Compressive Cooperative Sensing and Mapping in Mobile Networks

Yasamin Mostofi
Department of Electrical and Computer Engineering
University of New Mexico, Albuquerque, New Mexico 87131, USA
Email: ymostofi@ece.unm.edu

Abstract

In this paper we consider a mobile cooperative network that is tasked with building a map of the spatial variations of a parameter of interest. We propose a new theoretical framework that allows the nodes to build a map of the parameter of interest with a small number of measurements. By using the recent results in the area of compressive sensing, we show how the nodes can exploit the sparse representation of the parameter of interest in the transform domain, in order to build a map with minimal sensing. More specifically, we discuss three applications: 1) aerial mapping by cooperative Unmanned Autonomous Vehicles (UAVs) and through direct sensing, 2) mapping of the obstacles based on indirect sensing and wireless measurements, and 3) mapping of the communication signal strength to a fixed station, based on direct sensing. Our results demonstrate the potential of this framework. More specifically, for the cases of aerial mapping and mapping of the communication signal strength, we assert the framework by using real data.

Index Terms

mobile networks, compressive sensing, cooperative aerial mapping, mapping of obstacles, mapping of communication signal strength

I. INTRODUCTION

Mobile intelligent networks can play a key role in emergency response, surveillance and security, and battlefield operations. The vision of a multi-agent robotic network cooperatively

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learning and adapting in harsh unknown environments to achieve a common goal is closer than ever. In this paper, we are interested in the cases where a mobile cooperative network is tasked with collecting information from its environment. More specifically, we consider scenarios where the network is in charge of building a map of the spatial variations of a parameter (or a number of parameters) cooperatively, to which we refer to as cooperative mapping. Such problems can arise in several different applications. For instance, building a map of the indoor obstacles [3], ocean sampling [4] or aerial mapping [5] all fall into this category. A mobile network tasked with a certain exploratory mission faces an abundance of information. In such an information-rich world, there is simply not enough time to sample the whole environment due to the potential delay-sensitive nature of the application as well as other practical constraints. A group of Unmanned Air Vehicles (UAVs), for instance, may need to cooperatively build an aerial map of an area in a limited time, as is shown in Fig. 1. It is not practical to wait for the collective sampling of the vehicles to cover every single point in the terrain. A fundamental open question is then as follows: What is the minimal collective sensing needed to accurately build a map of the whole terrain despite the fact that significant parts of it will not be sampled? This is a considerably important problem as it enhances our ability to collect information and allows us to keep up with the high volume of information in the environment.

If we can understand the core information present in the data and can show that it has a dimension far less than the data itself, we can then reduce our sensing considerably. While considerable progress has been made in the area of mobile networks, a framework that allows the vehicles to reconstruct the parameter of interest based on a severely under-determined data set is currently missing. In most related work, only areas that are directly sensed are mapped. The rich literature on Simultaneous Localization and Mapping (SLAM) and its several variations fall into this category [6]–[9]. SLAM approaches mainly focus on reducing the uncertainty in the sensed landmarks by using a Kalman filter. Similarly, approaches based on generating an occupancy map also address sensing uncertainty [10]. Another set of approaches, suitable mainly for mapping obstacles, are based on the Next Best View (NBV) problem [3], [11]–[14]. In NBV approaches, the aim is to move to the positions “good” for sensing by guiding the vehicles to the perceived next safest area (area with the most visibility) based on the current map [3]. However, areas that are not sensed directly are not mapped in NBV. In order to map the areas that are not sensed directly, one can possibly use an interpolator [15]. However, when the sample
points are more clustered or the field of interest has localized features (such as an obstacle map), interpolation’s performance can degrade considerably. In [16], it is assumed that the spatial field of interest has a Gaussian distribution with known models for its average and auto-correlation. Similarly, in [4], [17], a linear estimator is used based on known first and second order statistics of the field. In practice, however, spatial models of the average and auto-correlation of the field may not be available. Even if they are known a priori, finding the parameters of the models typically requires gathering several samples of the field. Therefore, such approaches may not be suitable for developing a framework for mapping based on minimal sensing and without a priori information on field distribution.

In this paper, we present a compressive cooperative mapping framework for mobile exploratory networks. By compressive cooperative mapping, we refer to the cooperative mapping of a spatial function based on a considerably small observation set where a large percentage of the area of interest is not sensed directly. Our proposed theory and design tools are inspired by the recent breakthroughs in non-uniform sampling theory [18], [19]. The famous Nyquist-Shannon sampling theorem [20] revolutionized several different fields by showing that, under certain conditions, it is indeed possible to reconstruct a uniformly sampled signal perfectly. The new theory of compressive sampling (also known by other terms such as compressed sensing, compressive sensing or sparse sensing) shows that under certain conditions, it is possible to reconstruct a signal from a considerably incomplete set of observations, i.e. with a number of measurements much less than predicted by the Nyquist-Shannon theorem [18], [19]. This opens new and fundamentally different possibilities in terms of information gathering and processing in mobile networks. In this paper, we develop the fundamentals of cooperative sensing and mapping in mobile networks from a compressive sampling perspective.

While our proposed framework would be applicable to several mobile network applications, in this paper we focus on three main areas essential to the efficient and robust operation of a cooperative network. A mobile network is given an exploratory task, such as collective ground, aerial, or underwater mapping. It needs to build a map of the obstacles for navigation. It furthermore needs to maintain connectivity. In order to do so, an estimation of the spatial variations of the communication signal strength becomes crucial. Fig. 1 shows an example of cooperative aerial mapping. We show how our proposed framework can impact these three key areas, resulting in cooperative mapping with minimal sensing. In the area of obstacle mapping,
we furthermore propose a novel way of mapping obstacles non-invasively, i.e. without sensing them directly. By using wireless channel measurements and the compressive mapping framework, we show how the robots can map the obstacles before entering a building or a room. While we use the term “cooperative mapping” throughout the paper, most of the proposed framework is also applicable to the case where only one robot is tasked with building a spatial map of a field of interest.

