consensus that is independent of the initial state. However, the network can still be in accurate consensus for a long period of time (good enough for practical purposes), before it heads towards a memoryless steady-state, depending on its second largest eigenvalue. To characterize this, we derived an approximated expression for the second largest eigenvalue of the transition probability matrix of the network. The approximation matched the true value considerably well and showed how the communication noise variance and size of the network affect consensus behavior. We considered a fully connected network and the same communication noise variance for all the links in this paper. We are currently working on relaxing these assumptions. We are also extending our results to include both communication and sensing errors as well as time-varying link qualities.

**REFERENCES**