The paper is organized as follows. In Section II we discuss the compressibility of the signals of interest in mobile exploratory networks. In Section III we provide a brief introduction to the theory of compressive sensing. In Section IV we consider cooperative aerial mapping, mapping of obstacles as well as mapping of the spatial variations of the communication signal strength. We conclude in Section V. A list of key variables used in the paper is provided in Table 1.

**II. SIGNAL COMPRESSIBILITY IN COOPERATIVE MOBILE NETWORKS**

We first define what “sparse” and “compressible” signals refer to.

*Definition:* A *sparse* signal is a signal that can be represented with a small number of non-zero coefficients.

*Definition:* A *compressible* signal is a signal that has a transformation where most of its energy is in a very few coefficients, making it possible to approximate the rest with zero. In this paper,
we are interested in linear transformations.

The new theory of compressive sampling shows that, under certain conditions, a compressible signal can be reconstructed using very few observations. Most natural signals are indeed compressible. The best sparse representation of a signal depends on the application and can be inferred from analyzing similar data. Our analysis of aerial maps, obstacle maps (indoor or outdoor) as well as maps of communication signal strength, for instance, has shown them to have a considerably sparse representation. Fig. 2 shows two maps based on real data, an aerial map and an obstacle map. By applying a linear transformation to the signals, it can be seen that most of the signal’s energy is contained in a small percentage of the transform coefficients. For instance, 100\% of the energy is in less than 1\% of the wavelet coefficients for the obstacle map on the right. However, this energy is not necessarily confined to a consecutive set of transform coefficients, which makes reconstructing the signal based on a considerably small number of observations challenging. In general, Fourier transformation can provide a good compression for the spatial variations of the communication channel or a height map. For an obstacle map (see Fig. 2 b), wavelet transform or total variation (a difference-based approach) can provide an even better compression. It should be noted that in the compressive mapping of the obstacles, an object-based approach is not suitable. Instead, we consider the space of interest as a binary spatial function that takes on values of 0 or 1. It is also possible to make it non-binary and include the properties of the objects as we shall see in Section IV-B.
In this paper, we show how the new theory of compressive sampling can result in fundamentally different sensing approaches in mobile cooperative exploratory networks. While several signals of interest to the operation of a mobile network have sparse representations, the main challenge is to sense them in the right domain such that it is possible to reconstruct them with minimal sensing, as we show in the next sections. For instance, an obstacle map is sparse in the spatial domain. However, by sensing it in the spatial domain directly, it will not be possible to reconstruct it with a very small number of observations. Therefore, new sensing methods are required, as we will propose in the next sections.

III. COMPRESSION SAMPLING THEORY

The new theory of sampling is based on the fact that real-world signals typically have a sparse representation in a certain transformed domain. Exploiting sparsity, in fact, has a rich history in different fields. For instance, it can result in reduced computational complexity (such as in matrix calculations) or better compression techniques (such as in JPEG2000). However, in such approaches, the signal of interest is first fully sampled, after which a transformation is applied and only the coefficients above a certain threshold are saved. This, however, is not efficient as it puts a heavy burden on sampling the entire signal when only a small percentage of the transformed coefficients are needed to represent it. The new theory of compressive sampling, on the other hand, allows us to sense the signal in a compressed manner to begin with.

Consider a scenario where we are interested in recovering a vector $x \in \mathbb{R}^N$. We refer to the domain of vector $x$ as the primal domain. For 2D signals, vector $x$ can represent the columns of the matrix of interest stacked up to form a vector (a similar approach can be applied to higher-order signals). Let $y \in \mathbb{R}^K$ where $K \ll N$ represent the incomplete linear measurement of vector $x$ obtained by the sensors. We will have

$$y = \Phi x,$$

where we refer to $\Phi$ as the observation matrix. Clearly, solving for $x$ based on the observation set $y$ is an ill-posed problem as the system is severely under-determined ($K \ll N$). However, suppose that $x$ has a sparse representation in another domain, i.e. it can be represented as a linear combination of a small set of vectors:

$$x = \Gamma X,$$
where $\Gamma$ is an invertible matrix and $X$ is $S$-sparse, i.e. $|\text{supp}(X)| = S \ll N$ where $\text{supp}(X)$ refers to the set of indices of the non-zero elements of $X$ and $| \cdot |$ denotes its cardinality. This means that the number of non-zero elements in $X$ is considerably smaller than $N$. Then we will have

$$y = \Psi X,$$

where $\Psi = \Phi \times \Gamma$. We refer to the domain of $X$ as the sparse domain (or transform domain). If $S \leq K$ and we knew the positions of the non-zero coefficients of $X$, we could solve this problem with traditional techniques like least-squares. In general, however, we do not know anything about the structure of $X$ except for the fact that it is sparse (which we can validate by analyzing similar data). The new theory of compressive sensing allows us to solve this problem.

**Theorem 1** (see [18] for details and the proof): If $K \geq 2S$ and under specific conditions, the desired $X$ is the solution to the following optimization problem:

$$\min ||X||_{0}, \text{ such that } y = \Psi X,$$

where $||X||_{0} = |\text{supp}(X)|$ represents the zero norm of vector $X$.

Theorem 1 states that we only need $2 \times S$ measurements to recover $X$ and therefore $x$ fully. This theorem, however, requires solving a non-convex combinatorial problem, which is not practical. For over a decade, mathematicians have worked towards developing an almost perfect approximation to the $\ell_{0}$ optimization problem of Theorem 1 [21]-[22]. Recently, such efforts resulted in several breakthroughs.

More specifically, consider the following $\ell_{1}$ relaxation of the aforementioned $\ell_{0}$ optimization problem:

$$\min ||X||_{1}, \text{ subject to } y = \Psi X.$$

**Theorem 2**: (see [23], [18], [24], [25], [19] for details, the proof and other variations) Assume that $X$ is $S$-sparse. The $\ell_{1}$ relaxation can exactly recover $X$ from measurement $y$ if matrix $\Psi$ satisfies the Restricted Isometry Condition for $(2S, \sqrt{2} - 1)$, as described below.

**Restricted Isometry Condition (RIC)** [26]: Matrix $\Psi$ satisfies the RIC with parameters $(Z, \epsilon)$ for $\epsilon \in (0, 1)$ if

$$(1 - \epsilon)||c||_{2} \leq ||\Psi c||_{2} \leq (1 + \epsilon)||c||_{2}$$

for all $Z$-sparse vector $c$. 

7
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$N$</td>
<td>size of the original signal in the primal domain</td>
</tr>
<tr>
<td>$S$</td>
<td>size of the support of the signal in the sparse domain</td>
</tr>
<tr>
<td>$K$</td>
<td>number of measurements taken to estimate the signal</td>
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<tr>
<td>$x$</td>
<td>signal in the primal domain, an $N \times 1$ vector</td>
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<tr>
<td>$y$</td>
<td>$K \times 1$ measured vector of $x$ in the primal domain</td>
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<tr>
<td>$X$</td>
<td>$N \times 1$ vector representing a linear transform of $x$</td>
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<tr>
<td>$\Phi$</td>
<td>$K \times N$ observation matrix, s.t. $y = \Phi x$</td>
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<tr>
<td>$\Gamma$</td>
<td>$N \times N$ linear projection matrix, s.t. $x = \Gamma X$</td>
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<tr>
<td>$\Gamma^H$</td>
<td>Hermitian of $\Gamma$</td>
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<tr>
<td>$\Psi$</td>
<td>$K \times N$ matrix (defined as $\Psi = \Phi \times \Gamma$), s.t. $y = \Psi X$</td>
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**TABLE I**

**KEY NOTATIONS USED IN THIS PAPER**

The RIC is mathematically related to the uncertainty principle of harmonic analysis [26]. However, it has a simple intuitive interpretation, i.e. it aims at making every set of $Z$ columns of the matrix $\Psi$ as orthogonal as possible. Other conditions and extensions of Theorem 2 have also been developed [27], [28]. While it is not possible to define all the classes of matrices $\Psi$ that satisfy RIC, it is shown that random partial Fourier matrices [29] as well as random Gaussian [30]-[31] or Bernoulli matrices [32] satisfy RIC (a stronger version) with the probability $1 - O(N^{-M})$ if

$$K \geq B_M S \times \log^{O(1)} N,$$

where $B_M$ is a constant, $M$ is an accuracy parameter and $O(\cdot)$ is Big-O notation [18]. Eq. 7 shows that the number of required measurements could be considerably less than $N$.

While the recovery of sparse signals is important, in practice signals may rarely be sparse. Most signals, however, will be compressible. In practice, the observation vector $y$ will also be
corrupted by noise. The $\ell_1$ relaxation and the corresponding required RIC condition can be easily extended to the case of noisy observations with compressible signals [23].

The possibility presented by the new theory of sampling, i.e. recovering signals from a considerably incomplete data set has sparked new research in different fields. A good example of a new resulting technology is the recent development of a compressive imaging camera that efficiently captures a single-pixel image by producing a $\Psi$ with Bernoulli distribution using pseudorandom binary patterns [33]. Other applications include medical imaging [34], DNA decoding [35] and computer graphics [36] among others.

A. Basis Pursuit: Reconstruction Using $\ell_1$ Relaxation

The $\ell_1$ optimization problem of Eq. 5 can be posed as a linear programming problem [37]. The compressive sensing algorithms that reconstruct the signal based on $\ell_1$ optimization are typically referred to as “Basis Pursuit” [19]. Reconstruction through $\ell_1$ optimization has the strongest known recovery guarantees [26]. However, the computational complexity of such approaches can be high. SPARSA [38], GPSR [39] and AC [40] are a few examples of the continuing attempts to reduce the computational complexity of the convex relaxation approach.

B. Matching Pursuit: Reconstruction using Successive Interference Cancellation

The Restricted Isometry Condition implies that the columns of matrix $\Psi$ should have a certain near-orthogonality property. Let $\Psi = [\Psi_1, \Psi_2, \ldots, \Psi_N]$, where $\Psi_i$ represents the $i^{th}$ column of matrix $\Psi$. We will have $y = \sum_{j=1}^{N} \Psi_j X_j$, where $X_j$ is the $j^{th}$ component of vector $X$. Consider recovering $X_i$:

$$\frac{\Psi_i^H y}{\Psi_i^H \Psi_i} = X_i + \sum_{j=1, j \neq i}^{N} \frac{\Psi_i^H \Psi_j}{\Psi_i^H \Psi_i} X_j. \quad (8)$$

If the columns of $\Psi$ were orthogonal, then Eq. 8 would have resulted in the recovery of $X_i$. For an under-determined system, however, this will not be the case. Then there are two factors affecting recovery quality based on Eq. 8. First, how orthogonal is the $i^{th}$ column to the rest of the columns and second how strong are the other components of $X$. In other words, it is desirable to first recover the strongest component of $X$, subtract its effect from $y$, recover the second strongest component and continue the process. Adopting the terminology of CDMA
(Code Division Multiple Access) in communication literature, we refer to such approaches as *Successive Interference Cancellation*. In fact, if $X_i \neq 0$, one can think of $\Psi_i$ coding $X_i$. If the $i^{th}$ code is used as in Eq. 8, then ideally $X_j$ for $j \neq i$ can not be decoded properly and only $X_i$ can be recovered.

Such successive cancellation methods have been used in the context of CDMA systems in communication literature for recovering the signals of different users at the base station [41], [42]. While the context of the two problems may seem different, they share a very core fundamental form. Recently, Tropp et al. independently proposed using a version of successive interference cancellation in the context of compressive sampling and derived the conditions under which it can result in almost perfect recovery [43]. They refer to it as Orthogonal Matching Pursuit (OMP). Similar to Successive Interference Cancellation, the basic idea of OMP is to iteratively multiply the measurement vector, $y$, by $\Psi^H$, recover the strongest component, subtract its effect and continue again. Let $I_{\text{set}}$ denote the set of indices of the non-zero coefficients of $X$ that is estimated and updated in every iteration. Once the locations of the $S$ nonzero components of $X$ are found, we can solve directly for $X$ by using a least squares solver:

$$\hat{X} = \arg\min_{X: \text{supp}(X) = I_{\text{set}}} ||y - \Psi X||_2.$$  \hspace{1cm} (9)

A variation of OMP, Regularized Orthogonal Matching Pursuit (ROMP), was later introduced by Needell et al. [26]. The main difference in ROMP as compared to OMP is that in each iterative step, a set of indices (locations of vector $X$ with non-negligible components) are recovered at the same time instead of only one at a time, resulting in a faster recovery [26]. Other variations of this work (some under different names) have also appeared [26]- [44]. Algorithm 1 shows a summary of the steps involved in Matching Pursuit approaches. Function $\Upsilon$ in the second step is $\Upsilon(y^\text{new}) = y^\text{new}$ for OMP and ROMP. Consequently, $\Psi_u = \Psi$ in the third step in this case. Then, $X_{\text{proj}}$ represents the projection of the measurement $y$ to the columns of matrix $\Psi$. This projection serves as the base for deciding the indices that correspond to the significant coefficients of $X$. Let $x_{\text{proj}}$ represent the signal that corresponds to $X_{\text{proj}}$ in the primal domain: $x_{\text{proj}} = \Gamma X_{\text{proj}}$. In applications where $X$ represents the Fourier transformation of $x$, we have $\Gamma^{-1} = \Gamma^H$ (this is the case for several other applications). Furthermore, for the cases where the signal was point sampled in the primal domain (we will see such cases in the next section), the measurement matrix $\Phi$ will have exactly one 1 in every row and at most one 1 at every column, with the rest
Algorithm 1 A Summary of Matching Pursuit Approaches (OMP [43], ROMP [26] and I-ROMP [2])

Input: measured vector $y \in \mathbb{R}^K$, target sparsity $S$, and size of full signal $N$

Output: set of indices $I_{\text{set}} \subset \{1, \ldots, N\}$ of non-zero coefficients in $X$ with $|I_{\text{set}}| \leq S$, and $\hat{X}$, the estimated $X$.

Initialize: $I_{\text{set}} = \emptyset$ and $y^{\text{new}} = y$

1: while stop criteria not met do
2: $y_u^{\text{new}} = \Upsilon(y^{\text{new}})$
3: $X_{\text{proj}} = \Psi H y_u^{\text{new}}$
4: choose a subset of indices from $X_{\text{proj}}$ based on a utilized criteria for deciding the significant coefficients
5: update index set $I_{\text{set}}$
6: $\hat{X} = \arg\min_{X : \text{supp}(X) = I_{\text{set}}} ||y - \Psi X||_2$
7: $y^{\text{new}} = y - \Psi \hat{X}$
8: end while

of the elements zero. Therefore, we will have:

$$x_{\text{proj}} = \Phi^H y,$$ (10)

where $x_{\text{proj}}$ will be the same as $y$ at the measured points but will be zero elsewhere. Although $X_{\text{proj}}$ does not exactly serve as an estimate of $X$, it does serve as a base for deciding the index set. Therefore, it may not be a good starting point for several applications since it assumes that the signal is zero at unmeasured points in the primal domain. If $x$ represents the spatial variations of an aerial map, for instance, having zero in between the acquired measurements is not a realistic initial estimate. In other words, both OMP and ROMP do not consider the progression of the reconstructed signal in the primal domain and only process the signal in the sparse domain. In order to address this, we proposed Interpolated ROMP (I-ROMP) [2], an extension of ROMP [26] with a considerably better performance for certain applications. In I-ROMP, we take advantage of processing the signal in both the primal domain (domain of $x$) as well as the sparse domain (domain of $X$). At every iteration, we start by upsampling the
residual vector $y^{\text{new}}$ through interpolation:

$$y_u^{\text{new}} = \Upsilon_{\text{interp}}(y^{\text{new}}), \quad \Upsilon_{\text{interp}} : \mathbb{R}^K \rightarrow \mathbb{R}^N,$$

where function $\Upsilon_{\text{interp}}$ represents an upsampling function such as an interpolator. Then function $\Upsilon = \Upsilon_{\text{interp}}$ in the second step and $\Psi_u$ is the full $N \times N$ version of $\Psi$ matrix in the third step for I-ROMP. The rest of the steps remain the same. It should be noted that the target sparsity of Algorithm 1 is typically set to a percentage of the number of measurements gathered [45].

While $\ell_1$ relaxation of the previous part can solve the compressive sampling problem with performance guarantees, the computational complexity of the iterative greedy approaches of this part can be considerably less [43]. In the next section, we use both approaches when reconstructing the signal.

IV. COMPRESSION COOPERATIVE MAPPING IN MOBILE NETWORKS

In this section we show how the new theory of compressive sampling and reconstruction can result in the efficient mapping of a spatial function in mobile cooperative networks. In particular, we discuss three main areas, cooperative aerial mapping, mapping of the obstacles and mapping of the communication signal strength.

There are three main factors affecting how well a sampled function can be reconstructed using the compressive sampling framework: 1) the domain in which it was sampled, 2) its sparsity in a targeted transform domain and 3) the RIC of the resulting $\Psi$ matrix (see Table I). As an example, consider an obstacle map where non-zero values represent obstacles. An obstacle map is typically sparse in the spatial domain, i.e. it has localized features. As a result in the Fourier domain, it is not very compressible. Therefore, point sampling it in the spatial domain and using sparsity in the Fourier domain will not result in an efficient reconstruction. An obstacle map is also sparse in the wavelet domain. However, the $\Psi$ matrix that results from point sampling in the spatial domain and using wavelet transform will not meet the RIC condition, making this approach unsuitable. Therefore, in order to have a compressive and efficient reconstruction of an obstacle map, one should consider other possibilities for sensing and reconstruction, as we shall propose in this section. For spatial functions that are not sparse in the spatial domain, such as an aerial map, applying Fourier transformation can result in a considerably compressible function. As discussed earlier, it has been proved that the $\Psi$ matrix that results from point sampling in
the spatial domain and using Fourier transform for reconstruction also meets the RIC condition, making it suitable for compressive reconstruction. In the rest of the section, we first consider cooperative aerial mapping. We then proceed with compressive obstacle mapping and mapping of the communication signal strength.

A. Compressive Mapping of a Spatial Field of Interest: Cooperative Aerial Mapping

Consider a case where a group of Unmanned Air Vehicles (UAVs) are tasked with building an aerial map of a region, as shown in Fig. 1.\(^1\) Then \(x\) of Eq. 1 represents the aerial map of interest in the spatial domain, where the columns of the aerial map matrix are stacked up to form a vector. The vehicles make measurements in the spatial domain, i.e. vector \(y\) consists of the few measurements made by the vehicles. There is no requirement on a specific sampling pattern for the measurements. As such, the measurements could be gathered randomly. The Fourier transform of a height map can be considerably sparse, as is shown in Fig. 2 (left). Then, Fourier transformation can be used for sparse representation and reconstruction. Matrix \(\Gamma\) represents the inverse Fourier transformation matrix and matrix \(\Phi\) will be as follows:

\[
\forall i \ 1 \leq i \leq K, \exists j \ 1 \leq j \leq N \text{ such that } \Phi(i,j) = 1 \text{ and }
\forall i \ 1 \leq i \leq K, \forall j \neq j', \ 1 \leq j, j' \leq N, \text{ if } \Phi(i,j) = 1 \rightarrow \Phi(i,j') = 0, \quad (12)
\]

\(^1\)The approach of this section is equivalently relevant if there is only one UAV that is gathering the measurements over time.
Fig. 4. Demonstration of compressive reconstruction of part of the height map of Albuquerque Sandia Mountains with only 10.33% measurement – the original map (left), reconstruction using random point samples with normalized MSE of $4 \times 10^{-7}$ (middle) and reconstruction using random line trajectories of Fig. 5 with normalized MSE of $5.9 \times 10^{-6}$ (right). For clarity, refer to the original PDF for the color version of this image.

where it represents a matrix with only one 1 in every row. If there are redundant measurements, they may be more than one 1 in every column. Otherwise, there will be at most one 1 in every column. This matrix is the result of point sampling in the spatial domain. Fig. 3 (left) shows an aerial map of a portion of the Sandia Mountains in Albuquerque, NM. Fig. 3 (right) shows our reconstruction when only 30% of the area is sensed. We used I-ROMP of Algorithm 1 for reconstruction and exploited the sparse representation of the signal in the Fourier domain. The normalized Mean Square Error (MSE) of this reconstruction is $7.5 \times 10^{-8}$. It can be seen that the reconstructed map is almost identical to the real map. In this figure, the measurement points are randomly distributed over the area of interest. In practice, it would be hard for a number
of mobile units (or one) to randomly point sample the area. For instance, a number of UAVs would make measurements along their trajectories. Therefore, we next show the performance when measurements are gathered randomly over line trajectories. Fig. 4 shows compressive reconstruction of a smaller part of the Sandia height map with only 10.33% measurements. The middle figure shows the reconstruction when the measurements are randomly distributed over the area of interest for the original map shown in the left figure. The right figure, on the other hand, shows the case where the same number of measurements is gathered over random line trajectories. Fig. 5 shows an example of random line trajectories, which were used for the reconstruction of Fig. 4 (right). It can be seen that using random line measurements (instead of random point samples) results in a loss of performance as the normalized MSE increases from $4 \times 10^{-7}$ (for the middle figure) to $5.9 \times 10^{-6}$ (for the right figure). This is expected as the derivations and performance guarantees of compressive sampling theory are based on purely random samples. Still the MSE is considerably low for both cases and they provide useful information about the field of interest with very few measurements. In this case, we used I-ROMP for the reconstruction of the middle figure. However, when the samples are more localized as is the case for the right figure, I-ROMP’s performance degrades. This is due to the fact that I-ROMP combines spatial interpolation with compressive sensing. Therefore, as the sampling positions become more clustered, its performance degrades. We then used an $\ell_1$ relaxation approach for the reconstruction of the right figure. OMP also provides a very similar result with a slightly higher MSE.

Overall, the results indicate the potentials of compressive mapping framework for efficient sensing and reconstruction in mobile networks.

B. Compressive Cooperative Mapping of Obstacles

In this section we show how a group of mobile nodes can build a high-quality map of the obstacles with minimal sensing and without directly sampling a high percentage of the area. Accurate mapping of the obstacles is considerably important for the robust operation of a mobile network. Yet the high volume of the information presented by the environment makes it prohibitive to sense all the areas, making accurate mapping considerably challenging. In this part, we show how the nodes can cooperatively build a map of the obstacles based on a considerably small set of observations. We furthermore propose a non-invasive mapping strategy.
which is enabled by the theory of compressive sampling. By non-invasive mapping, we refer to a mapping technique that allows the vehicles to map the obstacles without sensing them directly. For instance it allows the robots to map the obstacles inside a building or a room before entering it. To the best of author’s knowledge, there is currently no framework for efficient non-invasive mapping of obstacles.

As was discussed earlier, a map of obstacles is sparse in the spatial or wavelet domain, which makes point sampling in the spatial domain unsuitable for reconstruction using the compressive mapping framework. In this section we propose a new sensing method for mapping of obstacles, which is non-invasive and only uses wireless channel measurements.

1) Compressive Non-Invasive Mapping of Obstacles – A New Possibility for Non-Invasive Mapping:

In this part we propose a new non-invasive technique for mapping of the obstacles. In general, devising non-invasive mapping strategies can be considerably challenging. Motivated by computed tomography approaches to medical imaging [46], geology [47], and computer graphics [48], we show how our proposed compressive mapping framework can result in a new
and efficient non-invasive sensing technique for mapping obstacles, based on wireless channel measurements. Consider the case where a number of vehicles want to build a map of the obstacles inside a building before entering it. Non-invasive mapping allows the nodes to assess the situation before entering the building and can be of particular interest in several applications such as an emergency response. In this part, we consider building a 2D map (our proposed approach can be extended to 3D maps as well). Figure 6 (left) shows a sample 2D map where a number of vehicles want to map the space before entering it. Let \( g(u, v) \) represent the binary map of the obstacles at position \((u, v)\) for \(u, v \in \mathbb{R}\). We will have

\[
g(u, v) = \begin{cases} 
1 & \text{if } (u, v) \text{ is an obstacle} \\
0 & \text{else}
\end{cases}
\]  

Consider communication from Transmitter 1 to Receiver 1, as marked in Fig. 6 (left). A fundamental parameter that characterizes the performance of a communication channel is the received signal power, which is measured at every receiver [49]. There are three time-scales associated with the spatio-temporal changes of the channel quality and therefore received signal strength [50], as indicated in Fig. 7. The slowest dynamic is associated with the signal attenuation due to the distance-dependent power fall-off (path loss). Then there is a faster variation referred to as shadow fading (shadowing), which is due to the impact of the blocking objects. This means that each obstacle along the transmission path leaves its mark on the received signal by attenuating it to a certain degree characterized by its properties. Finally, depending on the receiver antenna angle, multiple replicas of the transmitted signal can arrive at the receiver due to the reflection from the surrounding objects, resulting in multipath fading, a faster variation in the received signal power [51].

A communication from Transmitter 1 to Receiver 1 in Fig. 6 (left), therefore, contains implicit information of the obstacles along the communication path. Consider the dashed ray (line) that corresponds to distance \(t\) and angle \(\theta\) in Fig. 6 (left). This line is at distance \(t\) from the origin and is perpendicular to the line that is at angle \(\theta\) with the x-axis. Let \(P(\theta, t)\) represent the received signal power in the transmission along the ray that corresponds to distance \(t\) and angle \(\theta\), as shown in Fig. 6 (left). We will have [50], [51],

\[
P(\theta, t) = P_s(\theta, t)w(\theta, t),
\]  

(14)
where
\[ P_s(\theta, t) = \frac{\beta P_T}{(d(\theta, t))^\alpha} \times e^{\sum_i r_i(\theta, t)n_i(\theta, t)} \]  
represents the contribution of distance-dependent path loss and shadowing. For the path loss term, \( P_T \) represents the transmitted power, \( d(\theta, t) \) is the distance between the transmitter and receiver across that ray, \( \alpha \) is the degradation exponent and \( \beta \) is a constant that is a function of system parameters. For the shadowing (or shadow fading) term, \( r_i \) is the distance travelled across the \( i \)th object along the \((\theta, t)\) ray and \( n_i < 0 \) is the decay rate of the wireless signal within the \( i \)th object. Furthermore, the summation is over the objects across that line. As can be seen, shadowing characterizes wireless signal attenuation as it goes through the obstacles along the transmission path and therefore contains information about the objects along that line.

\( w(\theta, t) \) of Eq. 14, on the other hand, is a positive random variable with unit average which models the impact of multipath fading. There are a number of well-established models for the distribution of \( w(\theta, t) \) in the communication literature [49]. Nakagami power distribution or its special cases such as Rician power or exponential are common models.\(^2\) For more mathematical details on wireless channel modeling, readers are referred to [49]–[51]. We can then model \( \ln P(\theta, t) \) as follows
\[ \ln P(\theta, t) = \ln P_T + \beta_{dB} - \alpha \ln d(\theta, t) + \sum_i r_i(\theta, t)n_i(\theta, t) + w_{dB}(\theta, t). \]  
where \( \beta_{dB} = \ln \beta \) and \( w_{dB} = \ln w(\theta, t) \). Then we have
\[ h(\theta, t) \triangleq \ln P(\theta, t) - \ln P_T - (\beta_{dB} - \alpha \ln d(\theta, t)) \]
\[ = \sum_i r_i(\theta, t)n_i(\theta, t) + w_{dB}(\theta, t). \]  
\(^2\)Rician is a possible distribution for \( \sqrt{w(\theta, t)} \). Thus we use the term “Rician power” to indicate that we are referring to the corresponding distribution for \( w(\theta, t) \).
Path loss and shadowing represent the signal degradation due to the distance travelled and obstacles respectively and $w_{dB}(\theta, t)$ represents the impact of multipath fading. By using an integration over the line that corresponds to $\theta$ and $t$, we can express Eq. 17 as follows:

$$h(\theta, t) = \int \int_{\text{line}(\theta, t)} g_n(u, v) du dv + w_{dB}(\theta, t),$$

where

$$g_n(u, v) = \begin{cases} n(u, v) & \text{if } g(u, v) = 1 \\ 0 & \text{else} \end{cases}$$

with $g(u, v)$ representing the binary map of the obstacles (indicated by Eq. 13) and $n(u, v)$ denoting the decay rate of the signal inside the object at position $(u, v)$ (see $n_i(\theta, t)$ in Eq. 15). $g_n(u, v)$ then denotes the true map of the obstacles including wireless decay rate information.

By changing $t$ at a specific $\theta$, a projection is formed, i.e. a set of ray integrals, as is shown in Fig. 6 (left). Let $G_n(\theta_t, f)$ represent the 2D Fourier transform of $g_n$ expressed in the polar coordinates. Let $H_t(\theta, f)$ denote the 1D Fourier transform of $h(\theta, t)$ with respect to $t$: $H_t(\theta, f) = \int h(\theta, t) e^{-j2\pi ft} dt$. We have the following theorem.

**Fourier Slice Theorem [46]:** Consider the case where there is no multipath fading in Eq. 18, i.e. $w_{dB} = 0$. Then $H_t(\theta, f)$, the Fourier transformation of $h(\theta, t)$ with respect to $t$, is equal to the samples of $G_n(\theta_f, f)$ across angle $\theta_f = \theta$.

Consider the illustrated line at angle $\theta$ that passes through the origin in Fig. 6 (left). Two robots can move outside the room such that the straight line between them is perpendicular to
the illustrated line at angle $\theta$. Then the receiving robot measures $P(\theta, t)$, the received signal power at different $ts$. In practice, the parameters of the path loss component of the received signal in Eq. 16 can be estimated by using a few Line Of Sight (LOS) transmissions in the same environment. Therefore, the impact of path loss can be removed and the receiving robot can calculate $h(\theta, t)$. By making a number of measurements at different $ts$ for a given $\theta$, the Fourier Slice Theorem allows us to measure the samples of the Fourier transform of the map $g_n$ at angle $\theta$. By changing $\theta$, we can sample the Fourier transform of the map of the obstacles at different angles. We can then pose the problem in a compressive sampling framework. By measuring the received signal power across the rays, the vehicles can indirectly sample the Fourier transformation of the obstacle map. Then the sparsity in the spatial or wavelet domain could be used for reconstruction. Both domains will result in a matrix $\Psi$ with good RIC properties.

Let $x$ of Eq. 1 denote the vector representation of $G_n$ (2D Fourier transform of the obstacle map), where the columns of $G_n$ are stacked up to form a vector. Then $y$ represents the very few samples of $G_n$ acquired using the proposed framework, i.e. wireless channel measurements across a number of rays and the Fourier Slice Theorem. By utilizing the sparse representation of the map in the spatial or wavelet domain, the vehicles can solve for the map cooperatively, based on minimal measurements, and more importantly in a non-invasive manner. For reconstruction based on sparsity in the spatial domain, $X$ will be the vector representation of $g_n$, $\Phi$ is as denoted in Eq. 12 and $\Gamma$ is the Fourier transform matrix. Then the $\Psi$ matrix that results from point sampling in the frequency domain and reconstruction using sparsity in the spatial domain will meet the RIC condition [29]. Since the changes in the map are typically sparser than the map itself, another approach is to consider and minimize the variations in $X$. This approach is referred to as Total Variation (TV) [18]. Typically, an obstacle map is also considerably sparse in the wavelet domain (as shown in Fig. 2 (right)). Let $X$ represent the vector representation of the wavelet transform of the obstacle map. Then, $\Gamma = F \times W^{-1}$ and $\Phi$ is as described earlier. $F$ and $W$ represent the 2D Fourier and wavelet matrices such that when applied to a vector that is formed by stacking the columns of a 2D map, they result in the vector representation of the 2D Fourier and wavelet transform of the map respectively. Since $W^H = W^{-1}$, the RIC property remains the same as the case where $\Gamma = F$, which has been shown to meet the RIC condition [29].

Fig. 6 (middle and right) shows our results for non-invasive compressive mapping of the
obstacles of the left figure. For this result, no multipath fading is simulated. We will discuss and show the impact of multipath fading in the next part. It can be seen that with only 11.7% measurements (right figure), the map can be built almost perfectly. Note that 11.7% measurements implies that 11.7% of the 2D Fourier transformation of the map is only sampled. Even with 4% measurements (middle figure), the reconstruction is very close to the original. Fig. 8 shows the normalized MSE of the reconstruction of the obstacle map of Fig. 6 (left) as a function of the percentage of the measurements taken. It can be seen that a cooperative network can build a high-quality and non-invasive map of obstacles with a considerably small set of measurements.

2) Impact of Multipath Fading: So far we proposed a compressive mapping framework for non-invasive mapping of the obstacles. In this part we discuss the impact of multipath fading component of Eq. 14 on the proposed non-invasive reconstruction. Multipath fading can result in some noise in the observations, as can be seen from Eq. 18. We next add multipath fading with Rician distribution to each wireless reception, in a simulation environment. Rician fading is the most commonly considered distribution for multipath fading. This means that $w_{sr} = \sqrt{w(\theta,t)}$ has the following distribution (see Eq. 14):

$$p(w_{sr}) = \frac{w_{sr}}{\sigma^2} \exp\left(-\frac{(w_{sr}^2 + \nu^2)}{2\sigma^2}\right)I_0\left(\frac{w_{sr}\nu}{\sigma^2}\right),$$

where $I_0(.)$ is the zeroth order modified Bessel function of the first kind, $\nu^2$ is the power of the Line Of Sight (LOS) component and $2\sigma^2$ is the power of the multipath terms. Then the multipath fading parameter is characterized as follows: $\rho = \frac{\nu^2}{2\sigma^2}$, which describes how strong the impact
of multipath fading is. Fig. 9 shows the normalized MSE of reconstruction in the presence of Rician multipath fading with $\rho = 1$ and $\rho = 2.34$. As compared to Fig. 8, it can be seen that the error is higher as expected and is lower bounded by the effect of multipath fading.

We should note that the aforementioned modeling of a wireless channel can not possibly embrace all the propagation phenomena. Still, the results of this section are promising. We are currently working on the experimental implementation of the proposed approach to test its feasibility and performance. The effect of multipath fading can also be reduced by using directional antennas as well as averaging the received signal over a very small distance. It should be noted that the compressive sensing framework enables the possibility of non-invasive mapping in ways that was not feasible beforehand. By utilizing the proposed approach, the map can be built with a considerably small set of measurements. This then allows for more measurements to go towards averaging over multipath fading. In our previous work [53]–[57], we have also developed other multipath fading mitigation techniques in the context of mobile communications. Such approaches can also be utilized to develop a framework where the vehicles cooperatively and compressively learn the impact of all the obstacles (not only the ones along the communication path) and remove the effect of interference (caused by multipath) from their received signals. Such approaches can improve the quality of the proposed reconstruction technique even further.
C. Compressive Cooperative Mapping of Communication Signal Strength

In this section, we show how a group of mobile nodes can build a high-quality map of the communication signal strength cooperatively and with minimal sensing. Building a map of the communication signal strength is considerably important for the robust operation of several emerging networked robotics and control applications. In order to maintain connectivity, mobile nodes need to have an estimation of the communication signal strength in locations they have not yet visited. Currently, there is no framework for mapping the communication signal strength with very few measurements. In this section we show how the proposed compressive mapping framework can be utilized to estimate the spatial variations of the communication signal strength with very few observations. Our analysis of the spatial variations of several channel measurements has shown that a wireless channel is considerably compressible in the Fourier domain. For instance, the solid line of Fig. 10 shows the measurement of a channel along a street in San Francisco [58]. For this channel, 99.99% of its energy is in 4.6% of its Fourier coefficients. The dashed line shows the sparsified version of the channel, where only the strongest 4.6% of the Fourier coefficients are kept. The two curves are almost identical and thus the spatial variations of the channel are compressible, i.e. a small percentage of the Fourier coefficients.
coefficients suffices for capturing the signal. Then $x$ of Eq. 1 represents the spatial map of the channel (received signal power). The vehicles make measurements in the spatial domain, i.e. vector $y$ consists of the few measurements made by the vehicles. Consequently, matrix $\Phi$ will be as denoted in Eq. 12 and $\Gamma$ is the inverse Fourier matrix.

1) Reconstruction of a Sparsified Channel: Eq. 7 shows that for a sparse signal, if the number of measurements is above a certain level, the reconstruction can be perfect. In order to see this and get more insight into compressive sensing, we first consider reconstructing the sparsified version of the channel. As noted earlier, a wireless channel is compressible in the Fourier domain but not sparse. Therefore, our goal for reconstructing the sparsified version is to merely show the implication of Eq. 7. We will then proceed with reconstructing the channel itself.

The circle line of Fig. 11 shows the result of reconstructing the sparsified version of the signal of Fig. 10 based on a varying number of random observations, $K$. It should be noted that the channel measurements of Fig. 10 naturally contain the measurement noise. The size of the signal of Fig. 10 is $N = 1024$. Fig. 11 then shows the average of the normalized Mean Square Error (MSE), averaged over 1000 iterations with random samples, as a function of the number of measurements ($K$). The reconstruction method used for this figure is OMP. It can be seen that after a certain number of measurements are collected ($K = 270$ or 26% in this case), the construction becomes perfect (or bounded by computational errors) as predicted by Eq. 7.

Fig. 12 shows another real-world channel measurement in San Francisco over a longer distance...
along with its sparsified version. Due to the longer distance, this channel exhibits more non-
stationary behavior, as can be seen. The length of the channel is $N = 4096$ in this case. Our
Fourier analysis showed that more than 99.995% of the energy of this measured signal is in less
than 2.5% of its Fourier coefficients. Then the dashed line shows the sparsified version of the
channel, where only the strongest 2.5% of the Fourier coefficients are kept. The dashed line of
Fig. 11 shows the performance of compressive mapping in reconstructing the sparsified version
of this channel. While the size of this signal is 4 times that of Fig. 10, the number of required
observations ($K$) for perfect recovery is less than 2 times, as can be seen from Fig. 11. Overall,
it can be seen that the compressive sensing approach can reconstruct the signal with a small
number of measurements.

Fig. 12. Another channel measurement along a street in San Francisco (courtesy of Mark Smith [58]) and its sparsified version. The two curves are almost identical.

2) Reconstruction of Non-Sparsified Channels: The previous section showed the strength of
compressive sensing in reconstructing the spatial variations of the channel based on a consid-
erably small measurement set. However, we showed the results for the sparsified version of
the channel where it was possible to represent the signal with only a small number of Fourier
coefficients (the rest of the coefficients were zero as opposed to negligible). Fig. 13 shows the
reconstruction of the measured channels of Fig. 10 and 12. It can be seen that the normalized
MSE of both cases is considerably low. However, the error can not be as low as that of Fig. 11 due to the fact that the signal is compressible but not sparse. As expected, the channel of Fig. 12 has a higher MSE for the same number of measurements due to its longer length. We used OMP for these reconstructions. Based on our experience, ROMP’s performance is considerably worse than OMP and I-ROMP. Depending on the sampling positions, I-ROMP could possibly perform better than OMP. For instance, if the samples are purely random (as was the case in this part), I-ROMP performs slightly better than OMP. On the other hand, for the cases where the samples are along line trajectories, as was discussed in Section IV-A (see Fig. 4 and 5), I-ROMP’s performance degrades and OMP or Basis Pursuit-type approaches should be used. Another example is the case where the samples are mainly distributed in a part of the channel with no samples in other parts. For instance, if all the samples are in the first half of the channel with no samples in the rest of the signal, OMP performs considerably better than I-ROMP. Overall, our results show that the proposed compressive mapping framework can build the spatial variations of the channel based on a considerably incomplete observation set. While we considered 1D channels in this section, similar results can be achieved for 2D channels. Furthermore, similar to the results in Section IV-A, reconstruction of a 2D channel using line trajectories can be achieved in a similar manner.
D. A Note on the Decentralized Nature of Compressive Mapping

It should be noted that the nature of our proposed compressive mapping framework is reconstruction based on minimal sensing. Therefore, it naturally lends itself to decentralized approaches where every node can estimate the map of interest based on its own observations as well as the observations of whichever node it can receive information from. This is particularly important in mobile cooperative networks since they typically lack a leader and the underlying graph of the network is not necessarily fully connected.

V. CONCLUSIONS

In this paper we considered a mobile network that is tasked with building a map of the spatial variations of a parameter in its environment. We proposed a new framework that allows the nodes to build a map of the parameter of interest with a small number of measurements. By using the recent results in the area of compressive sampling, we showed how the nodes can exploit the sparse representation of the parameter of interest in order to build a map with minimal sensing, and without directly sensing a large percentage of the area. We showed the application of our proposed framework to aerial mapping, mapping of the obstacles as well as mapping of the communication signal strength. Our simulation results showed the superior performance of the proposed framework.

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